Classification of $C$—integrable multilinear equations defined on a three or four-points lattice

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(joint work with R. Hernandez Heredero and D. Levi)

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Outline of the Seminar

- **Part 1**: Functional techniques:
  - **Part 1.1**: Classification of *complex, multilinear* discrete equations defined on *three points* linearizable by a *point transformation*;
  - **Part 1.2**: Classification of *complex, multilinear* discrete equations defined on *three points* linearizable by a *Cole-Hopf transformation*;
  - **Part 1.3**: Classification of *complex, multilinear* discrete equations defined on a *square* linearizable by a *point transformation*;
  - **Part 1.4**: Classification of *real, quadratic* discrete equations defined on a *square* linearizable by a *Cole-Hopf transformation* to a *homogeneous* lin. eq..

- **Part 2**: Multiscale analysis:
  - Classification of linearizable *dispersive, real, multilinear* discrete equations defined on a *square*.

- **Bibliography**.
Equivalence problem: Given a class of eqs. (continuous or discrete)

\[ \mathcal{E}(u_{0,0}, u_{1,0}, u_{0,1}, \ldots; \alpha, \beta, \ldots) = 0, \quad u_{1,0}, u_{0,1}, \ldots: \text{der., sh.; } \alpha, \beta, \ldots: \text{par.} \]

find all possible transformations \( g(u_{0,0}, u_{1,0}, u_{0,1}, \ldots, \tilde{u}_{0,0}, \tilde{u}_{1,0}, \tilde{u}_{0,1}, \ldots) = 0 \), so that the equation is covariant

\[ \mathcal{E}\left(\tilde{u}_{0,0}, \tilde{u}_{1,0}, \tilde{u}_{0,1}, \ldots; \tilde{\alpha}, \tilde{\beta}, \ldots\right) = 0. \]

Particular cases:

- Invariance: \( \mathcal{E}(\tilde{u}_{0,0}, \tilde{u}_{1,0}, \tilde{u}_{0,1}, \ldots; \alpha, \beta, \ldots) = 0 \Rightarrow \) auto-Backlund, symmetry!
- Linearization: \( \mathcal{E}\left(\tilde{u}_{0,0}, \tilde{u}_{1,0}, \tilde{u}_{0,1}, \ldots; \tilde{\alpha}, \tilde{\beta}, \ldots\right) \) is linear \( \Rightarrow \) linearizability!

Linearization not as an equivalence problem: no linear eq. belongs to class \( \mathcal{E} \).

Example: Two-points multilin. eq.: \( au_0 + bu_1 + cu_0u_1 + d = 0, \ c \neq 0. \)
Translation \( \Rightarrow d = 0; \) Inversion \( \Rightarrow \tilde{a}w_0 + \tilde{b}w_1 + c = 0: \) always linearizable!
**Example:** Ordinary multilinear differential eq.: \( au + bu_x + cuu_x + d = 0, \ c \neq 0, \ ab - cd \neq 0 \) (irreducibility). Translation, scaling \( \Rightarrow \tilde{a}w + wu_x + 1 = 0. \)

- \( \tilde{a} = 0 \Rightarrow (w^2)_x + 2 = 0. \ w^2 \div v \Rightarrow v_x + 2 = 0: \text{linearizable!}; \)
- \( \tilde{a} \neq 0. \) Assume there exists a linearizing autonomous, point transformation

\[
\nu = f(w), \quad \frac{df(w)}{dw} \neq 0,
\]

to the linear equation

\[
\alpha \nu + \beta \nu_x + \gamma = 0, \ \beta \neq 0. \ \text{Point trans.} \Rightarrow \tilde{\nu}_x + \tilde{\gamma} = 0 \text{ most general.}
\]

\[
(1 + \tilde{a}w) \frac{df(w)}{dw} - w\tilde{\gamma} = 0 \Rightarrow f(w) = \frac{\tilde{\gamma}}{\tilde{a}^2} [1 + \tilde{a}w - \log (1 + \tilde{a}w)] + c.
\]

Choose: \( \tilde{\gamma} = \tilde{a}^2, \ c = -1: \text{linearizability to } \tilde{\nu}_x + \tilde{a}^2 = 0 \text{ by point, non inv.} \)

\( \tilde{\nu} = \tilde{a}w - \log (1 + \tilde{a}w)! \)
Part 1: Linearizability through direct functional techniques
(joint work with D. Levi)
Consider a class $\mathcal{E} (u_0, u_1, u_0, u_1) = 0$ of equations defined on three points where:

- $u_{0,0} \doteq u_{n,m}$ is a complex function;
- The equations in the class are autonomous;
- You can solve with respect to (at least one of) the three variables

\begin{align}
    u_{0,0} &= F (u_{1,0}, u_{0,1}), \quad F_{,u_{1,0}} \neq 0, \quad F_{,u_{0,1}} \neq 0, \quad (1) \\
    u_{1,0} &= G (u_{0,0}, u_{0,1}), \quad G_{,u_{0,0}} \neq 0, \quad G_{,u_{0,1}} \neq 0, \quad (2) \\
    u_{0,1} &= S (u_{0,0}, u_{1,0}), \quad S_{,u_{0,0}} \neq 0, \quad S_{,u_{1,0}} \neq 0; \quad (3)
\end{align}

Assume there exists a linearizing autonomous, point transformation

\begin{equation}
    \tilde{u}_{0,0} \doteq f_{0,0} (u_{0,0}), \quad \frac{df (x)}{dx} \neq 0, \quad (4)
\end{equation}

to the linear equation (this is the most general!)

\begin{equation}
\tilde{u}_{0,0} + b \tilde{u}_{1,0} + c \tilde{u}_{0,1} + d = 0, \quad b, c, d \in \mathbb{C}, \quad b, c \neq 0; \quad (5)
\end{equation}
Lineariz. three-points equations by a point trans.

