

Semiclassical Limit of focusing NLS

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with

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Outline

- ♦ Introduction
- ♦ Semiclassical limit of the focusing NLS equation before the time of gradient catastrophe
- ♦ Painlevé I near the gradient catastrophe
- ♦ *tritronquées* solution to Painlevé I
- ♦ Numerical methods
- ♦ Outlook

Nonlinear Schrödinger equation

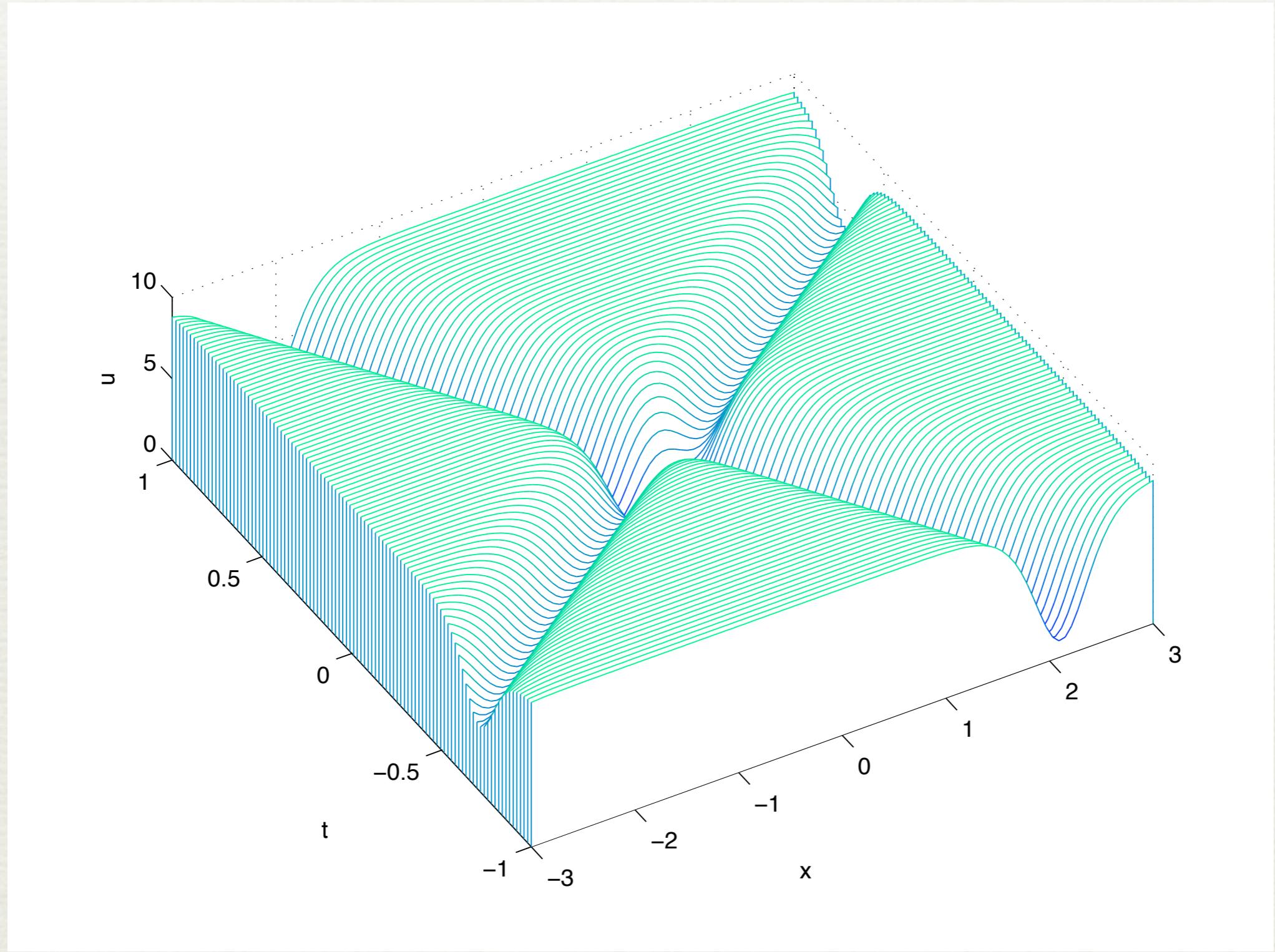
- NLS equation

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\psi_{xx} - \rho|\psi|^2\psi = 0$$

$\rho = -1$: focusing, $\rho = 1$: defocusing

- applications in nonlinear optics (fiber optics), hydrodynamics,...
- completely integrable (Zakharov, Shabat)

