Asymptotics for the Korteweg-de Vries equation in the small dispersion limit

Tom Claeys

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SISSA Trieste

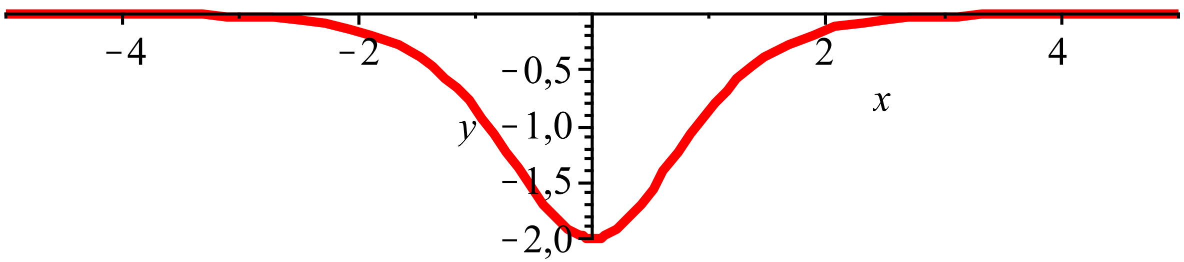
joint work with Tamara Grava

KdV equation

- Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0, \qquad \epsilon > 0$$

- initial data $u(x, t = 0, \epsilon) = u_0(x)$

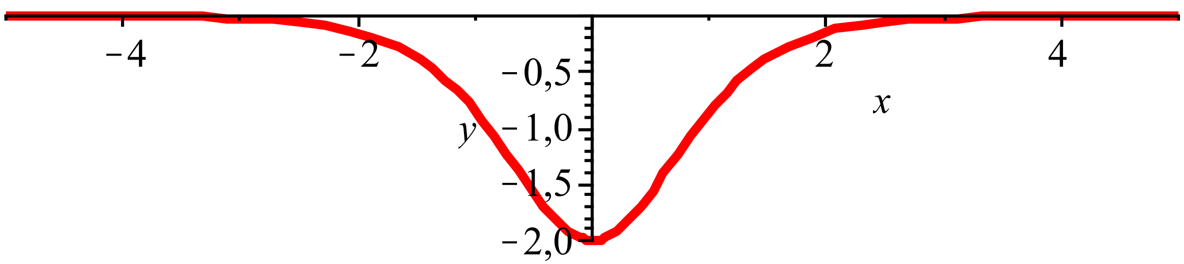


- small dispersion limit $\epsilon \to 0$
  - different asymptotics for $u(x, t, \epsilon)$ in various regions in (x, t)-plane
  - transitions between different regions

Hopf equation

- The KdV equation is a singular perturbation of the Hopf equation ($\epsilon = 0$)

$$u_t + 6uu_x = 0$$



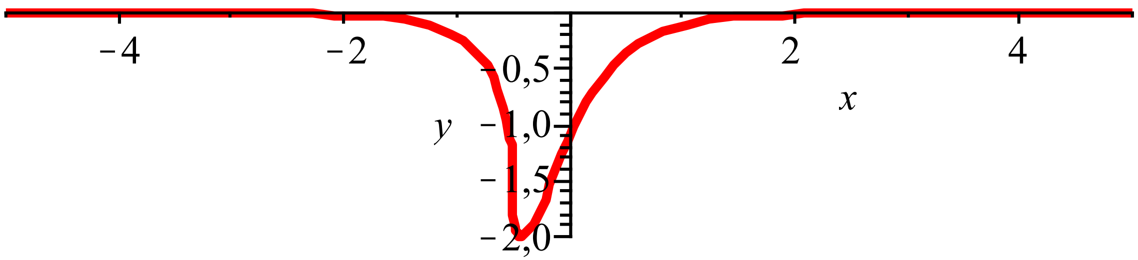
- With the same type of initial data, evolution in time of $u(x,t)$ is given by