Choose as independent variables $u_{1,0}$ and $u_{0,1}$ so that

$$f_{0,0} \big|_{u_{0,0} = F} + bf_{1,0} + cf_{0,1} + d = 0,$$  \hspace{1cm} (6)

must be \textit{identically} satisfied $\forall u_{1,0}$ and $u_{0,1}$, differentiate with respect to $u_{1,0}$ and $u_{0,1}$ and take the (principal value of the) logarithm

$$\log \left. \frac{df_{1,0}}{du_{1,0}} \right|_{u_{0,0} = F} - \log \left. \frac{df_{0,0}}{du_{0,0}} \right|_{u_{0,0} = F} = \log \left( -\frac{F, u_{1,0}}{\beta} \right) \pmod{2\pi i},$$  \hspace{1cm} (7a)

$$\log \left. \frac{df_{0,1}}{du_{0,1}} \right|_{u_{0,0} = F} - \log \left. \frac{df_{0,0}}{du_{0,0}} \right|_{u_{0,0} = F} = \log \left( -\frac{F, u_{0,1}}{\gamma} \right) \pmod{2\pi i};$$  \hspace{1cm} (7b)

Introduce a linear operator $\mathcal{A}$:

$$\mathcal{A} \doteq \frac{\partial}{\partial u_{1,0}} - \frac{F, u_{1,0}}{F, u_{0,1}} \frac{\partial}{\partial u_{0,1}},$$  \hspace{1cm} (8)

such that $A\phi(F) = 0$, $\phi$ arbitrary function of its argument, and apply it to (7)
Lineariz. three-points equations by a point trans.

\[
\frac{d}{du_{1,0}} \log \frac{df_{1,0}}{du_{1,0}} = \frac{W(u_{1,0}) \left[ F_{,u_{0,1}} ; F_{,u_{1,0}} \right]}{F_{,u_{1,0}} F_{,u_{0,1}}},
\]

(9a)

\[
\frac{d}{du_{0,1}} \log \frac{df_{0,1}}{du_{0,1}} = \frac{W(u_{0,1}) \left[ F_{,u_{1,0}} ; F_{,u_{0,1}} \right]}{F_{,u_{1,0}} F_{,u_{0,1}}},
\]

(9b)

where \( W(x) [f; g] \equiv fg_{,x} - gf_{,x} \), i.e. a Wronskian;

Left hand members of (9) depend only on one variable, while the right hand members depend on both \( u_{1,0} \) and \( u_{0,1} \):

First linearizability necessary condition!

- Require right-hand members be independent of the extra variable:

\[
\frac{\partial}{\partial u_{1,0}} \frac{\partial}{\partial u_{0,1}} \log \frac{F_{,u_{1,0}}}{F_{,u_{0,1}}} = 0;
\]

(10)
Solution:

\[
F,_{u_1,0} = \frac{d\mathcal{H}(u_{1,0})}{du_{1,0}} F,_{u_0,1}, \quad \frac{d\mathcal{H}(u_{1,0})}{du_{1,0}} \neq 0, \quad \frac{d\tilde{\mathcal{H}}(u_{0,1})}{du_{0,1}} \neq 0, \quad (11)
\]

\(\mathcal{H}(u_{1,0}), \tilde{\mathcal{H}}(u_{0,1})\) arbitrary functions of their argument;

Insert (11) into (9), integrate once, exponentiate and shift everything to \(u_{0,0}\):

\[
\frac{df_{0,0}}{du_{0,0}} = \rho \frac{d\mathcal{H}(u_{0,0})}{du_{0,0}}, \quad (12a)
\]

\[
\frac{df_{0,0}}{du_{0,0}} = \tilde{\rho} \frac{d\tilde{\mathcal{H}}(u_{0,0})}{du_{0,0}}, \quad (12b)
\]

\(\rho, \tilde{\rho} \neq 0\) arbitrary integration constants;
Lineariz. three-points equations by a point trans.

Second linearizability necessary condition!

- Require compatibility:

\[
\frac{d}{dx} A(x) = 0, \quad A(x) \equiv \frac{F_{,u_{1,0}}}{F_{,u_{0,1}}} \bigg|_{u_{1,0}=u_{0,1}=x} = \frac{d \mathcal{H}(x)}{dx} \neq 0; \tag{13}
\]

Do again everything choosing as indep. vars. \(u_{0,0}\) and \(u_{0,1}\) or \(u_{0,0}\) and \(u_{1,0}\) and introducing respectively the operators:

\[
\mathcal{B} \doteq \frac{\partial}{\partial u_{0,0}} - \frac{G_{,u_{0,0}}}{G_{,u_{1,1}}} \frac{\partial}{\partial u_{0,1}}, \quad \mathcal{C} \doteq \frac{\partial}{\partial u_{0,0}} - \frac{S_{,u_{0,0}}}{S_{,u_{1,0}}} \frac{\partial}{\partial u_{1,0}}; \tag{14}
\]

Extra linearizability necessary conditions!

- Independence of extra variable:

\[
\frac{\partial}{\partial u_{0,0}} \frac{\partial}{\partial u_{0,1}} \log \frac{G_{,u_{0,0}}}{G_{,u_{1,1}}} = \frac{\partial}{\partial u_{0,0}} \frac{\partial}{\partial u_{1,0}} \log \frac{S_{,u_{0,0}}}{S_{,u_{1,0}}} = 0; \tag{15}
\]
Lineariz. three-points equations by a point trans.