2-soliton, defocusing

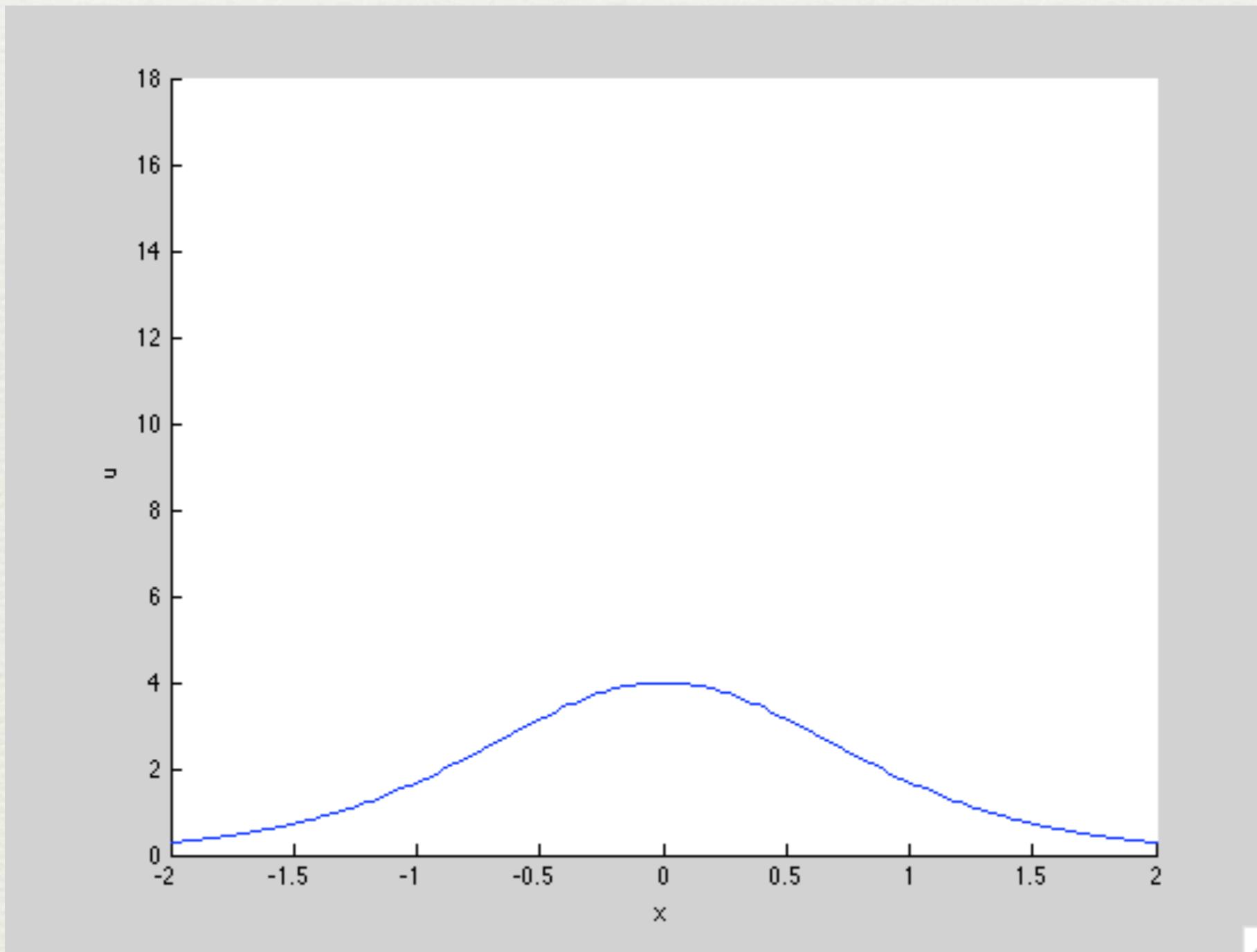


$$u = |\psi|^2$$

Breather

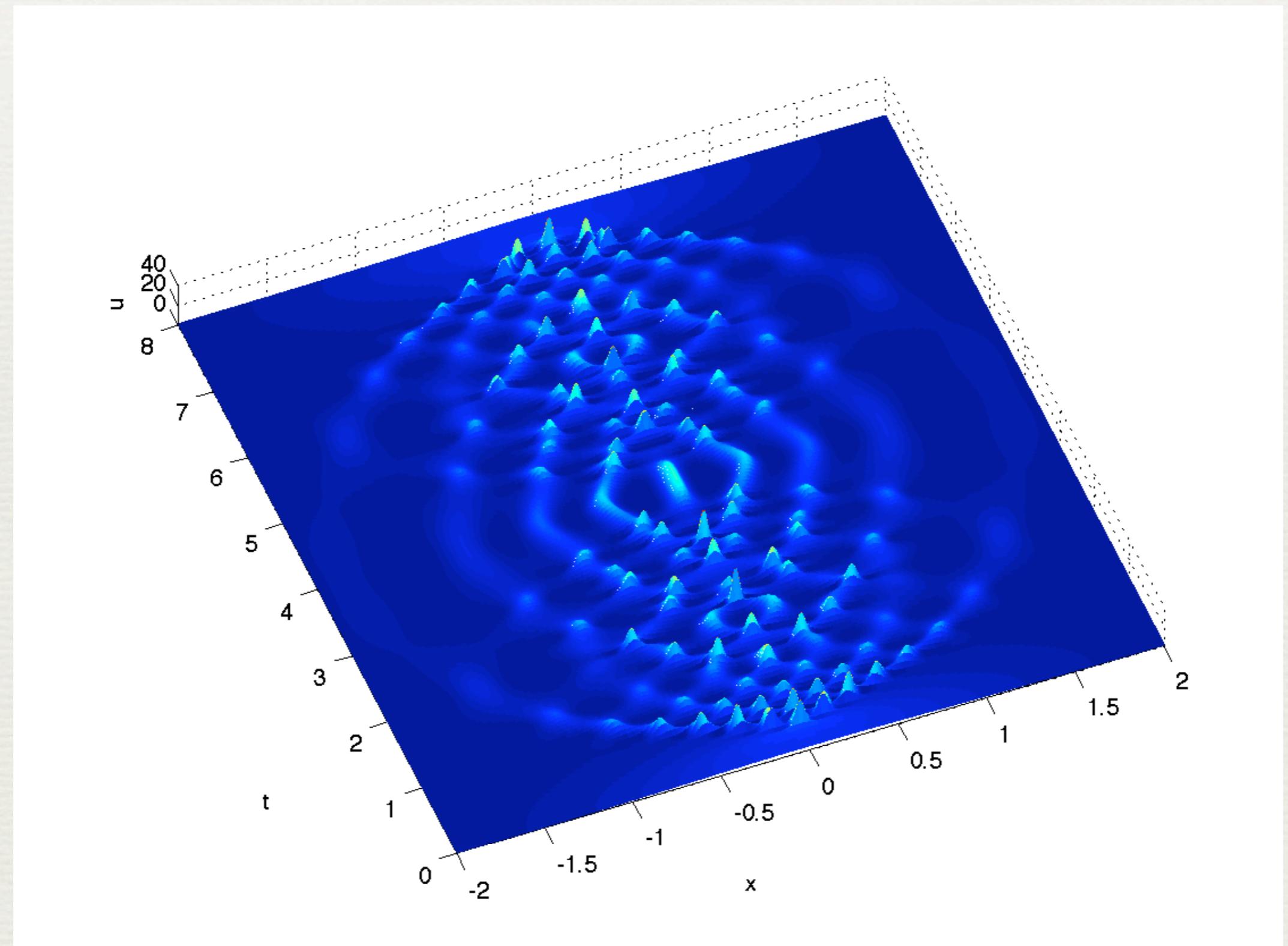
$$\psi = 4 \exp(it/2) \frac{\cosh(3x) + 3 \exp(4it) \cosh(x)}{\cosh(4x) + 4 \cosh(2x) + 3 \cos(4t)}$$