$$u(x,t) = u_0(\xi), \quad x = 6tu_0(\xi) + \xi.$$

  - solution moves to the left, minimum moves faster

Hopf equation

- solution moves to the left, minimum moves faster



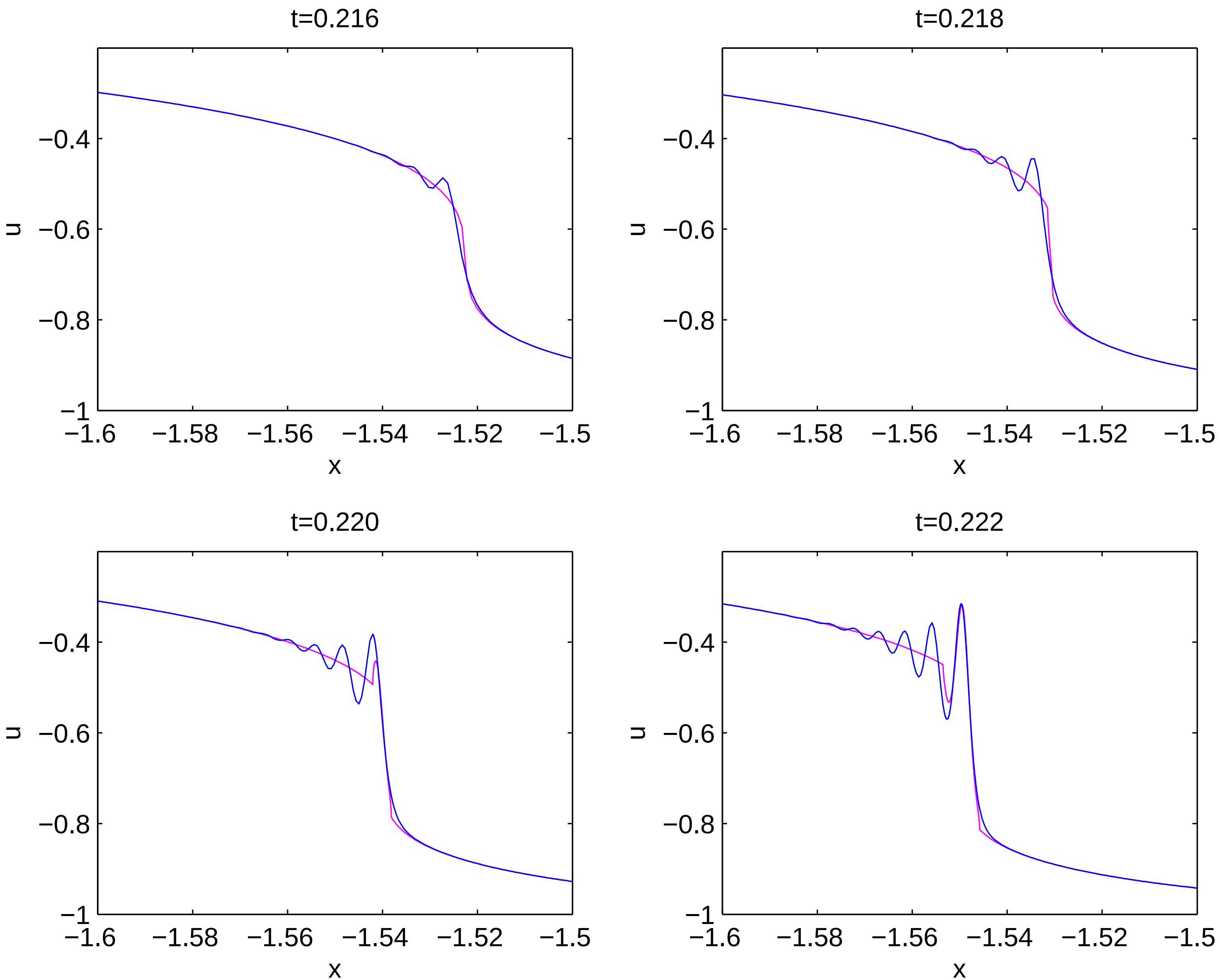
  - gradient catastrophe!
  - solution does not remain well-defined for

$$t > t_c = \frac{1}{\max_{\xi \in \mathbb{R}}[-6u_0'(\xi)]}$$

- What happens with KdV solution after time of gradient catastrophe?
  - interval $[x^-(t), x^+(t)]$ of rapid oscillations