Extra linearizability necessary conditions!

Compatibility:

\[
\frac{d}{dx} B(x) = 0, \quad B(x) \doteq \frac{G,u_{0,0}}{G,u_{0,1}} \bigg|_{u_{0,0}=u_{0,1}=x} = \frac{d\mathcal{J}(x)}{dx} \neq 0, \quad (16a)
\]

\[
\frac{d}{dx} C(x) = 0, \quad C(x) \doteq \frac{S,u_{0,0}}{S,u_{1,0}} \bigg|_{u_{0,0}=u_{1,0}=x} = \frac{d\mathcal{R}(x)}{dx} \neq 0, \quad (16b)
\]

\[
\frac{d}{dx} D(x) = 0, \quad D(x) \doteq \frac{d\mathcal{R}(x)}{d\mathcal{J}(x)} \neq 0, \quad (16c)
\]

\[
\frac{d}{dx} E(x) = 0, \quad E(x) \doteq \frac{d\mathcal{R}(x)}{d\mathcal{H}(x)} \neq 0. \quad (16d)
\]
All linearizability conditions must be \textit{identically satisfied} \( \forall x \! \); 

The three conds. (13, 16a, 16b), are expressed only in terms of \( F, \) \( G \) and \( S \): \textit{can always} be used!;

The two conds. (16c, 16d) can be used \textit{only when} one can calculate \textit{explicitly} \( \frac{d\tilde{R}}{dx}, \frac{dJ}{dx} \) and \( \frac{dH}{dx} \);

\begin{align*}
\text{Evolution eq. of the linearizing trans.} \\
\frac{d}{du_{1,0}} \log \frac{df_{1,0}}{df_{1,0}} = \frac{W(u_{1,0}) \left[ F_{,u_{0,1}}; F_{,u_{1,0}} \right]}{F_{,u_{1,0}} F_{,u_{0,1}}}, & \quad (17a) \\
\rho = \rho \mathcal{H}(u_{0,0}) + \sigma, \quad \sigma \text{ arb. const.} & \quad (17b)
\end{align*}

\[ \tilde{u}_{0,1} = g_{0,0}(u_{0,0}) \tilde{u}_{0,0}, \quad \frac{dg(x)}{dx} \neq 0, \quad (18) \]

to the linear equation (this is the *most general* for a three-points eq.!

\[ \tilde{u}_{0,0} + \tilde{u}_{1,0} + \tilde{u}_{0,1} = 0; \quad (19) \]

Insert (18) into (19):

\[ \tilde{u}_{1,0} = -\tilde{u}_{0,0} [1 + g_{0,0}(u_{0,0})]; \quad (20) \]

From compatibility: *nonlinear equation* for \( u_{0,0} \):

\[ \frac{1 + g_{0,0}(u_{0,0})}{g_{0,0}(u_{0,0})} = \frac{1 + g_{0,1}(u_{0,1})}{g_{1,0}(u_{1,0})}; \quad (21) \]
Choose as indep. vars. \( u_{1,0} \) and \( u_{0,1} \), take the (princip. val. of the) log., differentiate with resp. to \( u_{1,0} \) and \( u_{0,1} \), take again the (princip. val. of the) log. and apply the operator \( A \) def. in (8)

\[
\frac{d}{du_{1,0}} \log \frac{d}{du_{1,0}} \log g_{1,0} - \frac{d}{du_{1,0}} \log \frac{F_{,u_{1,0}}}{F_{,u_{0,1}}} = 0, (22a)
\]

\[
\frac{d}{du_{0,1}} \log \frac{d}{du_{0,1}} \log (1 + g_{0,1}) - \frac{d}{du_{0,1}} \log \frac{F_{,u_{0,1}}}{F_{,u_{1,0}}} = 0; (22b)
\]

Do again everything choosing as indep. vars. \( u_{0,0} \) and \( u_{0,1} \) or \( u_{0,0} \) and \( u_{1,0} \), introducing resp. the operators \( B \) or \( C \) def. by (14):

Lineariz. nec. conds. from indep. of extra variable!

\[
\frac{\partial}{\partial u_{1,0}} \frac{\partial}{\partial u_{0,1}} \log \frac{F_{,u_{1,0}}}{F_{,u_{0,1}}} = \frac{\partial}{\partial u_{0,0}} \frac{\partial}{\partial u_{0,1}} \log \frac{G_{,u_{0,0}}}{G_{,u_{0,1}}} = \frac{\partial}{\partial u_{0,0}} \frac{\partial}{\partial u_{1,0}} \log \frac{S_{,u_{0,0}}}{S_{,u_{1,0}}} = 0;
\]
Lineariz. three-points equations by a Cole-Hopf

Lineariz. nec. conds. from compatibility!

\[
\frac{\alpha_2}{\alpha_1} - A(x) = \frac{\alpha_5}{\alpha_6} A(x) \cdot C(x), \quad \alpha_i \neq 0 \text{ arb. consts.}, \quad (23a)
\]

\[
\frac{d}{dx} D(x) = 0, \quad D(x) \equiv \frac{d^2 R(x)}{dx^2} \neq 0, \quad (23b)
\]

\[
\frac{d}{dx} E(x) = 0, \quad E(x) \equiv \frac{d^2 \tilde{R}(x)}{dx^2} \neq 0, \quad (23c)
\]

\[
\frac{d}{dx} L(x) = 0, \quad L(x) \equiv \frac{d^2 \tilde{J}(x)}{dx^2} \neq 0, \quad (23d)
\]

\[
\frac{d}{dx} A(x) = \alpha_5 A(x) \frac{d}{dx} R(x). \quad (23e)
\]
The functions $A(x), C(x), \mathcal{H}(x), \tilde{\mathcal{H}}(x), J(x), \tilde{J}(x), R(x), \tilde{R}(x)$, are the same as previously def. for point trans..

