.

Asymptotic behavior of PVI functions: a Matching Method

Davide Guzzetti

This research has been completed at RIMS, Kyoto University, supported by the Kyoto Mathematics COE Fellowship

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right]\frac{dy}{dx}$$
$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right]. \qquad (PVI)$$

The generic solution has essential singularities and/or branch points in 0,1,∞. Other singularities are poles (depend on initial conditions).

A solution of PVI can be analytically continued to a meromorphic function on the universal covering of $\mathbf{P}^1\backslash\{0,1,\infty\}$.

$\diamond$ "New" transcendental functions. Solving (PVI)?

i) Determine the local (asymptotic) behavior of the transcendents at $x = 0, 1, \infty$

Example [Jimbo (1982)] (two integration constants a, σ). :

$$y(x) \sim a^{(0)} x^{1-\sigma^{(0)}}, \qquad x \to 0,$$
$$y(x) \sim 1 - a^{(1)}(1-x)^{1-\sigma^{(1)}}, \qquad x \to 1,$$
$$y(x) \sim a^{(\infty)} x^{\sigma^{(\infty)}}, \qquad x \to \infty,$$

ii) Solve the *connection problem*: find the relation between couples of integration constants at $x = 0, 1, \infty$.

Example:

$$(a^{(0)}, \sigma^{(0)}) \leftrightarrow (a^{(1)}, \sigma^{(1)}) \leftrightarrow (a^{(\infty)}, \sigma^{(\infty)}).$$

$\diamond$ Find "all" asymptotic behaviors? Problem of classifications of the solutions.

Monodromy Preserving Deformations

$\diamond$ To every Painlevé equation we can associate a suitable linear system of ODE:

$$\frac{d}{d\lambda}\begin{pmatrix}\psi\\ \varphi\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\psi\\ \varphi\end{pmatrix} \tag{1}$$

$$a_{ij} = a_{ij}(\lambda, x, y) \quad \text{rational of } \lambda, x, y$$

A fundamental solution is an invertible matrix solution $\Psi = \begin{pmatrix}\psi_1 & \psi_2\\ \varphi_1 & \varphi_2\end{pmatrix}$.

$\diamond$ $a_{ij}(\lambda, x, y)$'s have poles in the λ-plane. Ψ is multi-valued at the poles. The group of transformations of Ψ associated to the loops around the poles is the **monodromy group**.

$$\text{Ex:} \quad \frac{d\Psi}{d\lambda} = \frac{A}{\lambda}\Psi \quad \Rightarrow \quad \Psi = e^{A\ln\lambda}; \qquad \lambda \mapsto \lambda e^{2\pi i}, \quad \Psi \mapsto \Psi e^{2\pi i A}.$$

$\diamond$ x is a **monodromy preserving deformation:** there exists a fundamental solution with *monodromy independent of* x if and only if $y = y(x)$ satisfies the Painlevé equation. [Jimbo, Miwa, Ueno (1981)].

$\diamond$ The system (1) for (PVI) is Fuchsian:

$$\frac{d}{d\lambda}\begin{pmatrix}\psi\\ \varphi\end{pmatrix} = \left[\frac{A_0(x)}{\lambda} + \frac{A_1(x)}{\lambda-1} + \frac{A_{\mathbf{x}}(x)}{\lambda - x}\right]\begin{pmatrix}\psi\\ \varphi\end{pmatrix}, \tag{2}$$

$$A_0 + A_1 + A_x = -\frac{\theta_\infty}{2}\sigma_3, \quad \theta_\infty \neq 0.$$

$$\text{Eigenvalues } (A_i) = \pm\frac{1}{2}\theta_i, \quad i = 0, 1, x;$$

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad -\beta = \frac{1}{2}\theta_0^2, \quad \gamma = \frac{1}{2}\theta_1^2, \quad \left(\frac{1}{2} - \delta\right) = \frac{1}{2}\theta_x^2$$

$\diamond$ A Painlevé function is :

$$y(x) = \frac{x\ (A_0)_{12}}{x\ [(A_0)_{12} + (A_1)_{12}] - (A_1)_{12}}. \tag{3}$$

$\diamond$ $\lambda = 0, x, 1$ Fuchsian Singularities (poles). $\Psi = \begin{pmatrix}\psi_1 & \psi_2\\ \varphi_1 & \varphi_2\end{pmatrix}$ is multi-valued at the poles. M_0, M_x, M_1 **monodromy matrices**, generate the monodromy group.

$$\lambda \mapsto \lambda\, e^{2\pi i} \quad \Longrightarrow \quad \Psi \mapsto \Psi M_0,$$

$$\lambda - x \mapsto (\lambda - x)e^{2\pi i} \quad \Longrightarrow \quad \Psi \mapsto \Psi M_x,$$

$$\lambda - 1 \mapsto (\lambda - 1)e^{2\pi i} \quad \Longrightarrow \quad \Psi \mapsto \Psi M_1.$$

Connection Problem and Classification of the Solutions

$\diamond$ (For generic values of θ_0, θ_x, θ_1, θ_∞) *there is a one to one correspondence between a solution $y(x)$ and a point in the space of monodromy data.*

$$\theta_0, \theta_x, \theta_1, \theta_\infty, \qquad M_0,\ M_x,\ M_1 \quad \text{indipendent of } x;$$

$$\updownarrow$$

$$y(x) = y(x; \theta_0, \theta_x, \theta_1, \theta_\infty; \mathrm{tr}(M_0 M_x), \mathrm{tr}(M_0 M_1), \mathrm{tr}(M_1 M_x))$$

$\Longrightarrow$ (1) Write the *different* integration constants at $x = 0$, $x = 1$, $x = \infty$ as classical functions of the *same* monodromy data.