$$\psi_0 = 2 \operatorname{sech} x$$



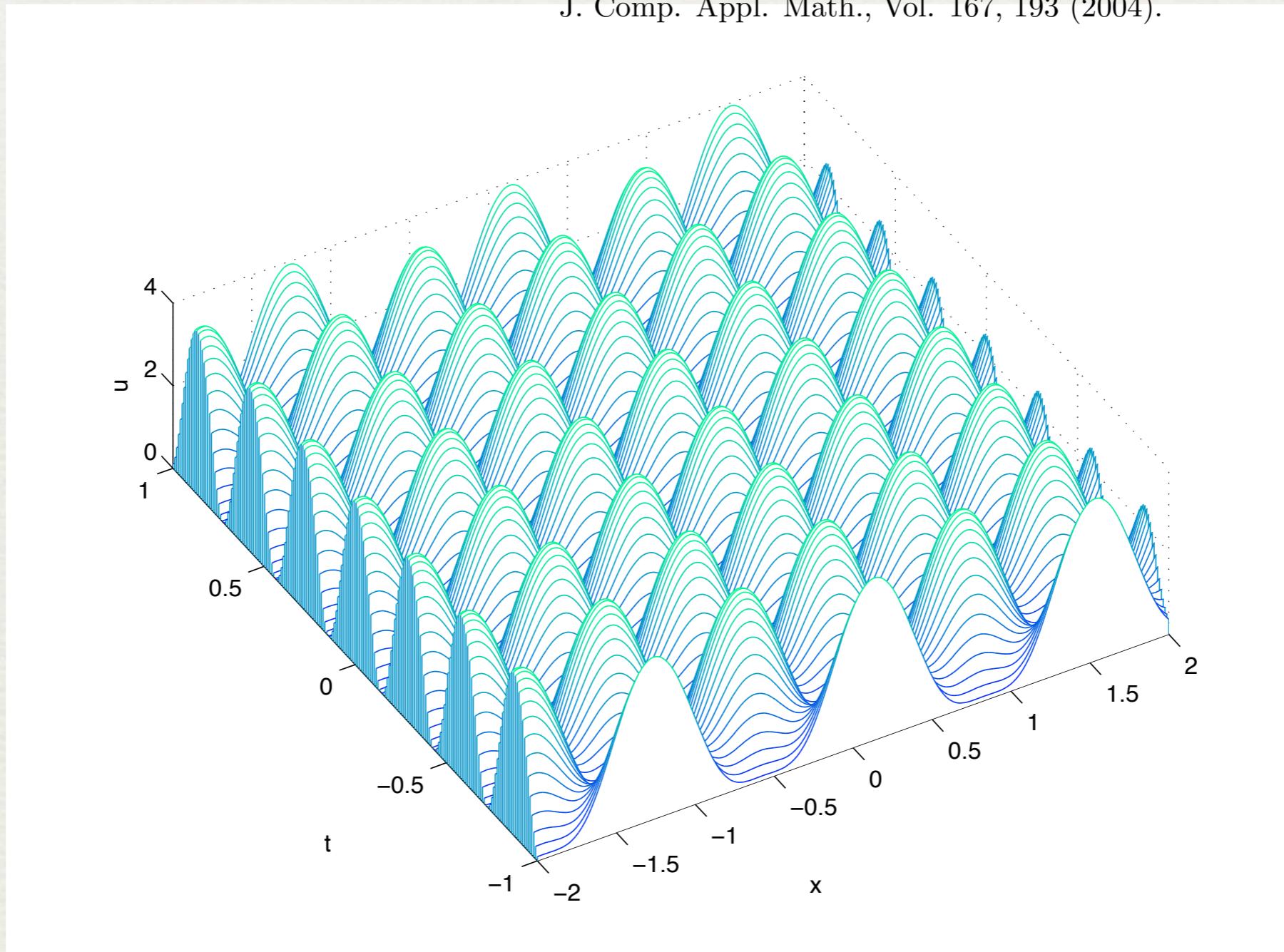
$$u = |\psi|^2$$

- exact solution for $\psi_0 = N \operatorname{sech} x$ in terms of ill conditioned determinant (Satsuma, Yajima 1974),
- rescaling $x \rightarrow x/\epsilon$, $t \rightarrow t/\epsilon$, here $\psi_0 = 2 \operatorname{sech} x$, $\epsilon = 0.2$,



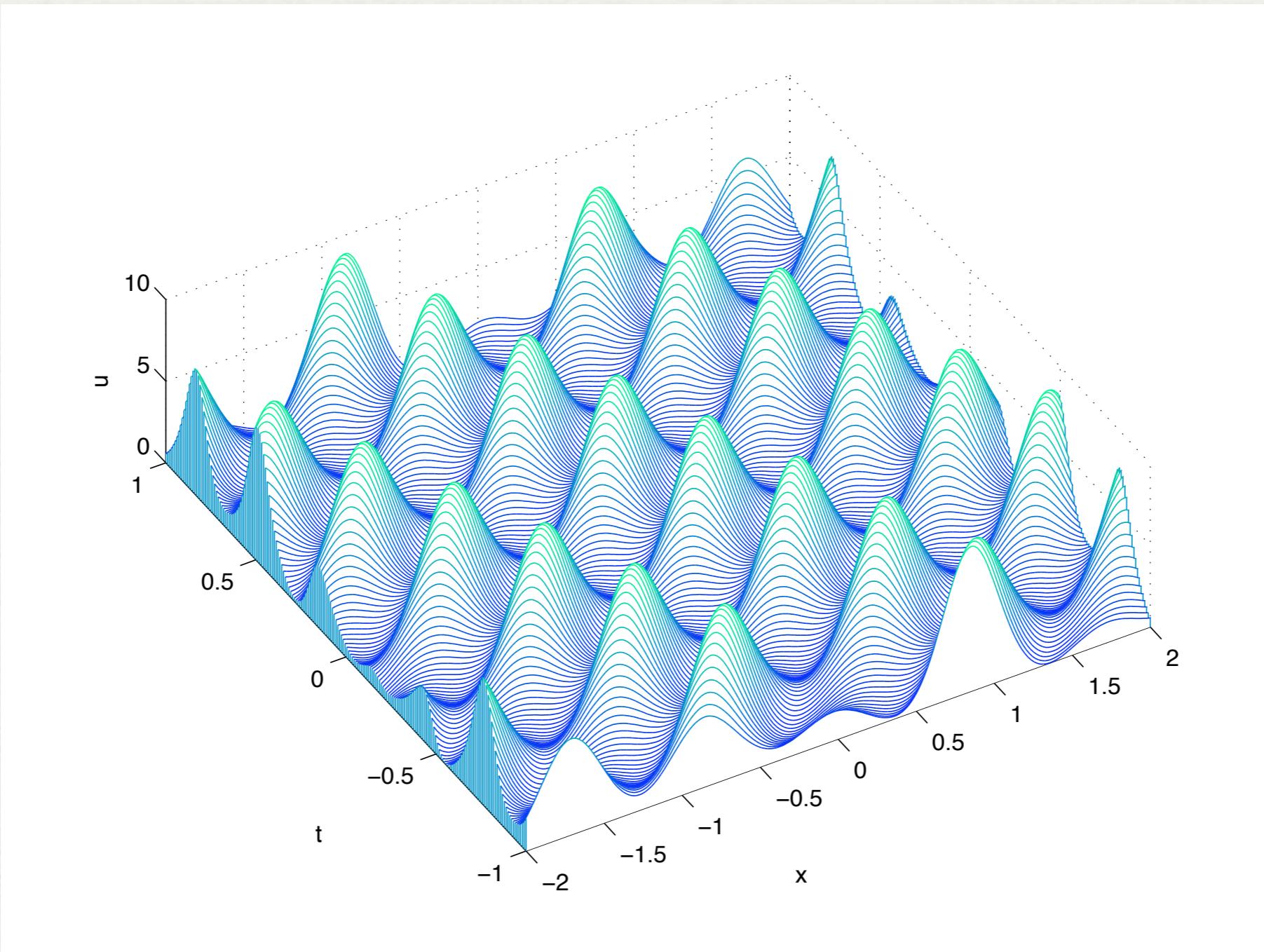
Hyperelliptic solution, $g=2$, defocusing

J. Frauendiener et C. K., 'Hyperelliptic theta functions and spectral methods: KdV and KP solutions', Lett. Math. Phys., Vol. 76, 249-267 (2006).
J. Frauendiener et C. K., 'Hyperelliptic theta functions and spectral methods', J. Comp. Appl. Math., Vol. 167, 193 (2004).