- All linearizability conditions must be *identically satisfied* $\forall x$!
- The cond. (23a), is expr. only in terms of $F$ and $S$: *can always* be used!
- The four conds. (23b, 23c, 23d, 23e) can be used *only when* one can calculate *explicitly* $d\tilde{R}/dx, d\tilde{J}/dx$ and $d\tilde{H}/dx$;

### Evolution eq. of the linearizing trans.

\[
\frac{d}{du_{1,0}} \log \frac{d}{du_{1,0}} \log g_{1,0} = \frac{W(u_{1,0}) \left[ F, u_{0,1} ; F, u_{1,0} \right]}{F, u_{1,0} F, u_{0,1}}, \tag{24a}
\]

\[
g(u_{0,0}) = \beta_1 e^{\alpha_1 \mathcal{H}(u_{0,0})}, \quad \beta_1 \neq 0 \text{ arb. const.} \tag{24b}
\]
Lineariz. multilinear three-points eqs.: structure

\textbf{Multilinear three-points equation:}

\[
\tilde{h} + \tilde{a}w_{0,0} + \tilde{b}w_{0,1} + \tilde{c}w_{1,0} + \tilde{d}w_{0,0}w_{1,0} + \\
+ \tilde{e}w_{0,0}w_{0,1} + \tilde{f}w_{1,0}w_{0,1} + \tilde{g}w_{0,0}w_{1,0}w_{0,1} = 0,
\]

\(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{f}, \tilde{g}\) and \(\tilde{h}\), complex parameters; \(w_{n,m}\) complex function;

\[
\frac{d\mathcal{H}(x)}{dx} = \frac{\kappa_1}{\tilde{e}\tilde{h} - \tilde{a}\tilde{b} + (\tilde{c}\tilde{e} - \tilde{b}\tilde{d} - \tilde{a}\tilde{f} + \tilde{g}\tilde{h})x + (\tilde{c}\tilde{g} - \tilde{d}\tilde{f})x^2},
\]

\[
\frac{d\tilde{\mathcal{H}}(x)}{dx} = \frac{\kappa_1}{\tilde{d}\tilde{h} - \tilde{a}\tilde{c} + (\tilde{b}\tilde{d} - \tilde{c}\tilde{e} - \tilde{a}\tilde{f} + \tilde{g}\tilde{h})x + (\tilde{b}\tilde{g} - \tilde{e}\tilde{f})x^2},
\]

\[
\frac{d\mathcal{J}(x)}{dx} = \frac{\kappa_2}{\tilde{f}\tilde{h} - \tilde{b}\tilde{c} + (\tilde{g}\tilde{h} - \tilde{c}\tilde{e} - \tilde{b}\tilde{d} + \tilde{a}\tilde{f})x + (\tilde{a}\tilde{g} - \tilde{d}\tilde{e})x^2},
\]

\[
\frac{d\tilde{\mathcal{J}}(x)}{dx} = \frac{\kappa_2}{\tilde{d}\tilde{h}} \frac{d\tilde{\mathcal{H}}(x)}{dx}, \quad \frac{d\mathcal{R}(x)}{dx} = \frac{\kappa_3}{\kappa_2} \frac{d\mathcal{J}(x)}{dx}, \quad \frac{d\tilde{\mathcal{R}}(x)}{dx} = \frac{\kappa_3}{\kappa_1} \frac{d\mathcal{H}(x)}{dx},
\]

\(\kappa_i, i = 1, \ldots, 3\) arbitrary complex parameters.
Point transformation: to impose only necessary conds. (13) and (16b); the remaining are \textit{identically satisfied} or are \textit{not independent}:

\[
\frac{d}{dx} A(x) = 0, \quad A(x) \equiv \left. \frac{F_{,u_{1,0}}}{F_{,u_{0,1}}} \right|_{u_{1,0}=u_{0,1}=x} = \frac{d\mathcal{H}(x)}{dx},
\]

\[
\frac{d}{dx} C(x) = 0, \quad C(x) \equiv \left. \frac{S_{,u_{0,0}}}{S_{,u_{1,0}}} \right|_{u_{0,0}=u_{1,0}=x} = \frac{d\mathcal{R}(x)}{dx};
\]

Cole-Hopf transformation: to impose only necessary conds. (23a) and (23e); the remaining are \textit{identically satisfied} or are \textit{not independent}:

\[
\frac{\alpha_2}{\alpha_1} - A(x) = \frac{\alpha_5}{\alpha_6} A(x) \cdot C(x),
\]

\[
\frac{d}{dx} A(x) = \alpha_5 A(x) \frac{d}{dx} R(x);
\]