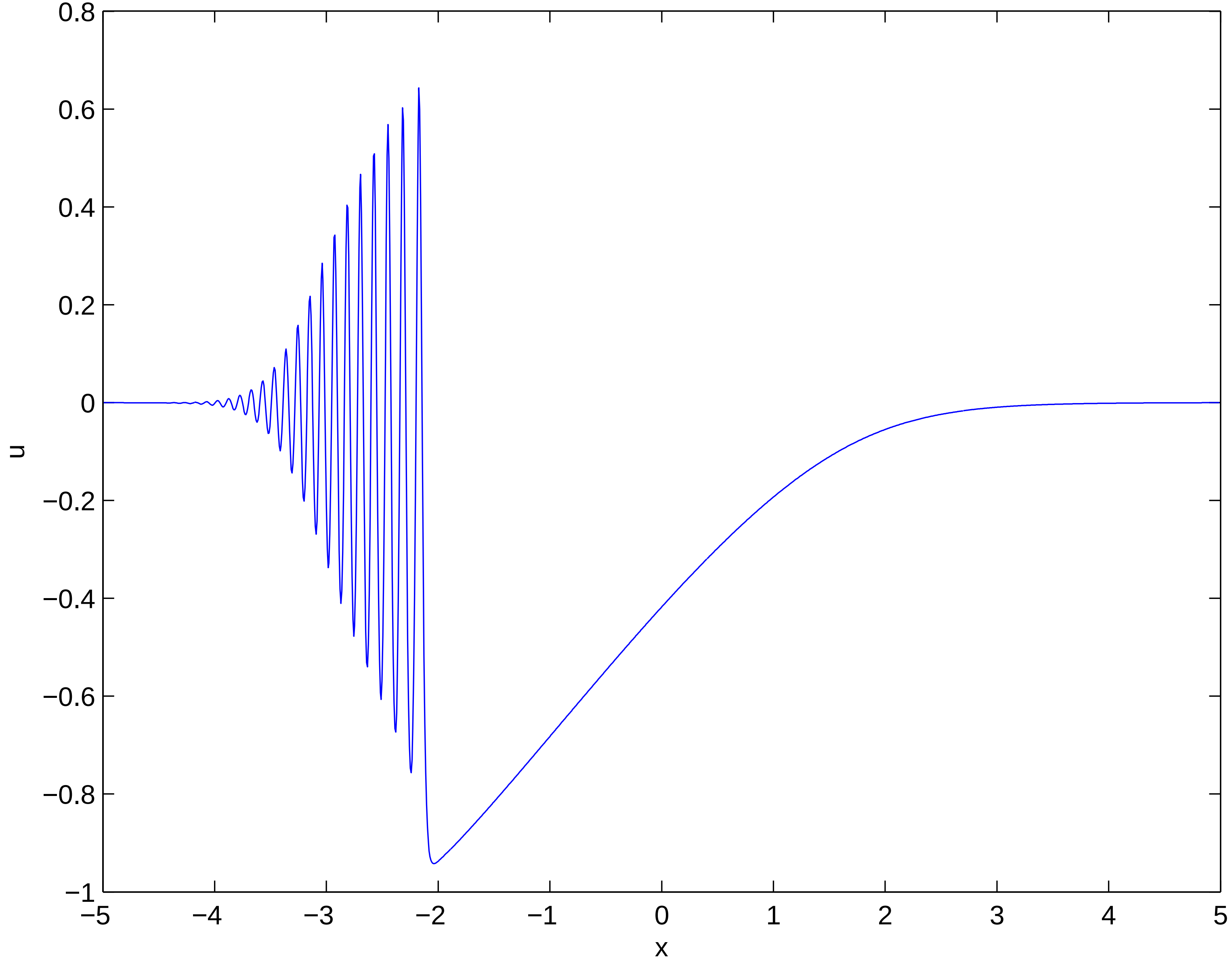
KdV equation

- Formation of oscillatory zone



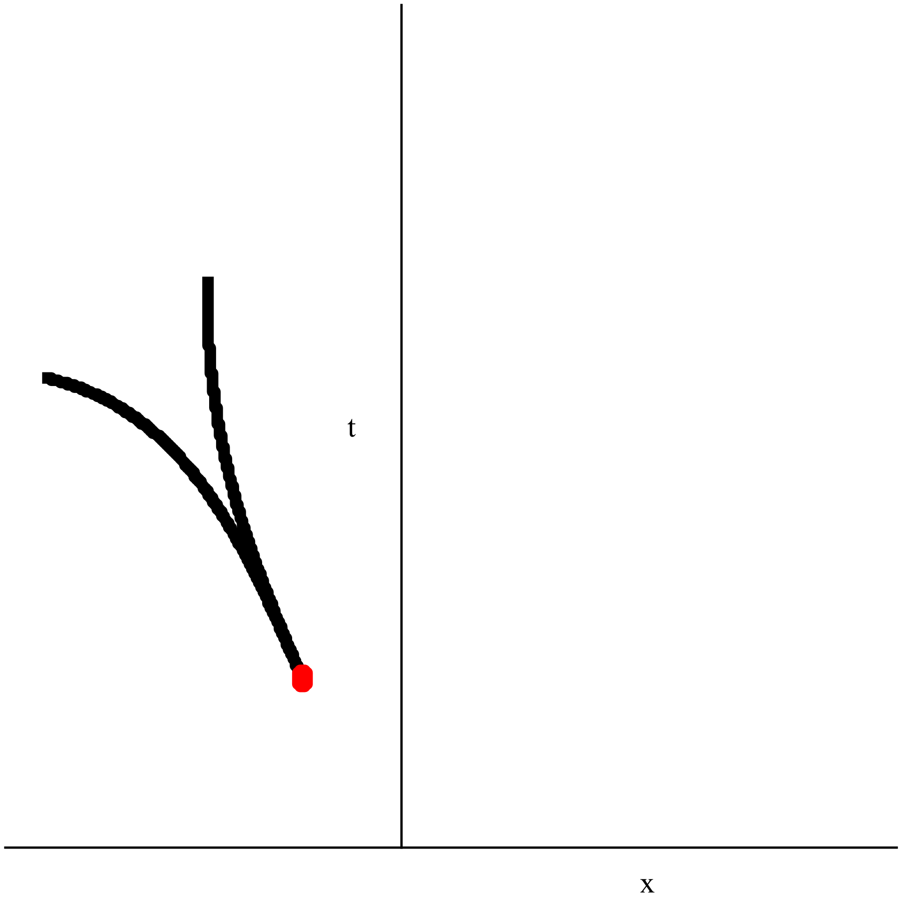
KdV equation

- Formation of oscillatory zone



KdV equation

- In the (x, t)-plane
  - region where the asymptotic behavior ($\epsilon \longrightarrow 0$) of $u(x, t, \epsilon)$ is determined by the Hopf equation
  - oscillatory region