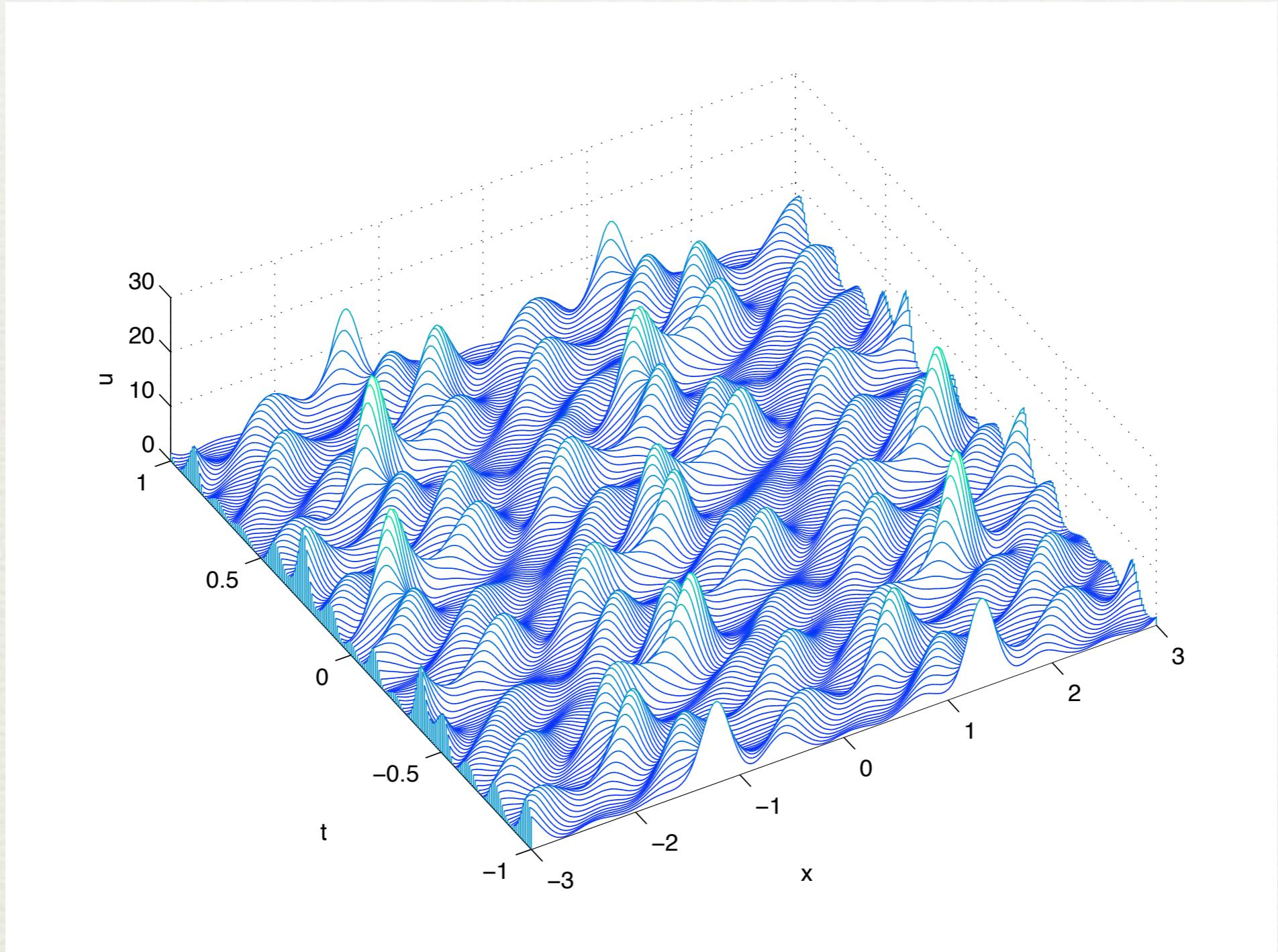
Hyperelliptic solution

$g=2$, focusing



Hyperelliptic solution

$g=4$, focusing



Hopf equation

- hyperbolic conservation law for $u(x, t)$, initial data $u_0(x)$

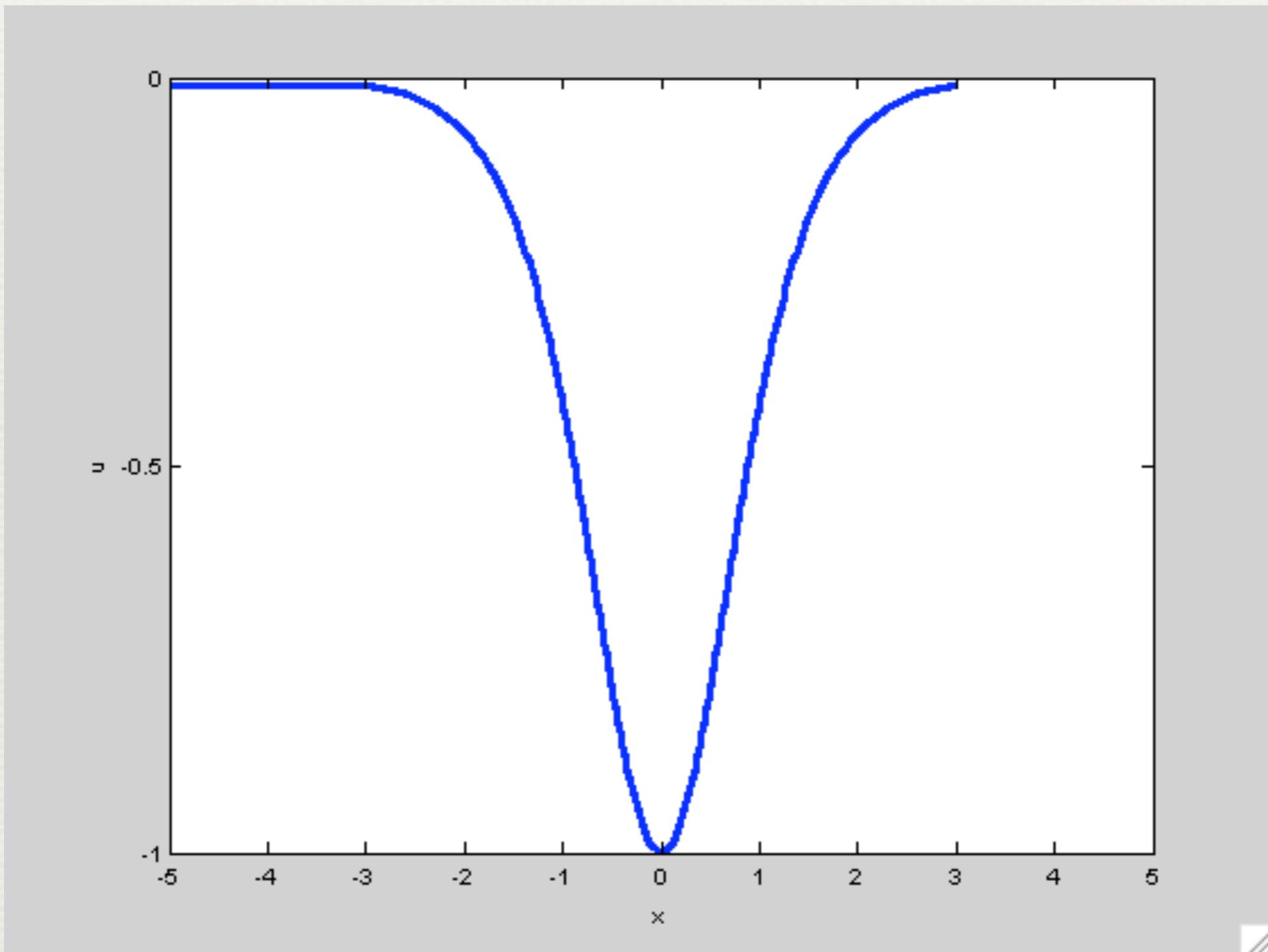
$$u_t + 6uu_x = 0, \quad u(x, 0) = u_0(x)$$