Translation and inversion \(\Rightarrow\) reduction to quadratic forms: \(\tilde{h} = \tilde{g} = 0\) or \(\tilde{a} + \tilde{b} + \tilde{c} = \tilde{d} + \tilde{e} + \tilde{f} = \tilde{g} = 0\) and \(\tilde{h} = 1\).
The only linearizable eqs. in our class are (up to Möbius and exchange \( n \leftrightarrow m \)):

\[
\begin{align*}
  w_{0,0}w_{1,0}w_{0,1} - 1 &= 0, \\
  w_{0,0}w_{0,1} - w_{1,0} &= 0, \\
  w_{1,0}w_{0,1} - w_{0,0} &= 0,
\end{align*}
\]

lineariz. by \( \tilde{u}_{0,0} = \log w_{0,0} \) (log princip. branch of complex log.), respec. to

\[
\begin{align*}
  \tilde{u}_{0,0} + \tilde{u}_{1,0} + \tilde{u}_{0,1} &= 2\pi iz, \quad z = -1, 0, 1, \\
  \tilde{u}_{0,0} - \tilde{u}_{1,0} + \tilde{u}_{0,1} &= 0, \\
  \tilde{u}_{0,0} - \tilde{u}_{1,0} - \tilde{u}_{0,1} &= 0,
\end{align*}
\]

and \( w_{1,0}w_{0,1} + \beta w_{0,0}w_{0,1} + \gamma w_{0,0}w_{1,0} + \delta w_{0,0}w_{1,0}w_{0,1} = 0, \beta, \gamma \neq 0, \delta = 0, 1, \)

lineariz. by \( \tilde{u}_{0,0} = 1/w_{0,0} \) to

\[
\begin{align*}
  \tilde{u}_{0,0} + \beta \tilde{u}_{1,0} + \gamma \tilde{u}_{0,1} + \delta &= 0.
\end{align*}
\]
Theorem of lineariz. by a Cole-Hopf trans.: The only linearizable eq. in our class is (up to Möbious and exchange $n \leftrightarrow m$):

$$(1 + w_{0,0}) w_{1,0} - (1 + w_{0,1}) w_{0,0} = 0,$$

lineariz. by $\tilde{u}_{0,1} = w_{0,0} \tilde{u}_{0,0}$ (eventually $\tilde{u}_{1,0} = w_{0,0} \tilde{u}_{0,0}$) to

$$\tilde{u}_{0,0} + \tilde{u}_{1,0} + \tilde{u}_{0,1} = 0.$$
Consider a class $\mathcal{E}(w_{0,0}, w_{1,0}, w_{0,1}, w_{1,1}) = 0$ of equations defined on the square where:

- $w_{0,0} \doteq w_{n,m}$ is a complex function;
- The equations in the class are autonomous;
- You can solve with respect to $w_{1,1}$ ($w_{0,0}$, $w_{1,0}$ and $w_{0,1}$ are now independent)

$$w_{1,1} = F(w_{0,0}, w_{1,0}, w_{0,1}), \quad F_{,w_{0,0}} \neq 0 \quad F_{,w_{1,0}} \neq 0 \quad F_{,w_{0,1}} \neq 0;$$

Assume there exists a linearizing autonomous, point transformation

$$\tilde{u}_{0,0} = f_{0,0}(w_{0,0}), \quad \frac{df(x)}{dx} \neq 0,$$  \hspace{1cm} (26)$$

to the linear equation (this is the most general!)

$$\tilde{u}_{0,0} + b\tilde{u}_{1,0} + c\tilde{u}_{0,1} + d\tilde{u}_{1,1} + e = 0, \quad b, c, d, e \in \mathbb{C}, \quad b, c, d \neq 0.$$  \hspace{1cm} (27)
Linearizable equations on the square by a point trans.  

Linearizability conditions and transformation

\[
\frac{F_{u_0,0}}{F_{u_1,0}}\bigg|_{u_0,0=u_1,0=x} = \frac{1}{b}, \quad \text{identically } \forall x, u_0,1, \tag{28a}
\]

\[
\frac{F_{u_0,0}}{F_{u_0,1}}\bigg|_{u_0,0=u_1,0=x} = \frac{1}{c}, \quad \text{identically } \forall x, u_1,0, \tag{28b}
\]

\[
\frac{F_{u_0,1}}{F_{u_1,0}}\bigg|_{u_1,0=u_0,1=x} = \frac{c}{b}, \quad \text{identically } \forall x, u_0,0, \tag{28c}
\]

\[
\frac{\partial}{\partial u_{0,1}} \frac{F_{u_0,0}}{F_{u_1,0}} = \frac{\partial}{\partial u_{1,0}} \frac{F_{u_0,0}}{F_{u_0,1}} = \frac{\partial}{\partial u_{0,0}} \frac{F_{u_0,1}}{F_{u_1,0}} = 0, \quad \text{identically } \forall u_0,0, u_1,0, u_0,1; \tag{28d}
\]

- Point transformation:

\[
\frac{d}{dw_{0,1}} \log \frac{df_{0,1}}{dw_{0,1}} = \frac{1}{F_{w_1,0} F_{w_0,1}} W(w_{0,1}) \left[ F_{w_1,0}; F_{w_0,1} \right]. \tag{29}
\]
Multilinear quad-eqs. by a point trans.: class. results

- Translation and inversion $\Rightarrow$ reduction to cubic forms: $a^{(0)} = a^{(4)} = 0$ or
  $\sum_j a_j^{(1)} = \sum_j a_j^{(2)} = \sum_j a_j^{(3)} = a^{(4)} = 0$ and $a^{(0)} = 1$;