This solves the connection problem.

Example: M.Jimbo: *Publ.RIMS* **18** (1982):

$$y(x) \sim a^{(0)} x^{1-\sigma}, \qquad x \to 0,$$
$$y(x) \sim 1 - a^{(1)} (1-x)^{1-\sigma^{(1)}}, \qquad x \to 1,$$
$$y(x) \sim a^{(\infty)} x^{\sigma^{(\infty)}}, \qquad x \to \infty,$$

$$a^{(i)} = a^{(i)}(\theta_0, \theta_x, \theta_1, \theta_\infty; \mathrm{tr}(M_0 M_x), \mathrm{tr}(M_0 M_1), \mathrm{tr}(M_1 M_x)), \quad i = 0, 1, \infty.$$

$$\mathrm{tr}(M_0 M_x) = 2\cos(\pi \sigma^{(0)}),$$
$$\mathrm{tr}(M_1 M_x) = 2\cos(\pi \sigma^{(1)}), \quad \mathrm{tr}(M_0 M_1) = 2\cos(\pi \sigma^{(\infty)})$$

$\Longrightarrow$ (2) We can *classify Painlevé functions in terms of monodromy data*

Do we know all the asymptotic behaviors? $\leftrightarrow$ Construct the asymptotic behaviors corresponding to each point in the space of monodromy data.

A **Matching method**: a **constructive method** to compute the leading term(s) of $y(x)$, and the associated monodromy data.

- Constructive procedure: no assumption about the asymptotics of $y(x)$.

- It is based on the isomonodromy deformation theory. It allows to compute the *monodromy data* associated to a solution, and thus solve the connection problem.

- I developed it in 2006, it produced new asymptotic behaviors. I conjecture that "all" the possible behaviors have been constructed.

[1] D.Guzzetti: *Matching Procedure for the Sixth Painlevé Equation.* J.Phys.A: Math.Gen. **39**, (2006), 11973-12031.

[2] D. Guzzetti: *On the Logarithmic Asymptotics of the Sixth Painlevé Equation.* J. Phys. A: Math. Theor. 41, (2008), 205201.

Remarks:

◇ There are other ways to obtain local analyticity properties of the solutions and their asymptotic behaviors. For ex: the *elliptic representation* of PVI.

[3] D.Guzzetti: *On the Critical Behavior, the Connection Problem and the Elliptic Representation of a Painlevé 6 Equation .* Mathematical Physics, Analysis and Geometry, **4**, (2001), 293-377.

[4] D.Guzzetti: *The Elliptic Representation of the General Painlevé 6 Equation.* Communications in Pure and Applied Mathematics, **55**, (2002), 1280-1363.

◇ A.D.Bruno, I.V, Goryuchkina constructed local expansions with their *power geometric technique* (Dokl.Math. **76**, 851–5).

Results obtained with the Matching Method

Remark: We consider $x \to 0$. From basic expansions we obtain all the expansions at $x = 0, 1, \infty$, by the action of the symmetries.

$$(\diamond) \qquad \gamma \mapsto -\beta, \quad -\beta \mapsto \gamma, \qquad y \mapsto 1-y, \quad x \mapsto 1-x,$$

$$(\diamond\diamond) \qquad \alpha \mapsto -\beta, \quad \beta \mapsto -\alpha, \qquad 1 \mapsto \frac{1}{y}, \quad 1 \mapsto \frac{1}{x},$$

$$(\diamond\diamond\diamond) \qquad 1-2\delta \mapsto 2\gamma, \quad 2\gamma \mapsto 1-2\delta, \quad -\beta \mapsto \alpha, \quad \alpha \mapsto -\beta, \quad y \mapsto \frac{x}{y}.$$

1) With two integration constants a, σ:

$$y(x) = x^{1-\sigma} \sum_{\substack{n,m \in \mathbf{Z} \\ n+m \geq 0,\ m \geq 0}} c_{nm}(a)\ x^{n(1-\sigma)+m\sigma} =$$

$$\sim \begin{cases} a x^{1-\sigma}, & 0 < \Re\sigma < 1; \\ a x^{1+\sigma}, & -1 < \Re\sigma < 0; \\ x \left\{ B \sin^2\left(\frac{i\sigma}{2} \ln x + a\right) + c(a) \right\}, & \Re\sigma = 0; \end{cases}$$

$$y(x) = \left[\frac{x}{2} + \frac{1}{\sin^2\left(\frac{i(\sigma-1)}{2} \ln x + a + \sum_{k \geq 1} c_k(a) x^{(1-\sigma)k}\right)} \right] (1 + O(x)),$$

$$\Re\sigma = \pm 1$$

The series are convergent for $|x|$ small and $\arg(x)$ bounded.

$$a = a(\theta_0, \theta_x, \theta_1, \theta_\infty; \operatorname{tr}(M_0 M_x), \operatorname{tr}(M_1, M_x))$$

$$\operatorname{tr}(M_0 M_x) = 2\cos(\pi\sigma)$$

$$\sigma \neq \pm 1, 0, \qquad \operatorname{tr}(M_0 M_x) \neq \pm 2$$

2) With one integration constant a:

$$y(x)=\sum_{N=0}^{\infty} y_N(x)(ax^{\omega})^N, \qquad y_N(x) \text{ Taylor series at } x=0.$$