- solution with the method of characteristics

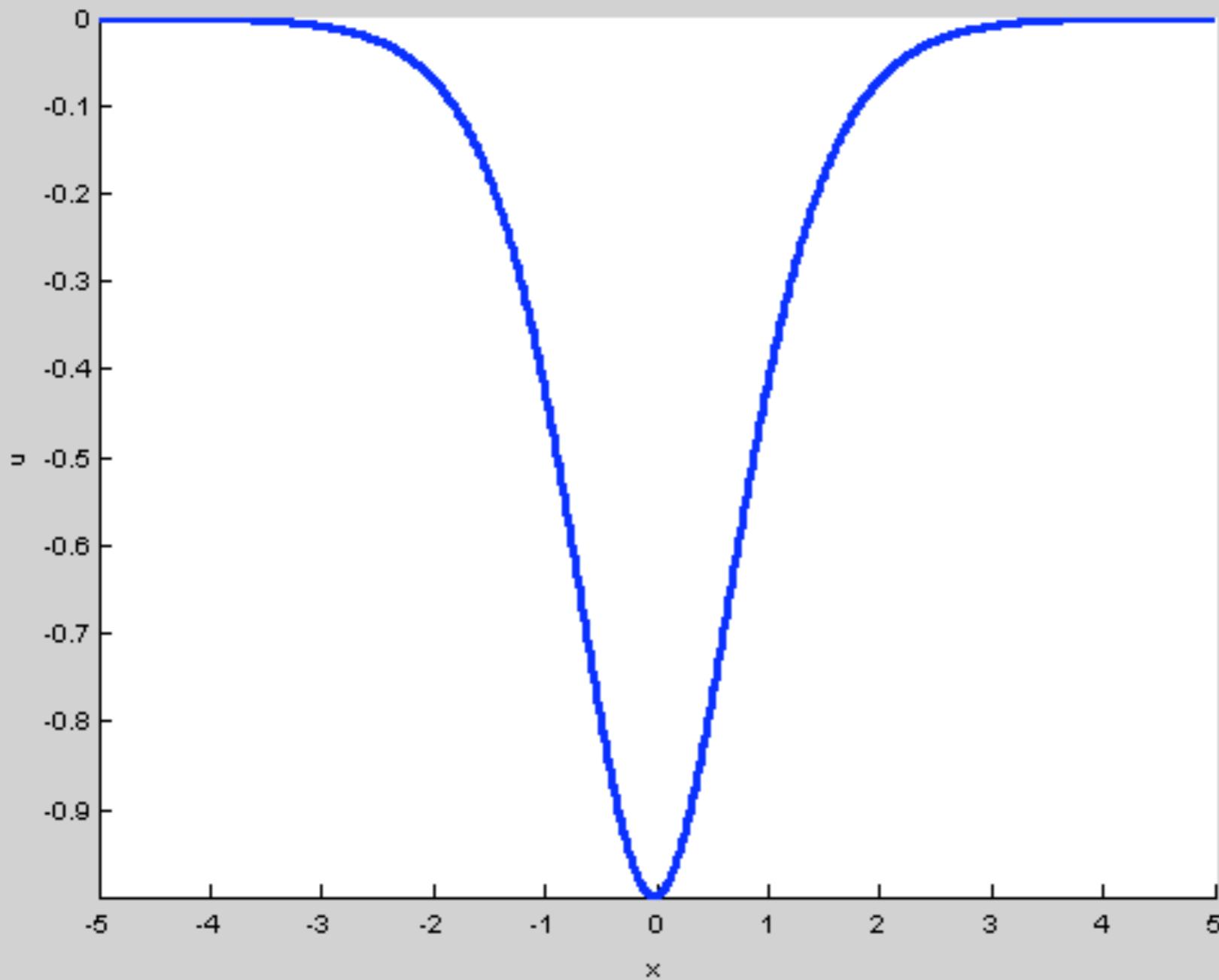
$$u(x, t) = u_0(\xi), \quad x = 6tu_0(\xi) + \xi$$

- critical time $t_c = \frac{1}{\min_{\xi \in \mathbb{R}} [-6u'_0(\xi)]}$,
gradient catastrophe,
 $t > t_c$: solution multivalued (shock)