KdV equation

- exterior of the cusp
  - solution can be approximated using Hopf equation as $\epsilon \to 0$
- interior of the cusp
  - solution can be approximated by elliptic functions as $\epsilon \to 0$ *(Lax-Levermore '83, Venakides '90, Fei-Ran Tian '93, Deift-Venakides-Zhou '97)*
- near the border of the two regions, these approximations are not valid anymore
  - leading edge
  - trailing edge
  - the point and time of gradient catastrophe

KdV equation

- Critical asymptotics near the boundaries
  - leading edge: Hastings-McLeod solution of Painlevé II
  - point of gradient catastrophe: special solution to a higher order Painlevé I equation
  - trailing edge: no Painlevé type behavior - discontinuous transition

Painlevé equations

- Painlevé II equation

$$Q_{xx} = xQ + 2Q^3.$$

  - unique solution (Hastings-McLeod solution) with boundary conditions

$$Q(x) \sim \text{Ai}(x), \qquad \text{as } x \to +\infty,$$
$$Q(x) \sim \sqrt{\frac{-x}{2}}, \qquad \text{as } x \to -\infty.$$

  - this solution is smooth on the real line

Painlevé equations

- Fourth order analogue of the Painlevé I equation

$$x = tU - \left(\frac{1}{6}U^3 + \frac{1}{24}(U_x^2 + 2UU_{xx}) + \frac{1}{240}U_{xxxx}\right).$$

  - unique real solution $U(x,t)$ satisfying the following conditions:
    $\rightarrow$ $U(x,t)$ has no poles for $x, t \in \mathbb{R}$,
    $\rightarrow$ U has the following asymptotic behavior

$$U(x,t) = \mp 6^{1/3}|x|^{1/3} + \mathcal{O}(x^{-1/3}) \qquad \text{as } x \rightarrow \pm\infty.$$

(Brézin-Marinari-Parisi ’90, Dubrovin ’06, TC-Vanlessen ’07)

Gradient catastrophe

Theorem *(TC-Grava)*

- Take a double scaling limit where we let $\epsilon \to 0$ and at the same time $x \to x_c$ and $t \to t_c$ in such a way that

$$x - x_c - 6u_c(t - t_c) = \mathcal{O}(\epsilon^{6/7}), \qquad t - t_c = \mathcal{O}(\epsilon^{4/7}).$$

Then we have

$$u(x, t, \epsilon) = u_c + \left(\frac{2\epsilon^2}{k^2}\right)^{1/7} U\left(\frac{x - x_c - 6u_c(t - t_c)}{(8k\epsilon^6)^{\frac{1}{7}}}, \frac{6(t - t_c)}{(4k^3\epsilon^4)^{\frac{1}{7}}}\right) + O\left(\epsilon^{4/7}\right).$$

(conjectured by Dubrovin in much more general settings)

Leading edge

- Leading edge x^- determined by equations

$$x^-(t) = 6tu(t) + f^-(u(t))$$
$$6t + \theta(v(t); u(t)) = 0$$
$$\partial_v \theta(v(t); u(t)) = 0,$$

  with f^- the inverse of the decreasing part of u_0, and
  $\theta(v; u) = \dfrac{1}{2\sqrt{u-v}} \int_v^u \dfrac{f_-'(\xi)d\xi}{\sqrt{\xi - v}}$.
  - Whitham equations
  - Lax-Levermore minimization problem
- $u(t)$ is leading order approximation of $u(x, t, \epsilon)$
- $v(t)$ is critical point of g-function

Leading edge

Theorem *(TC-Grava)*

- Fix $t_c < t < T$. Take a double scaling limit where we let $\epsilon \to 0$ and at the same time $x \to x^-$ in such a way that