The coefficients are certain rational functions of $\sqrt{2\alpha}$, $\sqrt{2\beta}$, $\sqrt{2\gamma}$, $\sqrt{1-2\delta}$.

$$\omega \in \mathbf{N}, \text{ or } \omega=\pm(\sqrt{2\alpha}\pm\sqrt{2\gamma}),$$

$$\operatorname{tr}(M_0M_x)=-2\cos(\pi\omega).$$

$\diamond$ Sub-cases: $\omega \in \mathbf{N}$ or $a=0 \Longrightarrow$ Transcendents with (convergent) Taylor expansion at $x=0$. The *basic* expansions are:

i) Parameter $a=0$, $\sqrt{2\gamma}\pm\sqrt{2\alpha}\not\in \mathbf{Z}$, $\alpha\neq 0$:

$$y(x)=\pm\frac{\sqrt{\gamma}\pm\sqrt{\alpha}}{\sqrt{\alpha}}+\sum_{n=2}^{\infty} b_n(\alpha,\beta,\gamma,\delta)\; x^n.$$

$$\operatorname{tr}(M_0M_x)=-2\cos(\pi[\sqrt{2\gamma}\pm\sqrt{2\alpha}]), \qquad <M_0M_x,M_1> \text{ reducible.}$$

ii) $\omega=1$, $\alpha\neq 0$, $\sqrt{-2\beta}\pm\sqrt{1-2\delta}\in \mathbf{Z}$:

$$y(x)=\frac{1}{\pm\sqrt{2\alpha}}+ax\; +\sum_{n=2}^{\infty} b_n(a;\alpha,\beta)x^n.$$

$$\operatorname{tr}(M_0M_x)=2, \quad M_0M_x=I, \qquad <M_0M_x,M_1> \text{ reducible.}$$

iii) $\alpha = 0$, $\omega = 0$:

$$y(x) = a + (1-a)(\delta - \beta)\ x\ + \sum_{n=2}^{\infty} b_n(a;\beta,\delta)x^n.$$

$$\mathrm{tr}(M_0 M_x) = -2, \qquad < M_0 M_x, M_1 > \ \text{reducible.}$$

---

3) Logarithmic behavior, with one integration constant a.

The *basic* (asymptotic) expansions are:

$$y(x) \sim x\left[\frac{1-2\delta+2\beta}{4}\ln^2 x + a\ln x + \frac{a+2\beta}{2\beta+1-2\delta}\right] +$$
$$+x^2 \cdot \{\text{Polynom in } \ln x\} + x^3 \cdot \{\text{Polynom in } \ln x\} + ...,$$

$$2\delta - 1 \neq 2\beta;$$

$$y(x) \sim x\left[a \pm \sqrt{-2\beta}\ln x\right] + x^2 \cdot \{\text{Polynom in } \ln x\} + ...,$$

$$2\delta - 1 = 2\beta.$$

$$a = a(\theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0 M_1), \mathrm{tr}(M_1 M_x))$$

$$\mathrm{tr}(M_0 M_x) = 2$$

in general $< M_0, M_x, M_1 >$ is irreducible

For the second solution, $< M_0, M_x >$ is reducible.

The symmetry $(\diamond\diamond\diamond)$ acting on the logarithmic expansions gives the following behaviors for $x \longrightarrow 0$:

$$y(x) = \frac{2}{(\gamma-\alpha)\ln^2 x}\left[1 - \frac{2a}{\gamma-\alpha}\frac{1}{\ln x} + O\left(\frac{1}{\ln^2 x}\right)\right], \qquad \alpha \neq \gamma;$$

$$y(x) = \frac{1}{\pm\sqrt{2\alpha}\ln x}\left[1 \mp \frac{a}{\sqrt{2\alpha}}\frac{1}{\ln x} + O\left(\frac{1}{\ln^2 x}\right)\right], \qquad \alpha = \gamma.$$

$$\mathrm{tr}(M_0 M_x) = -2$$

$\diamond$ *Remark:* In the examples, the monodromy groups are irreducible. In some cases, there are reducible subgroups.

$\diamond$ *Example* of one non-generic case

$$\alpha = \gamma = 0, \quad \delta = \frac{1}{2}, \quad \beta = -2p^2; \qquad p \in \mathbf{Z}$$

There is a 1-parameter (a) family of logarithmic solutions.

$$y(x) \sim \begin{cases} x\left[1 - p^2(\ln x + a)^2\right], & x \longrightarrow 0, \\ 1 - p^2\left(\ln\frac{1}{x} + \rho_\infty\right)^2, & x \longrightarrow \infty, \\ 1 - \frac{1}{p^2(\ln(1-x)+\rho_1)^2}, & x \longrightarrow 1. \end{cases}$$

where:

$$\rho_\infty = \frac{\pi(4\ln 2 - 1 + a)}{\pi - i(4\ln 2 - 1 + a)} - 2\ln 2 + 1, \quad \rho_1 = \frac{\pi^2}{4\ln 2 - 1 + a} - \ln 2 + 1.$$

$$M_0 = I, \quad \mathrm{tr}(M_0 M_x) = 2, \quad \mathrm{tr}(M_0 M_1) = 2, \quad \mathrm{tr}(M_1 M_x) = -2.$$

Monodromy independent of a

See also Mazzocco, *Math. Ann* (2001): *Chazy solutions*

Matching Procedure

Consider $x \to 0$, $\arg x$ bounded.