Theorem of lineariz. by a point trans.: The only eqs. lineariz. by a point trans. are (up to Möbius and exchange $n \leftrightarrow m$):

\[
\begin{align*}
  w_{0,0} w_{1,0} w_{0,1} w_{1,1} & - 1 = 0, \\
  w_{0,0} - w_{1,0} w_{0,1} w_{1,1} & = 0, \\
  w_{1,0} - w_{0,0} w_{0,1} w_{1,1} & = 0, \\
  w_{1,1} - w_{0,0} w_{1,0} w_{0,1} & = 0, \\
  w_{0,0} w_{1,0} - \theta w_{0,1} w_{1,1} & = 0, \quad \theta \neq 0, \\
  w_{0,0} w_{1,1} - \theta w_{1,0} w_{0,1} & = 0, \quad \theta \neq 0, \\
  w_{0,1} w_{1,1} (w_{1,0} + \beta w_{0,0}) + w_{0,0} w_{1,0} (\gamma w_{1,1} + \delta w_{0,1}) + \varepsilon w_{0,0} w_{1,0} w_{0,1} w_{1,1} & = 0, \\
  \varepsilon = 0, 1, \quad \beta, \gamma, \delta \neq 0 \in \mathbb{C};
\end{align*}
\]
They are lineariz. respectively to the eqs.:

\[\tilde{u}_{0,0} + \tilde{u}_{1,0} + \tilde{u}_{0,1} + \tilde{u}_{1,1} = 2\pi i z, \quad z = -1, 0, 1, 2,\]
\[\tilde{u}_{0,0} - \tilde{u}_{1,0} - \tilde{u}_{0,1} - \tilde{u}_{1,1} = 2\pi i z, \quad z = 0, 1,\]
\[\tilde{u}_{0,0} - \tilde{u}_{1,0} + \tilde{u}_{0,1} + \tilde{u}_{1,1} = 2\pi i z, \quad z = 0, 1,\]
\[\tilde{u}_{0,0} + \tilde{u}_{1,0} + \tilde{u}_{0,1} - \tilde{u}_{1,1} = 2\pi i z, \quad z = 0, 1,\]
\[\tilde{u}_{0,0} + \tilde{u}_{1,0} - \tilde{u}_{0,1} - \tilde{u}_{1,1} = \log \theta + 2\pi i z, \quad z = 0, 1,\]
\[\tilde{u}_{0,0} - \tilde{u}_{1,0} - \tilde{u}_{0,1} + \tilde{u}_{1,1} = \log \theta,\]
\[\tilde{u}_{0,0} + \beta \tilde{u}_{1,0} + \gamma \tilde{u}_{0,1} + \delta \tilde{u}_{1,1} + \varepsilon = 0.\]

The first 6 are lineariz. by \(\tilde{u}_{0,0} = \log w_{0,0}\) (\(\log\) princip. branch of complex log.), the last by an inversion \(\tilde{u}_{0,0} = 1/w_{0,0}\).
Part 1.4: Lineariz. multilin. quad-equations by a Cole-Hopf

Assume there exists a linearizing autonomous, Cole-Hopf transformation

\[ \tilde{u}_{0,1} = g_{0,0} (w_{0,0}) \tilde{u}_{0,0} \quad \text{or} \quad \tilde{u}_{1,0} = g_{0,0} (w_{0,0}) \tilde{u}_{0,0}, \quad \frac{d g (x)}{dx} \neq 0, \]

Theorem of lineariz. by a Cole-Hopf for real, quadratic to a linear homogeneous eq.:

There exists no equation in this class linearizable by a Cole-Hopf

\[ \tilde{u}_{0,1} = g (w_{0,0}) \tilde{u}_{0,0} \quad \text{or} \quad \tilde{u}_{1,0} = g (w_{0,0}) \tilde{u}_{0,0}; \]

Conjecture of lineariz. over \( \mathbb{C} \) by a Cole-Hopf:

The most general quad-eq. in this class is (up to Möbious and exchange \( n \leftrightarrow m \))

\[ (1 + w_{0,0}) (1 + \delta w_{1,1}) w_{1,0} - (1 + w_{0,1}) (1 + \delta w_{1,0}) w_{0,0} = 0, \quad \delta \neq 0, \]

(Hietarinta eq.) linearizable by a Cole-Hopf \( \tilde{u}_{0,1} = w_{0,0} \tilde{u}_{0,0} \) to the eq.

\[ \tilde{u}_{0,0} + \tilde{u}_{1,0} + \tilde{u}_{0,1} + \delta \tilde{u}_{1,1} = 0. \]
Part 2:

Linearizability through multiscale analysis
(joint work with R. Hernandez Heredero and D. Levi)
Part 2: Multiscale analysis of lineariz. $Q_\pm$ equations

- Multilin., real, quad-graph, dispersive eqs. of class $Q_\pm$:

\[a_1 (u_{n,m} \pm u_{n+1,m+1}) + a_2 (u_{n+1,m} \pm u_{n,m+1}) +
\]
\[+ (\alpha_1 - \alpha_2) u_{n,m} u_{n+1,m} + (\alpha_1 + \alpha_2) u_{n,m+1} u_{n+1,m+1} +
\]
\[+ (\beta_1 - \beta_2) u_{n,m} u_{n,m+1} + (\beta_1 + \beta_2) u_{n+1,m} u_{n+1,m+1} +
\]
\[+ \gamma_1 u_{n,m} u_{n+1,m+1} + \gamma_2 u_{n+1,m} u_{n,m+1} +
\]
\[+ (\xi_1 - \xi_3) u_{n,m} u_{n+1,m} u_{n,m+1} + (\xi_1 + \xi_3) u_{n,m} u_{n+1,m} u_{n+1,m+1} +
\]
\[+ (\xi_2 - \xi_4) u_{n+1,m} u_{n,m+1} u_{n+1,m+1} + (\xi_2 + \xi_4) u_{n,m} u_{n,m+1} u_{n+1,m+1} +
\]
\[+ \zeta u_{n,m} u_{n+1,m} u_{n,m+1} u_{n+1,m+1} = 0;
\]