$$x - x^- = \mathcal{O}(\epsilon^{2/3}).$$

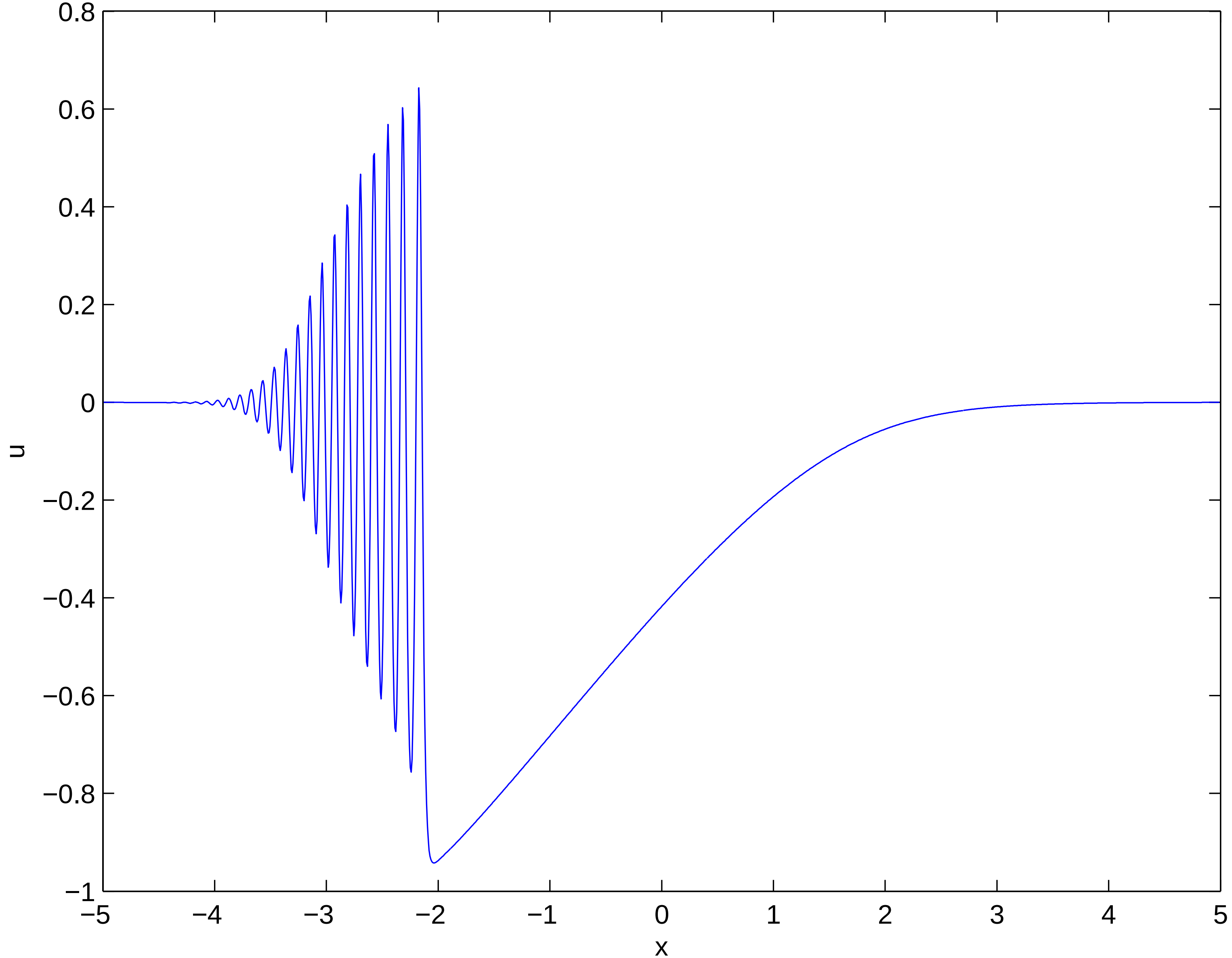
Under 'generic conditions', we have

$$u(x,t,\epsilon) = u(t) \\ - c_2\epsilon^{1/3} Q(-c_1\epsilon^{-2/3}(x - x^-)) \cos(2\omega\epsilon^{-1}) + O(\epsilon^{2/3}).$$

  - in accordance with numerical results of Grava and Klein

Leading edge

- Oscillatory zone



Trailing edge

- Fix $t_c < t < T$. Take a double scaling limit where we let $\epsilon \to 0$ and at the same time $x \to x^+$ in such a way that

$$y := c_0 \frac{x - x^+}{\epsilon \ln \epsilon}$$

remains bounded. Under 'generic conditions', we have

  - $u(x,t,\epsilon) = u + \mathcal{O}(\epsilon^{1/2})$ for $y \leq -\frac{1}{2}$,
  - $u(x,t,\epsilon) = u + c_1 \frac{\alpha_k}{(1+\alpha_k)^2} + \mathcal{O}(\epsilon^{1/2})$ for $k - \frac{1}{2} \leq y \leq k$,
  - $u(x,t,\epsilon) = u + c_1 \frac{\beta_k}{(1+\beta_k)^2} + \mathcal{O}(\epsilon^{1/2})$ for $k \leq y \leq k + \frac{1}{2}$,

where

$$\alpha_k = c_{3,k} \epsilon^{\frac{1}{2}+y-k}, \quad \beta_k = c_{4,k} \epsilon^{\frac{1}{2}+k-y}, \quad \alpha_{k+1}\beta_k = 1.$$

Riemann-Hilbert problem

Proofs of the results rely on

- a Riemann-Hilbert problem characterizing solutions to the KdV equation
- small dispersion asymptotics for the associated reflection coefficient
- an asymptotic analysis of the RH problem
  - contour deformation
  - construction of global and local parametrices