Example: $u_0 = -\operatorname{sech}^2 x$



Dissipative regularization

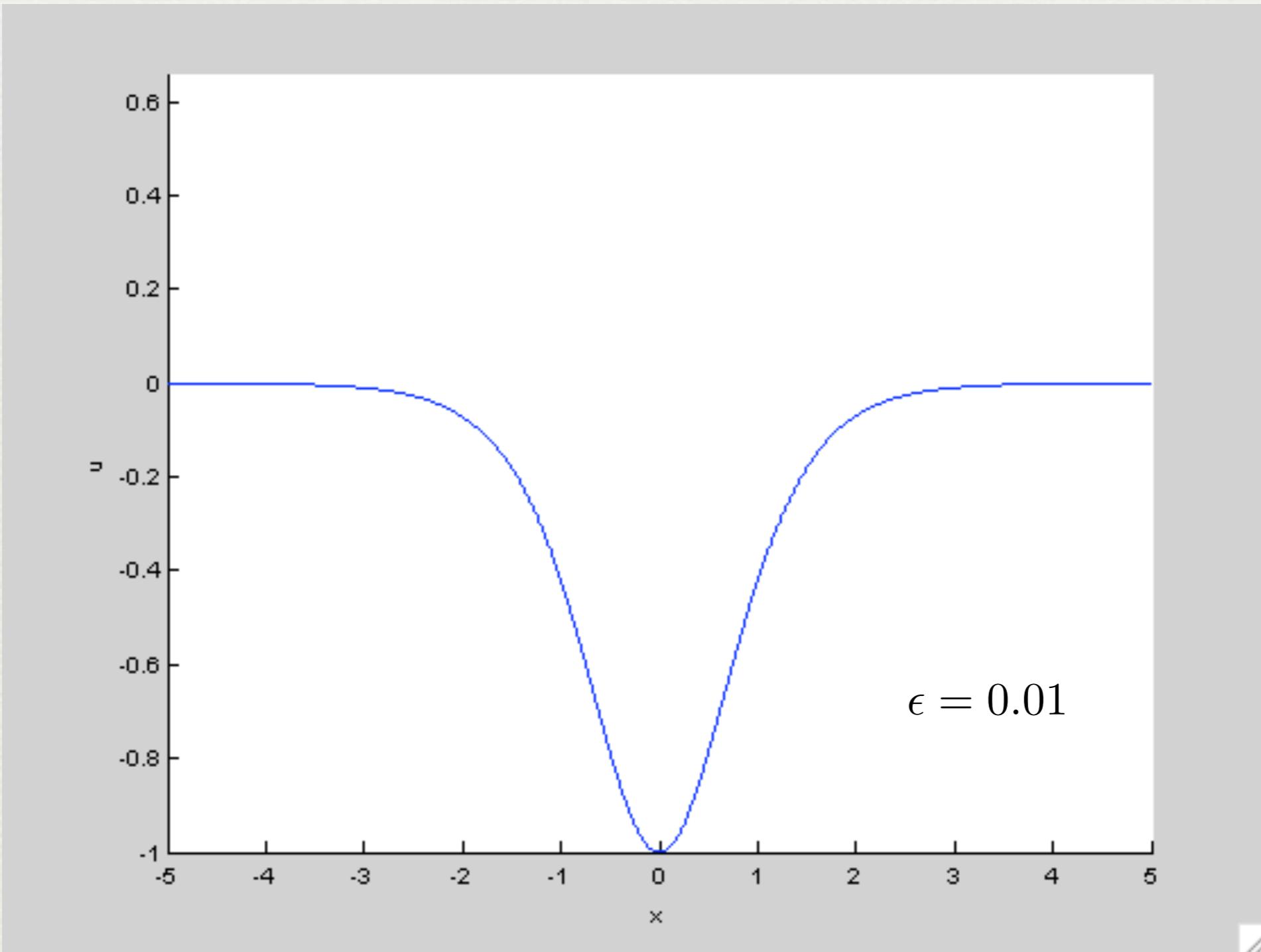


$$u_t + 6uu_x = \epsilon u_{xx}$$