- $Q_+$ linear dispersion relation:

\[\omega (\kappa) = \arctan \left[ \frac{(a_1^2 - a_2^2) \sin (\kappa)}{(a_1^2 + a_2^2) \cos (\kappa) + 2a_1 a_2} \right];
\]

- Equivalence class: Subgroup of real Möbius transformations

\[u_{n,m} \Rightarrow \frac{u_{n,m}}{A u_{n,m} + B}.
\]
Multiscale analysis of lineariz. $Q_+$ equations

Seek for a solution:

$$u_{n,m}(\varepsilon) = \sum_{j=1}^{+\infty} \sum_{\alpha=-j}^{j} \varepsilon^j u_j^{(\alpha)}(n_1, m_1, m_2, \ldots) e^{i\alpha(\kappa n - \omega m)}.$$ 

Reality conditions: $u_n^{(-\alpha)} = \bar{u}_n^{(\alpha)}, \forall \alpha$.

Expansion Parameters

1. $0 \leq \varepsilon \ll 1$: perturbative parameter around plane wave solution of $Q_+$;
2. $n_1 \equiv \varepsilon n$: slow “space” variable;
3. $m_j \equiv \varepsilon^j m, j \geq 1$ slow “times” variables;
Multiscale analysis of lineariz. $Q_+$ equations

- $u_j^{(\alpha)}(n_1, m_1, m_2, \ldots) \in C^{(\infty)}$ around $(n_1, m_1, m_2, \ldots) \Rightarrow$ shift operators as series of derivatives w. r. t. slow variables

$$T_n = T_n \sum_{j=0}^{+\infty} \varepsilon^j A^{(j)}_{n_1}, \quad T_m = T_m \sum_{j=0}^{+\infty} \varepsilon^j A^{(j)}_{m_1,m_2,\ldots}.$$

Expansion Operators

1. $A^{(j)}_{n_1} = (\partial_{n_1})^j / j!$ with $\partial_{n_1}$ derivative operator w. r. t. $n_1$;
2. $A^{(j)}_{m_1,m_2,\ldots}$ differential operators in $\partial^\alpha_{m_1}, \partial^\beta_{m_2}, \ldots$
Lowest orders classification: $\varepsilon^3$, $\varepsilon^4$

- Order $\varepsilon^3$, $\alpha = 1$: NLS equation

\[ i\partial_{m_2} u_1^{(1)} = \rho_1 \partial_{n_1}^2 u_1^{(1)} + \rho_2 u_1^{(1)}|u_1^{(1)}|^2, \]

$\rho_1$, $\rho_2$, rational expressions of $Q_+$ parameters and of $\cos(\kappa)$, $\sin(\kappa)$.

$\varepsilon^3$-linearizability conditions

$\rho_2 = 0$, $\forall \kappa \Rightarrow$ system of eleven nonlinear algebraic relations between $Q_+$ parameters $\Rightarrow$ 6 cases.

$\varepsilon^4$-linearizability conditions

No conditions $\Rightarrow$ all 6 $\varepsilon^3$-linearizable cases are $\varepsilon^4$-linearizable.
Lowest order classification: $\varepsilon^4$ C-integrability

**Theorem of $\varepsilon^4$-linearizability:** The only $\varepsilon^4$-linearizable eqs. in our class are: (5 free par. $a_1$, $a_2$, $\beta_1$, $\gamma_1$, $\zeta$)

- **Case L1:**

  \[
  \begin{aligned}
  \alpha_2 = \beta_2 &= 0, \quad \alpha_1 = \beta_1, \quad \gamma_1 + \gamma_2 = 2\beta_1, \\
  \xi_1 = \xi_2 &= \frac{(a_1+a_2)^2\gamma_1\gamma_2+(3a_1-2a_2)a_2\gamma_1\beta_1-a_1(2a_1-3a_2)\gamma_2\beta_1}{4a_1a_2(a_1+a_2)}, \\
  \xi_3 = \xi_4 &= \frac{-(a_1-a_2)(a_1+a_2)^2\gamma_1\gamma_2-a_2(-a_1^2-5a_2a_1+2a_2^2)\gamma_1\beta_1+a_1(2a_1^2-5a_2a_1-a_2^2)\gamma_2\beta_1}{4a_1a_2(a_1+a_2)^2};
  \end{aligned}
  \]

- **Case L2:**

  \[
  \begin{aligned}
  \alpha_2 = \beta_2 &= 0, \quad \alpha_1 = \beta_1, \\
  (3a_1 - 2a_2) a_2^2 \gamma_1 + a_1^2 (2a_1 - 3a_2) \gamma_2 &= 4a_1 (a_1 - a_2) a_2 \beta_1, \\
  \xi_1 = \xi_2 &= \frac{(a_1+a_2)(a_2^2\gamma_1^2-a_1^2\gamma_2^2)+2a_2(a_1^2-2a_2^2)\gamma_1\beta_1+2a_1(2a_1^2-a_2^2)\gamma_2\beta_1-6a_1(a_1-a_2)a_2\beta_1}{4a_1a_2(a_1-a_2)^2}, \\
  \xi_3 = \xi_4 &= \frac{2a_1a_2(a_1+a_2)(\gamma_1-\gamma_2)\beta_1+(a_1-a_2)(a_2\gamma_1-a_1\gamma_2)^2+2a_1a_2(a_2-a_1)\beta_1^2}{4a_1a_2(a_1+a_2)^2};
  \end{aligned}
  \]
Lowest order classification: $\varepsilon^4$ C-integrability