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{A_1}{\lambda - 1}\right]\Psi$$

Idea: Simplify the system when $x \to 0$. Keep only dominant terms.

Reduction to systems which can be solved in terms of special functions.

i) $|\lambda| \geq |x|^{\delta_{OUT}}$, $\delta_{OUT} > 0$:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{A_x}{\lambda}\sum_{n=1}^{N_{OUT}}\left(\frac{x}{\lambda}\right)^n + \frac{A_1}{\lambda - 1}\right]\ \Psi_{OUT}$$

ii) $|\lambda| \leq |x|^{\delta_{IN}}$, $\delta_{IN} > 0$:

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} - A_1\sum_{n=0}^{N_{IN}}\lambda^n\right]\ \Psi_{IN},$$

The simplest case is **Fuchsian**:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{A_1}{\lambda - 1}\right]\ \Psi_{OUT}$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x}\right]\ \Psi_{IN}.$$

!! Our method is based on the **Matching** condition: there exist fundamental solutions which match in the overlapping regions:

$$\Psi_{OUT}(\lambda,x)\sim\Psi_{IN}(\lambda,x),\quad x\to0,\qquad |x|^{\delta_{OUT}}\le|\lambda|\le|x|^{\delta_{IN}}.$$

$$\Psi(\lambda,x)\sim\Psi_{OUT}(\lambda,x),\qquad \lambda\to\infty,\qquad |\lambda|\ge|x|^{\delta_{OUT}}.$$

$$\Psi(\lambda,x)\sim\Psi_{IN}(\lambda,x),\qquad \lambda\to0 \text{ or } x,\qquad |\lambda|\le|x|^{\delta_{IN}}.$$

◇ Fuchsian truncations are not enough to obtain all the results explained above. **Non-Fuchsian** truncations are necessary.

Some of the results above are obtained from:

$$\frac{d\Psi_{OUT}}{d\lambda}=\left[\frac{x\ A_x}{\lambda^2}+\frac{A_0+A_x}{\lambda}+\frac{A_1}{\lambda-1}\right]\ \Psi_{OUT}$$

$$\frac{d\Psi_{IN}}{d\lambda}=\left[\frac{A_0}{\lambda}+\frac{A_x}{\lambda-x}-A_1\right]\ \Psi_{IN},$$

Part I: Matching $\Psi_{OUT} \sim \Psi_{IN} \Longrightarrow$ Leading Term(s) of $y(x)$

Determine the asymptotic behavior of $A_0(x)$, $A_1(x)$, $A_x(x)$

$$\Longrightarrow \quad y(x) = \frac{x\ (A_0)_{12}}{x\ [(A_0)_{12} + (A_1)_{12}] - (A_1)_{12}}.$$

We are in the domain

$$|x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}}, \quad x \to 0, \quad \lambda \to 0, \quad \lambda/x \to \infty.$$

i) Write general parameterization of $A_0 + A_x$ and $A_0,\ A_x,\ A_1$ in terms of: $\theta_0,\ \theta_x,\ \theta_1,\ \theta_\infty$, and (four) more scalar *unknown functions* of x. One of them is the eigenvalue of $A_0 + A_x$.

ii) Find the *local behaviors* of: $\Psi_{OUT}(\lambda, x)$ for $\lambda \to 0$,
$\Psi_{IN}(\lambda, x)$ for $\frac{\lambda}{x} \to \infty$,
in terms of the entries of A_0, A_1, A_x.

iii) Impose:

1) that the OUT and IN systems are *isomonodromic* $\Longrightarrow$ Two unknown functions are constant (integration constants).

2) The *matching* condition:

$$\Psi_{OUT}(\lambda, x) \sim \Psi_{IN}(\lambda, x)$$

It gives the leading terms of the remaining unknown function of x.

$$\Downarrow$$

We obtain the leading terms in $x \to 0$ of $A_0,\ A_x,\ A_1 \Rightarrow y(x)$. They depend on $\theta_0, \theta_x, \theta_1, \theta_\infty$, and two integration constants.

Example 1: Fuchsian reduction

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{A_1}{\lambda - 1}\right] \Psi_{OUT}, \qquad \frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x}\right] \Psi_{IN}.$$

By elementary linear algebra we have:

$$A_1 = \begin{pmatrix} \frac{\sigma^2-\theta_\infty^2-\theta_1^2}{4\theta_\infty} & -r_1 \\ \frac{[\sigma^2-(\theta_1-\theta_\infty)^2][\sigma^2-(\theta_1+\theta_\infty)^2]}{16\theta_\infty^2}\ \frac{1}{r_1} & -\frac{\sigma^2-\theta_\infty^2-\theta_1^2}{4\theta_\infty} \end{pmatrix},$$

$$A_0 + A_x = \begin{pmatrix} \frac{\theta_1^2-\sigma^2-\theta_\infty^2}{4\theta_\infty} & r_1 \\ -\frac{[\sigma^2-(\theta_1-\theta_\infty)^2][\sigma^2-(\theta_1+\theta_\infty)^2]}{16\theta_\infty^2}\ \frac{1}{r_1} & -\frac{\theta_1^2-\sigma^2-\theta_\infty^2}{4\theta_\infty} \end{pmatrix}.$$