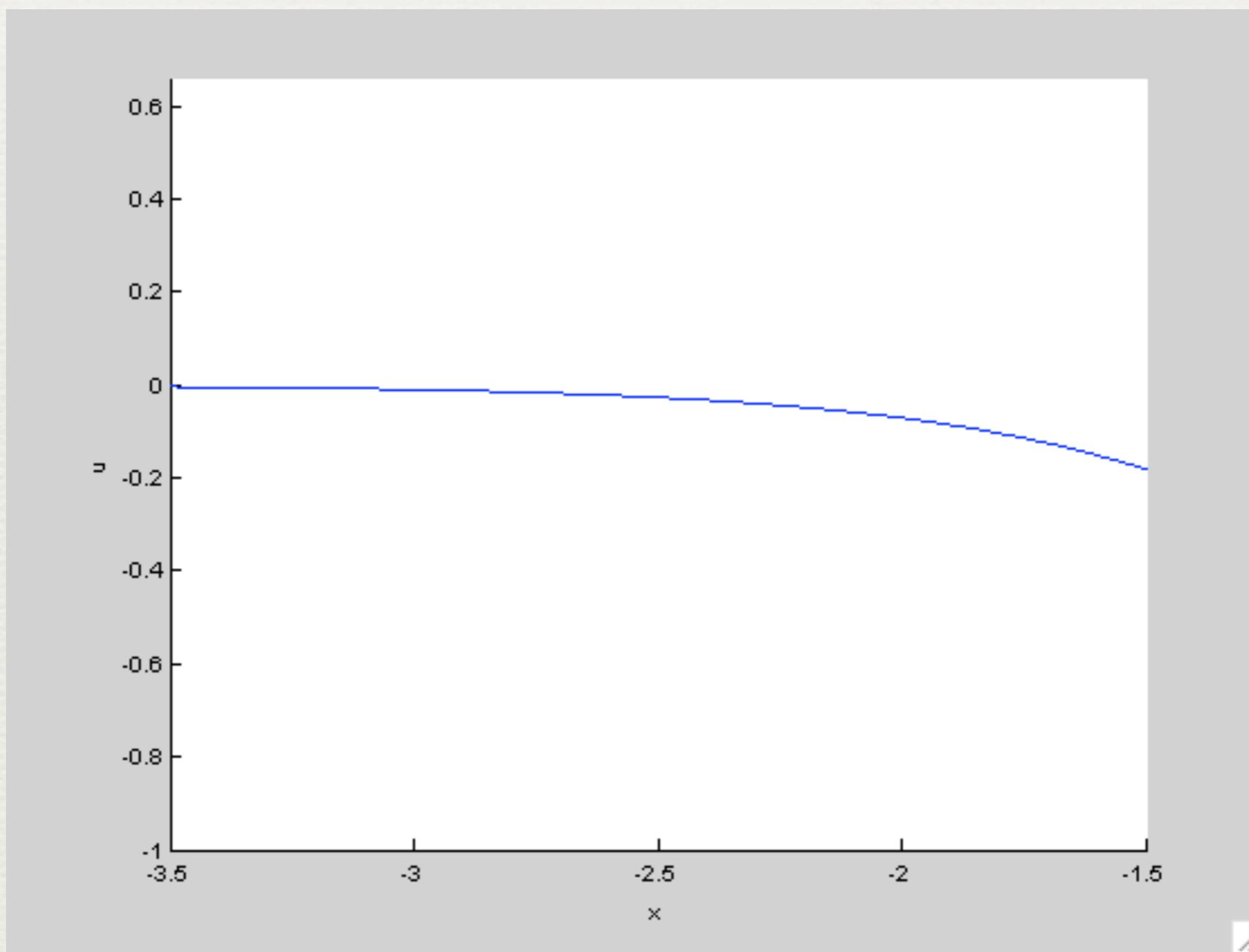
$$\epsilon = 0.1$$

KdV equation

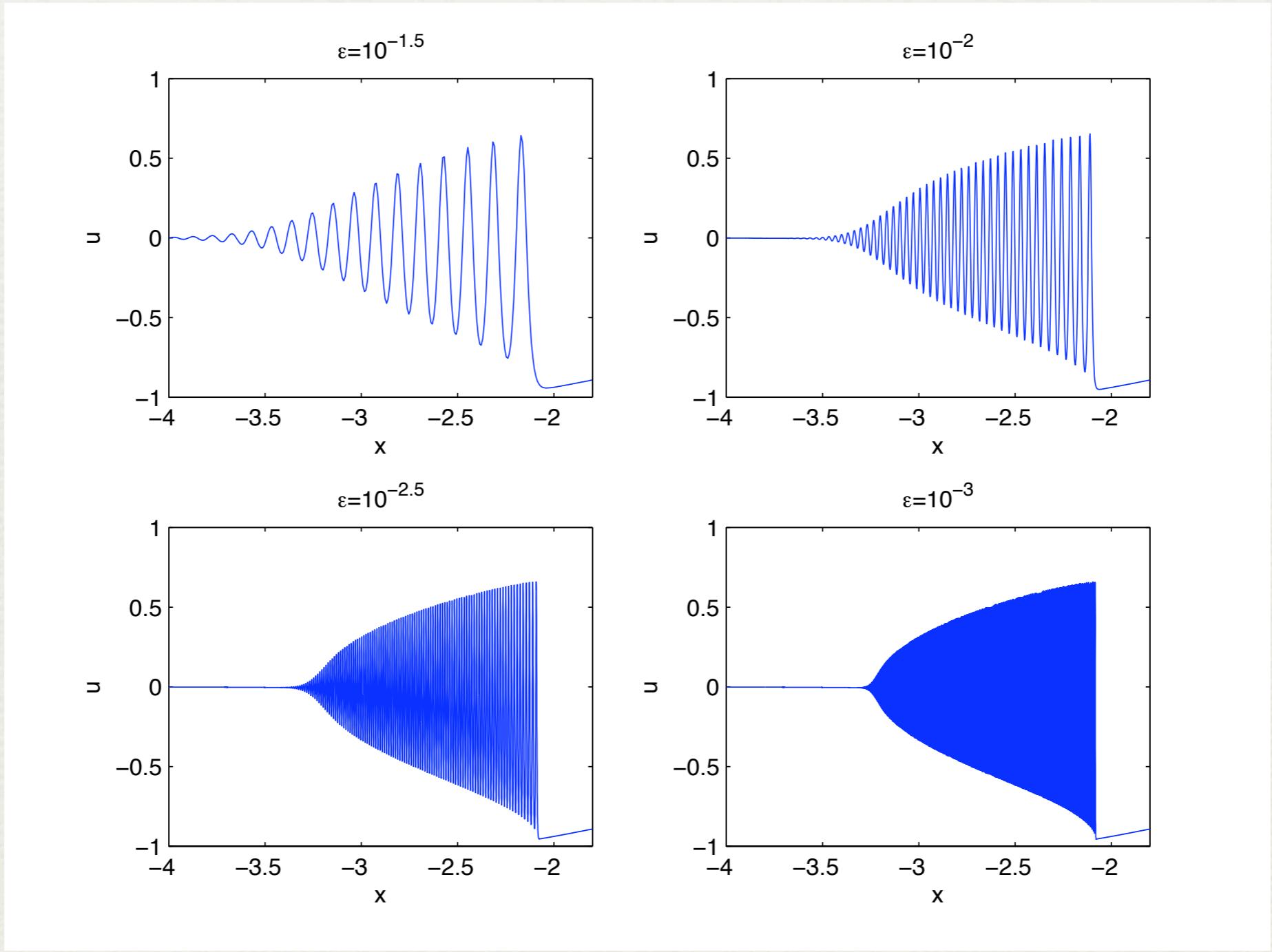
$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0$$