Theorem of $\varepsilon^4$-linearizability (cont.): (4 free param. $a_2, \alpha_1, \beta_1, \zeta$)

- **Case L3:**
  \[
  \begin{align*}
  \alpha_2 &= \beta_2 = 0, \quad a_1 = 2a_2, \\
  \gamma_1 &= \frac{2}{3} (\alpha_1 + \beta_1), \quad \gamma_2 = \frac{1}{3} (\alpha_1 + \beta_1), \\
  \xi_1 &= \xi_2 = -\frac{-5\alpha_1\beta_1+\alpha_1^2+\beta_1^2}{6a_2}, \quad \xi_3 = \xi_4 = -\frac{(\alpha_1-2\beta_1)(2\alpha_1-\beta_1)}{18a_2};
  \end{align*}
  \]

- **Case L4:**
  \[
  \begin{align*}
  \alpha_2 &= \beta_2 = 0, \quad 2a_1 = a_2, \\
  \gamma_1 &= \frac{1}{3} (\alpha_1 + \beta_1), \quad \gamma_2 = \frac{2}{3} (\alpha_1 + \beta_1), \\
  \xi_1 &= \xi_2 = -\frac{-5\alpha_1\beta_1+\alpha_1^2+\beta_1^2}{6a_1}, \quad \xi_3 = \xi_4 = \frac{(\alpha_1-2\beta_1)(2\alpha_1-\beta_1)}{18a_1};
  \end{align*}
  \]
Lowest order classification: $\varepsilon^4$ $C$-integrability

**Theorem of $\varepsilon^4$-linearizability:** (cont.): (4 free param. $a_2, \beta_1, \beta_2, \zeta$)

- **Case L5:**
  \[
  \begin{aligned}
  \alpha_2 &= \beta_2, \quad \alpha_1 = \beta_1, \\
  \xi_1 &= \frac{3\beta_1^2 - 2\beta_2\beta_1 + \beta_2^2}{6a_1}, \\
  \xi_3 &= -\frac{\beta_1^2 - 6\beta_2\beta_1 + 7\beta_2^2}{18a_1}, \\
  2a_1 &= a_2, \quad \gamma_1 = \frac{2\beta_1}{3}, \quad \gamma_2 = \frac{4\beta_1}{3}, \\
  \xi_2 &= \frac{3\beta_1^2 + 2\beta_2\beta_1 + \beta_2^2}{6a_1}, \\
  \xi_4 &= -\frac{\beta_1^2 + 6\beta_2\beta_1 + 7\beta_2^2}{18a_1}; \\
  \end{aligned}
  \]

- **Case L6:**
  \[
  \begin{aligned}
  \alpha_2 &= -\beta_2, \quad \alpha_1 = \beta_1, \\
 \xi_1 &= \frac{3\beta_1^2 + 2\beta_2\beta_1 + \beta_2^2}{6a_2}, \\
  \xi_3 &= \frac{\beta_1^2 + 6\beta_2\beta_1 + 7\beta_2^2}{18a_2}, \\
  a_1 &= 2a_2, \quad \gamma_1 = \frac{4\beta_1}{3}, \quad \gamma_2 = \frac{2\beta_1}{3}, \\
  \xi_2 &= \frac{3\beta_1^2 - 2\beta_2\beta_1 + \beta_2^2}{6a_2}, \\
  \xi_4 &= \frac{\beta_1^2 - 6\beta_2\beta_1 + 7\beta_2^2}{18a_2}; \\
  \end{aligned}
  \]

- All cases invariant under restricted Möbius.
Theorem of $\varepsilon^5$-linearizability: The only $\varepsilon^5$-linearizable case out of the previous six $\varepsilon^4$-linearizable is: (4 free param. $a_1, a_2, \gamma_1, \zeta$)

\[
\begin{align*}
\alpha_1 &= \beta_1 = \frac{(a_1 + a_2) \gamma_1}{2a_1}, & \alpha_2 &= \beta_2 = 0, & \gamma_2 &= \frac{a_2 \gamma_1}{a_1}, \\
\xi_1 &= \xi_2 = \frac{3 (a_1 + a_2) \gamma_1^2}{8a_1^2}, & \xi_3 &= \xi_4 = \frac{(a_1 - a_2) \gamma_1^2}{8a_1^2}.
\end{align*}
\]

- Canonical form (restr. Möbius): $v + v_{1,2} + \varepsilon (v_1 + v_2) + \zeta' v v_1 v_2 v_{1,2} = 0$, $\varepsilon \doteq \frac{a_2}{a_1} \neq 0$, ±1, $\zeta' \doteq -2 \left[ (a_1 + a_2) \gamma_1^3 - 4a_1^3 \zeta \right] / a_1$. Rescaling $\Rightarrow \zeta' = 0, 1$;

- Theorem: $Q_+$ is linearizable by a real Möbius iff $\zeta' = 0 \Rightarrow \zeta = \frac{(a_1 + a_2) \gamma_1^3}{4a_1^3}$.

The linearizing transformation is

\[
u_{n,m} = \frac{\alpha v_{n,m} + \beta}{\gamma v_{n,m} + \delta}, \quad \beta = 0, \quad \gamma = -\frac{\alpha \gamma_1}{2a_1};
\]

- Functional methods $\Rightarrow$ Theorem: if $\zeta \neq \frac{(a_1 + a_2) \gamma_1^3}{4a_1^3}$, no one-point, two-points or Cole-Hopf linearizes $Q_+$. 