$$K(x)^{-1}A_0K(x) = \begin{pmatrix} \frac{\theta_0^2-\theta_x^2+\sigma^2}{4\sigma} & r \\ -\frac{[\sigma^2-(\theta_0-\theta_x)^2][\sigma^2-(\theta_0+\theta_x)^2]}{16\sigma^2}\ \frac{1}{r} & -\frac{\theta_0^2-\theta_x^2+\sigma^2}{4\sigma} \end{pmatrix},$$

$$K(x)^{-1}A_xK(x) = \begin{pmatrix} \frac{\sigma^2+\theta_x^2-\theta_0^2}{4\sigma} & -r \\ \frac{[\sigma^2-(\theta_0-\theta_x)^2][\sigma^2-(\theta_0+\theta_x)^2]}{16\sigma^2}\ \frac{1}{r} & -\frac{\sigma^2+\theta_x^2-\theta_0^2}{4\sigma} \end{pmatrix}.$$

Unknown functions: $r_1,\ r,\ K$

Eigenvalues of $A_0 + A_x \ = \ \pm\frac{\sigma}{2}$.

- ◇ Isomonodromy $\implies$ σ, r_1, r are constants

- ◇ We determine $K(x)$ by matching.

Case $\sigma \notin \mathbf{Z}$. Standard theory of Fuchsian systems gives:

$$\Psi_{OUT} = \sum_{n=0}^{\infty} G_n \lambda^n \begin{pmatrix} \lambda^{\frac{\sigma}{2}} & 0 \\ 0 & \lambda^{-\frac{\sigma}{2}} \end{pmatrix}, \qquad \lambda \to 0$$

$$G_0 := \begin{pmatrix} 1 & 1 \\ \frac{(\theta_\infty+\sigma)^2-\theta_1^2}{4\theta_\infty r_1} & \frac{(\theta_\infty-\sigma)^2-\theta_1^2}{4\theta_\infty r_1} \end{pmatrix}$$

$$\Psi_{IN} = \left[K(x) + \sum_{n=1}^{\infty} K_n \frac{x^n}{\lambda^n} \right] \begin{pmatrix} (\lambda/x)^{\frac{\sigma}{2}} & 0 \\ 0 & (\lambda/x)^{-\frac{\sigma}{2}} \end{pmatrix}, \qquad \frac{\lambda}{x} \to \infty.$$

We obtain $K(x)$ through the matching $\Psi_{OUT} \sim \Psi_{IN}$:

$$G_0 \begin{pmatrix} \lambda^{\frac{\sigma}{2}} & 0 \\ 0 & \lambda^{-\frac{\sigma}{2}} \end{pmatrix} \sim K(x) \begin{pmatrix} \lambda^{\frac{\sigma}{2}} & 0 \\ 0 & \lambda^{-\frac{\sigma}{2}} \end{pmatrix} \begin{pmatrix} x^{-\frac{\sigma}{2}} & 0 \\ 0 & x^{\frac{\sigma}{2}} \end{pmatrix}.$$

$$\Downarrow$$

$$K(x) \sim \begin{pmatrix} 1 & 1 \\ \frac{(\theta_\infty+\sigma)^2-\theta_1^2}{4\theta_\infty r_1} & \frac{(\theta_\infty-\sigma)^2-\theta_1^2}{4\theta_\infty r_1} \end{pmatrix} \begin{pmatrix} x^{\frac{\sigma}{2}} & 0 \\ 0 & x^{-\frac{\sigma}{2}} \end{pmatrix}.$$

$$\Downarrow$$

So we compute A_0, A_x, A_1 and

$$y(x) = \frac{x \ (A_0)_{12}}{x \ [(A_0)_{12} + (A_1)_{12}] - (A_1)_{12}}.$$

$$y(x) \sim \frac{1}{r} \frac{[\sigma^2 - (\theta_0 + \theta_x)^2][(\theta_0 - \theta_x)^2 - \sigma^2]}{16\sigma^3} x^{1-\sigma} + \frac{\theta_0^2 - \theta_x^2 + \sigma^2}{2\sigma^2} x - \frac{r}{\sigma} x^{1+\sigma}$$

Keep only leading term, according to $\Re\sigma > 0$ or < 0.

Example 2: One example of non Fuchsian reduction

* Non-generic case $\theta_1 \pm \theta_\infty = 0$, $\theta_x \pm \theta_0 = 0$, $\text{Eigenv}(A_0 + A_x) = 0$.

$$\lim_{x\to 0}(A_0(x) + A_x(x)) = 0,$$

$$A := \lim_{x\to 0} A_x(x) = \text{ a constant matrix with eigenvalues } \pm \frac{\theta_x}{2}$$

$$\Downarrow$$

$$A_0 + A_x = \begin{pmatrix} a(x) & b(x)\, r \\ c(x)\, \frac{1}{r} & -a(x) \end{pmatrix}, \quad \lim_{x\to 0} a(x) = \lim_{x\to 0} b(x) = \lim_{x\to 0} c(x) = 0,$$

$$A = \begin{pmatrix} s + \frac{\theta_x}{2} & -r \\ \frac{(s+\theta_x)\ s}{r} & -s - \frac{\theta_x}{2} \end{pmatrix}, \quad r, s \in \mathbf{C}, \quad r \neq 0.$$

$$A_1(x) = -\frac{\theta_\infty}{2}\,\sigma_3 \ - (A_0 + A_x) \ \longrightarrow \ -\frac{\theta_\infty}{2}\,\sigma_3, \quad x \to 0.$$

◇ Problem: determine $a(x)$, $b(x)$, $c(x)$ asymptotically.