Riemann-Hilbert problem

Proofs of the results rely on the Riemann-Hilbert problem for KdV:
find a function M satisfying

(a) $M : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic

(b)
$$M_+(\lambda) = M_-(\lambda) \begin{pmatrix} 1 & r(\lambda;\epsilon)e^{2i\alpha(\lambda;x,t)/\epsilon} \\ -\bar{r}(\lambda;\epsilon)e^{-2i\alpha(\lambda;x,t)/e} & 1-|r(\lambda;\epsilon)|^2 \end{pmatrix}, \quad \text{for } \lambda < 0,$$
$$M_+(\lambda) = M_-(\lambda) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{for } \lambda > 0,$$

with α given by $\alpha(\lambda;x,t) = 4t(-\lambda)^{3/2} + x(-\lambda)^{1/2}$.

(c) $M(\lambda;x,t,\epsilon) \sim \begin{pmatrix} 1 & 1 \\ i\sqrt{-\lambda} & -i\sqrt{-\lambda} \end{pmatrix}$ for $\lambda \to \infty$.

Riemann-Hilbert problem

- if $r(\lambda;\epsilon)$ is the reflection coefficient for the Schrödinger equation with potential u_0, the KdV solution $u(x,t,\epsilon)$ can be recovered from

$$u(x,t;\epsilon) = -2i\epsilon\partial_x M_{1,11}(x,t;\epsilon),$$

where $M_{11}(\lambda;x,t,\epsilon) = 1 + \frac{M_{1,11}(x,t;\epsilon)}{\sqrt{-\lambda}} + \mathcal{O}(1/\lambda)$ as $\lambda \to \infty$.

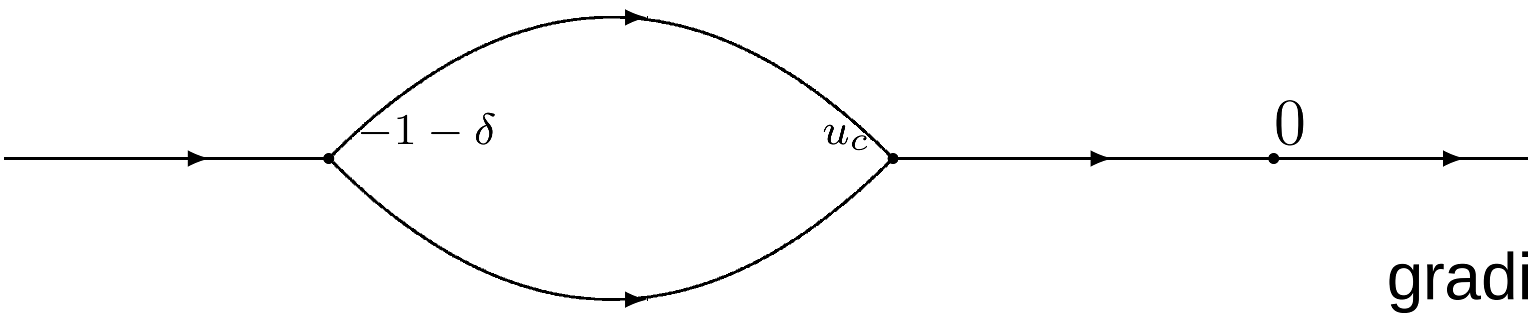
- there are good approximations known for $r(\lambda;\epsilon)$ as $\epsilon \longrightarrow 0$
  - exponential decay for $\lambda < -1$
  - oscillatory behavior for $-1 < \lambda < 0$
  - transition for $\lambda \approx -1$

(Ramond '96)

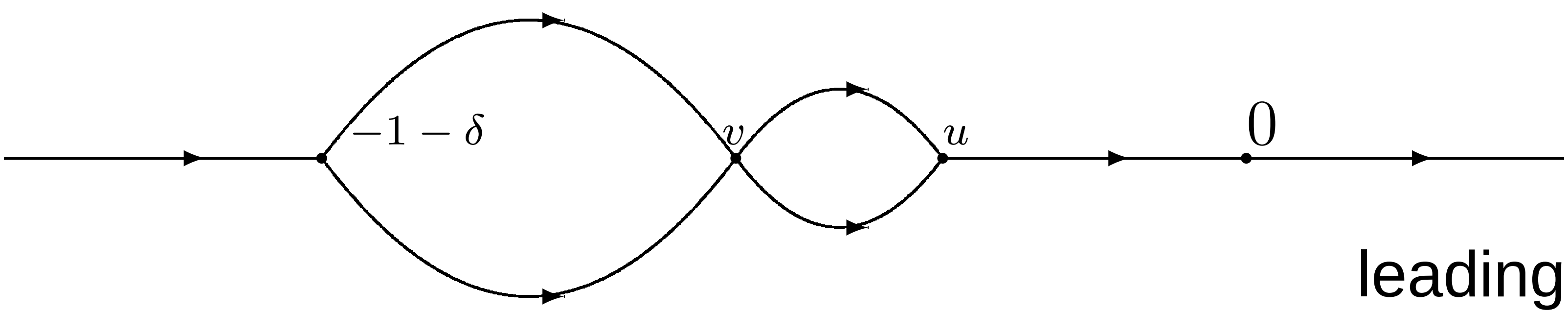
Riemann-Hilbert problem

- using those approximations, we can perform the Deift/Zhou steepest-descent method on the RH problem *(cf. Deift-Venakides-Zhou '97)*
  - construction of G-function $\longrightarrow$ Lax-Levermore minimization problem
  - contour deformation: opening of lenses