◇ We need non-fuchsian reduction:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{xA}{\lambda^2} + \frac{A_0 + A_x}{\lambda} - \frac{\theta_\infty}{2}\frac{\sigma_3}{\lambda - 1}\right] \Psi_{OUT},$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{\theta_\infty}{2}\sigma_3\right] \Psi_{IN}.$$

Let $G^{-1}AG = -\frac{\theta_x}{2}\sigma_3$.

$\diamond$ Isomonodromy $\Longrightarrow$ $\mathrm{diag}(A_0+A_x)=0$, $\mathrm{diag}(G^{-1}(A_0+A_x)G)=0$.

$$\Updownarrow$$

$$a(x)=o(b(x),c(x)), \quad c(x)\sim s(s+\theta_x)b(x)$$

$\diamond$ Local behaviors for $\lambda\to 0$:

$$\Psi_{OUT}=G\left[I+G_1^{OUT}\lambda+...\right]\ \exp\left\{x\ \frac{\theta_x}{2}\sigma_3\ \frac{1}{\lambda}\right\}\ G^{-1}=$$

$$=\ I\ +GG_1^{OUT}G^{-1}\lambda+\frac{\theta_x}{2}G\sigma_3G^{-1}\ \frac{x}{\lambda}+O\left(\lambda^2,x,\frac{x^2}{\lambda^2}\right)$$

$\diamond$ Local behavior for $\lambda/x\to\infty$:

$$\Psi_{IN}=\left[I+G_1^{IN}\ \frac{x}{\lambda}+...\right]\ \exp\left\{\frac{\theta_\infty}{2}\lambda\right\}=$$

$$=I\ +\frac{\theta_\infty}{2}\sigma_3\ \lambda\ +G_1^{IN}\ \frac{x}{\lambda}+O\left(\lambda^2,x,\frac{x^2}{\lambda^2}\right)$$

G_1^{IN}, G_1^{IN} are computed as functions of s,r,b.

$\diamond$ Matching:

$$G_1^{OUT}(x)\sim\frac{\theta_\infty}{2}G^{-1}\sigma_3\ G,\qquad G_1^{IN}(x)\sim\frac{\theta_x}{2}G\sigma_3G^{-1}$$

$$\Updownarrow$$

$$b(x)\sim -x\ \theta_\infty,\qquad x\to 0$$

$\diamond$ Final result:

$$y(x)=\frac{1}{1-\theta_\infty}+\frac{\theta_\infty(2s+\theta_x+1)}{2(\theta_\infty-1)}\ x\ +O(x^2),$$

Part II: Matching $\Psi \sim \Psi_{OUT}$ and $\Psi \sim \Psi_{IN} \Longrightarrow$ Monodromy, Connection Problem

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{A_1}{\lambda - 1}\right]\Psi$$

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{A_x}{\lambda}\sum_{n=1}^{N_{OUT}}\left(\frac{x}{\lambda}\right)^n + \frac{A_1}{\lambda - 1}\right]\ \Psi_{OUT}$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} - A_1\sum_{n=0}^{N_{IN}}\lambda^n\right]\ \Psi_{IN},$$

If three fundamental solutions satisfy:

$$\Psi_{OUT}(\lambda, x) \sim \Psi_{IN}(\lambda, x), \qquad |x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}}, \quad x \to 0.$$

$$\Psi(\lambda, x) \sim \Psi_{OUT}(\lambda, x), \qquad \lambda \to \infty \quad (|\lambda| \geq |x|^{\delta_{OUT}})$$

$$\Psi(\lambda, x) \sim \Psi_{IN}(\lambda, x), \qquad \lambda \to 0,\ x \quad (|\lambda| \leq |x|^{\delta_{IN}})$$

then (consequence of isomonodromy):

$$M_1 = M_1^{OUT}, \qquad M_\infty = M_\infty^{OUT}$$

$$M_0 = M_0^{IN}, \qquad M_x = M_x^{IN}$$

◇ Standard normalization for Ψ:

$$\Psi(\lambda) = \left[I + O\left(\frac{1}{\lambda}\right)\right]\ \lambda^{-\frac{\theta_\infty}{2}\sigma_3}\lambda^{R_\infty}, \quad \lambda \to \infty;$$

For suitable normalization: $\Psi_{OUT}(\lambda, x) \sim \Psi_{IN}(\lambda, x)$, $x \to 0$.

◇ Choose normalization C s.t.

$$\Psi_{OUT}(\lambda, x)\ C = \left[I + O\left(\frac{1}{\lambda}\right)\right]\ \lambda^{-\frac{\theta_\infty}{2}\sigma_3}\lambda^{R_\infty} \sim \Psi(\lambda, x), \quad \lambda \to \infty$$

$\Psi_{OUT}(\lambda, x)\ C$, $\Psi_{IN}\ C$ realize the matching:

$$\Psi_{OUT}(\lambda, x)\ C \sim \Psi(\lambda, x), \qquad \lambda \to \infty$$

$$\Psi_{IN}(\lambda, x)\ C \sim \Psi(\lambda, x), \qquad \lambda \to 0,\ x$$

◇ We must be able to compute the monodromy of Ψ_{OUT}, Ψ_{IN} exactly:

This is possible, in our cases, because the systems for $\Psi_{OUT}(\lambda, x)$, $\Psi_{IN}(\lambda, x)$ are solvable in terms of *special functions*, both in the Fuchsian and non-Fuchsian cases.

Example1:

$$y(x) \sim a x^{1-\sigma}.$$

$$2\cos(\pi\sigma) = \mathrm{tr}(M_0 M_x), \qquad a = \frac{1}{r}\,\frac{[\sigma^2-(\theta_0+\theta_x)^2][(\theta_0-\theta_x)^2-\sigma^2]}{16\sigma^3}$$

$$r = \frac{(\theta_0-\theta_x+\sigma)(\theta_0+\theta_x-\sigma)(\theta_\infty+\theta_1-\sigma)}{4\sigma(\theta_\infty+\theta_1+\sigma)}\;\frac{1}{\mathbf{F}},$$

where

$$\mathbf{F} := \frac{\Gamma(1+\sigma)^2\Gamma\left(\frac{1}{2}(\theta_0+\theta_x-\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0-\sigma)+1\right)}{\Gamma(1-\sigma)^2\Gamma\left(\frac{1}{2}(\theta_0+\theta_x+\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0+\sigma)+1\right)}\times$$

$$\times\frac{\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1-\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty-\sigma)+1\right)}{\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1+\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty+\sigma)+1\right)}\;\frac{V}{U},$$

and:

$$U := \left[\frac{i}{2}\sin(\pi\sigma)\mathrm{tr}(M_1M_x) - \cos(\pi\theta_x)\cos(\pi\theta_\infty) - \cos(\pi\theta_0)\cos(\pi\theta_1)\right]e^{i\pi\sigma} +$$

$$+\frac{i}{2}\sin(\pi\sigma)\mathrm{tr}(M_0M_1) + \cos(\pi\theta_x)\cos(\pi\theta_1) + \cos(\pi\theta_\infty)\cos(\pi\theta_0)$$

$$V := 4\sin\frac{\pi}{2}(\theta_0+\theta_x-\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_x+\sigma)\;\sin\frac{\pi}{2}(\theta_\infty+\theta_1-\sigma)\sin\frac{\pi}{2}(\theta_\infty-\theta_1+\sigma).$$

First computed in M.Jimbo: *Publ. RIMS* **18** (1981).

Example 2:

$$y(x) = \frac{1}{1-\theta_\infty} + a\ x\ + O(x^2), \qquad a = \frac{\theta_\infty(2s+\theta_x+1)}{2(\theta_\infty-1)}$$

$$M_0 = G\ \exp\{i\pi\theta_x\sigma_3\}\ G^{-1}, \qquad M_1 = \exp\{-i\pi\theta_\infty\sigma_3\}$$

$$M_x = G\ \exp\{-i\pi\theta_x\sigma_3\}\ G^{-1}, \qquad M_\infty = \exp\{-i\pi\theta_\infty\sigma_3\}$$

$$G = \begin{pmatrix} 1 & 1 \\ \frac{s+\theta_x}{r} & \frac{s}{r} \end{pmatrix}.$$

$$s = \frac{\theta_x[2\cos(\pi(\theta_\infty+\theta_x)) - \mathrm{tr}(M_1 M_0)]}{2[\cos(\pi(\theta_\infty-\theta_x)) - \cos(\pi(\theta_\infty+\theta_x))]}.$$

θ_x, $\theta_\infty \notin \mathbf{Z}$.

Example 3:

$$y(x) \ = \ a \ + \ \frac{1-a}{2}(1+\theta_0^2-\theta_x^2)\ x \ + \ O(x^2), \qquad a =: \frac{1}{1-s}$$

$$M_0 = (C_{\infty 0})^{-1} \ \exp\{i\pi\theta_0\sigma_3\}\ C_{\infty 0}, \qquad M_\infty = \begin{pmatrix} -1 & 0 \\ 2\pi i\,(1-s) & -1 \end{pmatrix}$$

$$M_x = (C_{\infty 0})^{-1}(C_{01})^{-1} \ \exp\{i\pi\theta_x\sigma_3\}\ C_{01}C_{\infty 0}, \qquad M_1 = \begin{pmatrix} 1 & 0 \\ 2\pi i s & 1 \end{pmatrix}.$$

$$C_{\infty 0} = \ 2 \ \begin{pmatrix} 0 & \frac{\Gamma(-\theta_0)\ e^{-i\pi\{\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{3}{2}\}}}{\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{3}{2}\right)\Gamma\left(-\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{3}{2}\right)} \\ \frac{\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{1}{2}\right)\Gamma\left(-\frac{\theta_0}{2}+\frac{\theta_x}{2}-\frac{1}{2}\right)}{\Gamma(1-\theta_0)e^{-i\pi\{\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{3}{2}\}}} & \frac{\Gamma(\theta_0)\ e^{-i\pi\{-\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{3}{2}\}}}{\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{3}{2}\right)\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{3}{2}\right)} \end{pmatrix}.$$

$$C_{01} = \begin{pmatrix} \frac{\Gamma(-\theta_x)\Gamma(1+\theta_0)}{\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{3}{2}\right)\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{1}{2}\right)} & \frac{\Gamma(-\theta_x)\Gamma(1-\theta_0)}{\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{3}{2}\right)\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{1}{2}\right)} \\ \frac{\Gamma(\theta_x)\Gamma(1+\theta_0)}{\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{3}{2}\right)\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{1}{2}\right)} & \frac{\Gamma(\theta_x)\Gamma(1-\theta_0)}{\Gamma\left(-\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{3}{2}\right)\Gamma\left(-\frac{\theta_0}{2}+\frac{\theta_x}{2}-\frac{1}{2}\right)} \end{pmatrix}$$

$$s = \frac{\mathrm{tr}(M_1M_0) - 2\cos(\pi\theta_0)}{4\pi\sin(\pi\theta_0)}\frac{(C_{\infty 0})_{21}}{(C_{\infty 0})_{22}}.$$

$\theta_0,\, \theta_x \notin \mathbf{Z}$.