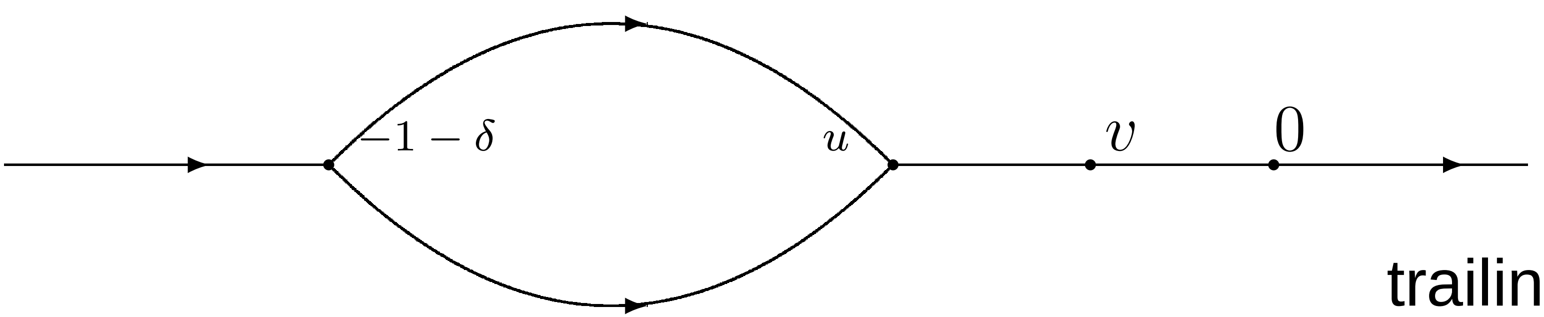
Riemann-Hilbert problem



gradient catastrophe



leading edge

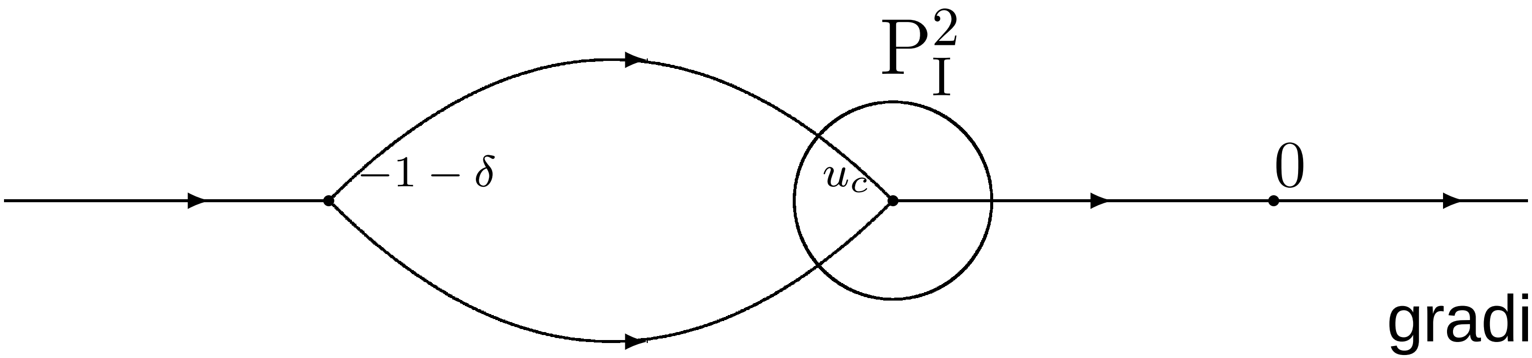


trailing edge

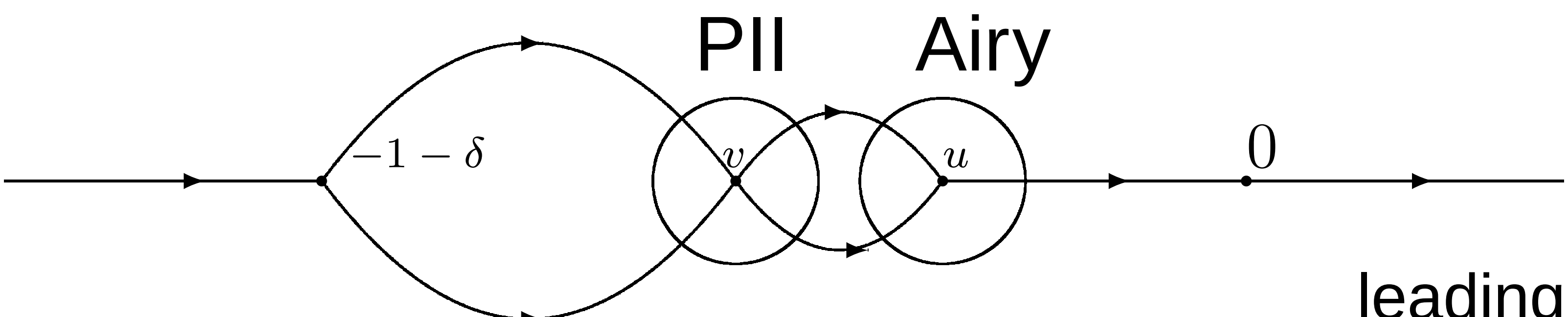
Riemann-Hilbert problem

- on deformed contour, jump matrices are uniformly close to constant matrices as $\epsilon \to 0$
  - except near u_c in the first picture
  - except near u and v in the second and third picture
- ignoring special points and small jumps $\longrightarrow$ explicit solution of RH problem
- local parametrices near special points

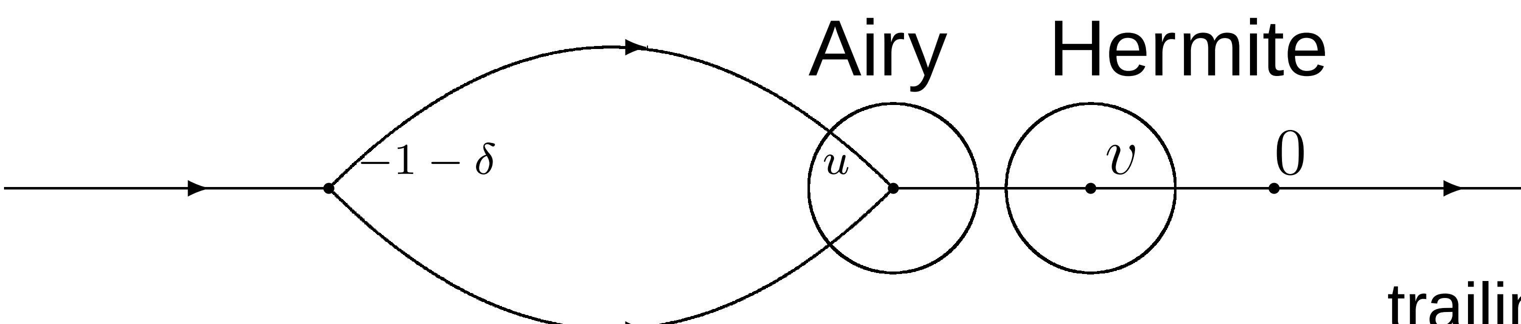
Riemann-Hilbert problem: local parametrices



gradient catastrophe



leading edge



trailing edge

Riemann-Hilbert problem

- Hermite parametrix in last picture using the matrix

$$\Psi(\zeta;k) = \begin{pmatrix} \frac{\pi^{1/4}\sqrt{k!}}{2^{k/2}} H_k(\zeta) & \frac{\pi^{1/4}\sqrt{k!}}{2\cdot 2^{k/2}\pi i} \int_{\mathbb{R}} \frac{H_k(u)e^{-u^2}}{u-\zeta}\, du \\ -2\pi i \frac{2^{(k-1)/2}}{\pi^{1/4}\sqrt{(k-1)!}} H_{k-1}(\zeta) & -\frac{2^{(k-1)/2}}{\pi^{1/4}\sqrt{(k-1)!}} \int_{\mathbb{R}} \frac{H_{k-1}(ku)e^{-u^2}}{u-\zeta}\, du \end{pmatrix} e^{-\frac{\zeta^2}{2}\sigma_3}$$

  - degree of Hermite polynomials depends on the value of

$$y := c_0 \frac{x - x^+}{\epsilon \ln \epsilon}$$

  - k is the closest positive integer to y
  - shift from k to $k+1$ when y is a half integer
    $\rightarrow$ this transition describes the pulses

Riemann-Hilbert problem

- Hermite polynomials do not appear in asymptotics for $u(x, t, \epsilon)$, only the residue of $\Psi(\zeta)\zeta^{-k\sigma_3}$ at infinity
  - sub-leading terms in expansion of u comes from Hermite parametrix
- Airy parametrix: only residue at infinity contributes
  - contribution only of order $\mathcal{O}(\epsilon^{2/3})$
- Outside parametrix: explicit construction
  - leading order contribution

Universality?

- Similar critical asymptotic regimes for other equations?
  - Riemann-Hilbert techniques leave space for generalizations
  - e.g. different time dependence of reflection coefficient