Example 4:

$$y(x)=\frac{\theta_1-\theta_\infty+1}{1-\theta_\infty}+\frac{\theta_1[(\theta_1-\theta_\infty)^2+\theta_x^2-\theta_0^2+2\theta_1-2\theta_\infty]}{2(1-\theta_\infty)(\theta_\infty-\theta_1)(\theta_1-\theta_\infty+2)}\ x\ +O(x^2).$$

$$M_0=C_{0\infty}\ \exp\{i\pi\theta_0\sigma_3\}\ C_{0\infty}^{-1},$$

$$M_x=C_{0\infty}\ C_{01}^{-1}\ \exp\{i\pi\theta_x\sigma_3\}\ C_{01}\ C_{0\infty}^{-1}.$$

$$M_1=\exp\{-i\pi\theta_1\sigma_3\},\quad M_\infty=\exp\{-i\pi\theta_\infty\sigma_3\}.$$

$$C_{01}:=$$

$$\begin{bmatrix}\frac{\Gamma(-\theta_x)\Gamma(1+\theta_0)}{\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{\theta_1}{2}-\frac{\theta_\infty}{2}+1\right)\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)} & \frac{\Gamma(-\theta_x)\Gamma(1-\theta_0)}{\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{\theta_\infty}{2}+\frac{\theta_1}{2}+1\right)\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{\theta_1}{2}+\frac{\theta_\infty}{2}\right)}\\ \frac{\Gamma(\theta_x)\Gamma(1+\theta_0)}{\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{\theta_1}{2}-\frac{\theta_\infty}{2}+1\right)} & \frac{\Gamma(\theta_x)\Gamma(1-\theta_0)}{\Gamma\left(\frac{\theta_x}{2}-\frac{\theta_0}{2}+\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)\Gamma\left(\frac{\theta_x}{2}-\frac{\theta_0}{2}+\frac{\theta_1}{2}-\frac{\theta_\infty}{2}+1\right)}\end{bmatrix}$$

$$C_{0\infty}:=$$

$$\begin{bmatrix}\frac{\Gamma\left(1+\frac{\theta_1}{2}-\frac{\theta_\infty}{2}\right)\Gamma(1+\theta_0)e^{i\frac{\pi}{2}[\theta_0+\theta_x+\theta_\infty-\theta_1]}}{\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{\theta_1}{2}-\frac{\theta_\infty}{2}+1\right)\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{\theta_1}{2}-\frac{\theta_\infty}{2}+1\right)} & \frac{\Gamma\left(1+\frac{\theta_1}{2}-\frac{\theta_\infty}{2}\right)\Gamma(1-\theta_0)e^{i\frac{\pi}{2}[\theta_x-\theta_0+\theta_\infty-\theta_1]}}{\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{\theta_\infty}{2}+\frac{\theta_1}{2}+1\right)\Gamma\left(\frac{\theta_x}{2}-\frac{\theta_0}{2}+\frac{\theta_1}{2}-\frac{\theta_\infty}{2}+1\right)}\\ -\frac{\Gamma\left(\frac{\theta_\infty}{2}-\frac{\theta_1}{2}-1\right)\Gamma(1+\theta_0)e^{i\frac{\pi}{2}[\theta_0+\theta_x+\theta_1-\theta_\infty]}}{\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)} & -\frac{\Gamma\left(\frac{\theta_\infty}{2}-\frac{\theta_1}{2}-1\right)\Gamma(1-\theta_0)e^{i\frac{\pi}{2}[\theta_x-\theta_0+\theta_1-\theta_\infty]}}{\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{\theta_1}{2}+\frac{\theta_\infty}{2}\right)\Gamma\left(\frac{\theta_x}{2}-\frac{\theta_0}{2}+\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)}\end{bmatrix}$$

$\theta_\kappa\notin\mathbf{Z}$, $\kappa=0,1,x,\infty$.

Example 4 was computed also in K.Kaneko *Proc.Japan.Acad* **82** (2006). Heun's type scalar equation.

Conclusion

◇ The matching procedure yields the leading terms of the asymptotic behavior of $y(x)$

◇ The leading terms are expressed as functions of of monodromy data $\Rightarrow$ Solution of the connection problem.

◇ It is a constructive procedure.

◇ It allowed us to compute several new solutions (generic and non generic cases).
- Two parameter solutions.
- One parameter solutions, including logarithmic type.
- Taylor solutions.

* Conjecture: Applying symmetry transformations to the basic solutions we obtain almost all the asymptotic behaviors (classification problem). These, together with previous results obtained with an elliptic representation, should provide all the asymptotic behaviors (?).