Oscillatory zone



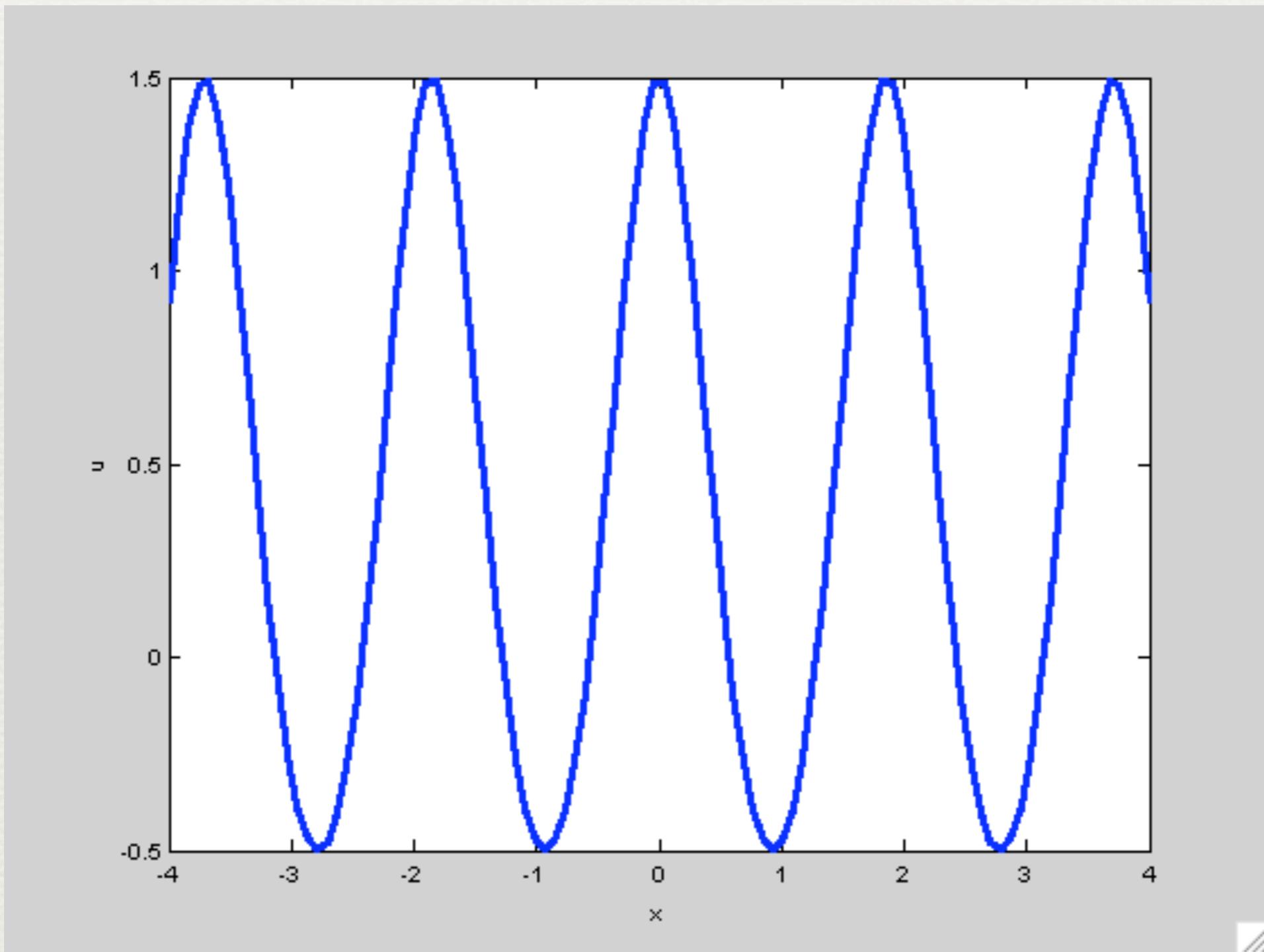
Different values of ϵ



Whitham equations and asymptotic solution

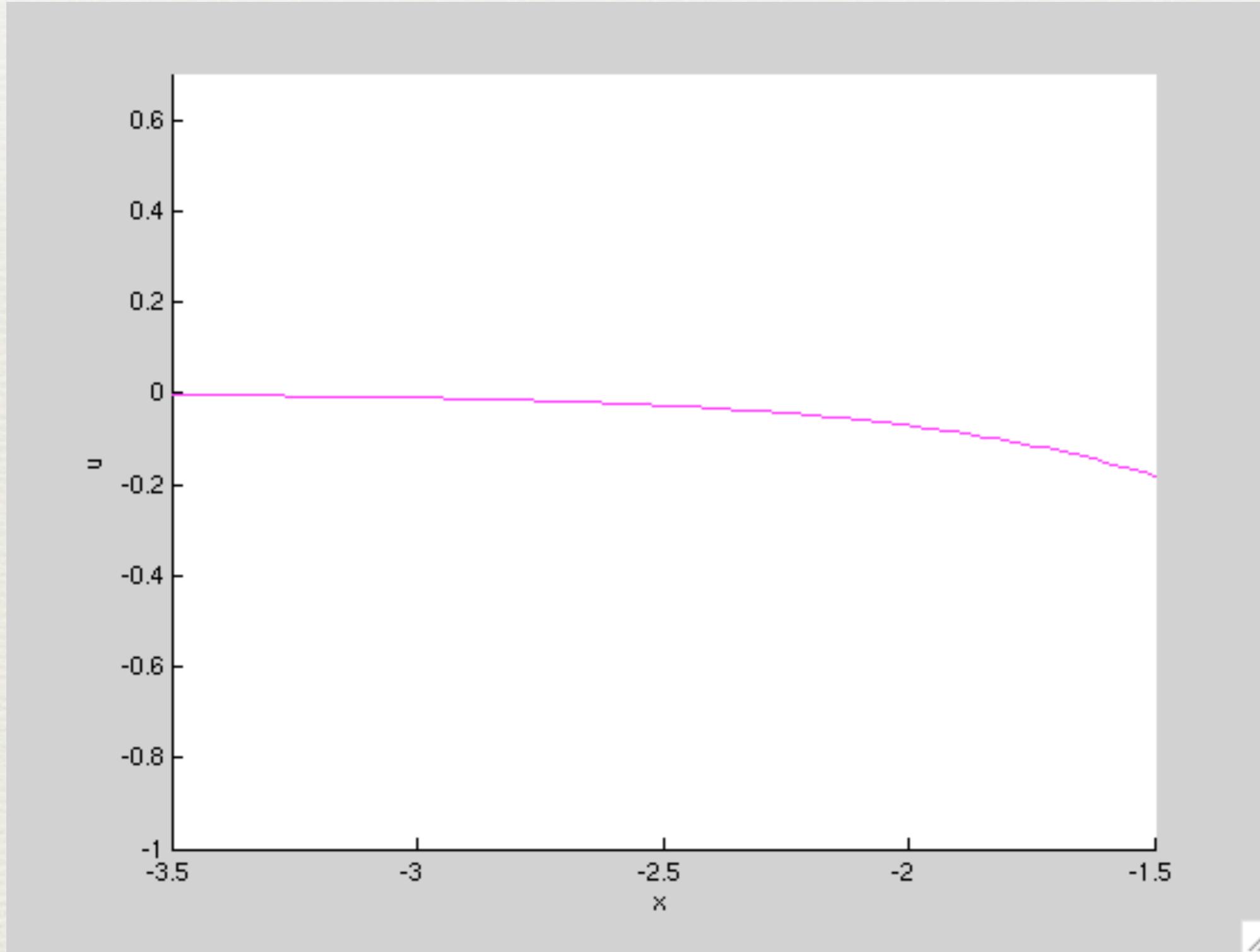
- ♦ qualitatitve picture (Gurevitch, Pitaevskii 1973):
 - ♦ for $t < t_c$: solution to the Hopf equation
 - ♦ for $t > t_c$: oscillatory zone (roughly where Hopf solution is multi-valued)
 - ♦ outside: Hopf solution
 - ♦ inside: exact elliptic solution to KdV, with branch points dependent on x, t according to the Whitham equations

Elliptic solution



- ♦ Whitham: averaging of KdV solution in the multi-valued region of the Hopf solution such that 3 conserved quantities of KdV are unchanged
- ♦ Lax, Levermore, Venakides: rigorous proof
- ♦ Deift, Venakides, Zhou: phase of the elliptic solution in dependence of the coordinates
- ♦ Fei-Ran Tian: non-monotonous initial data

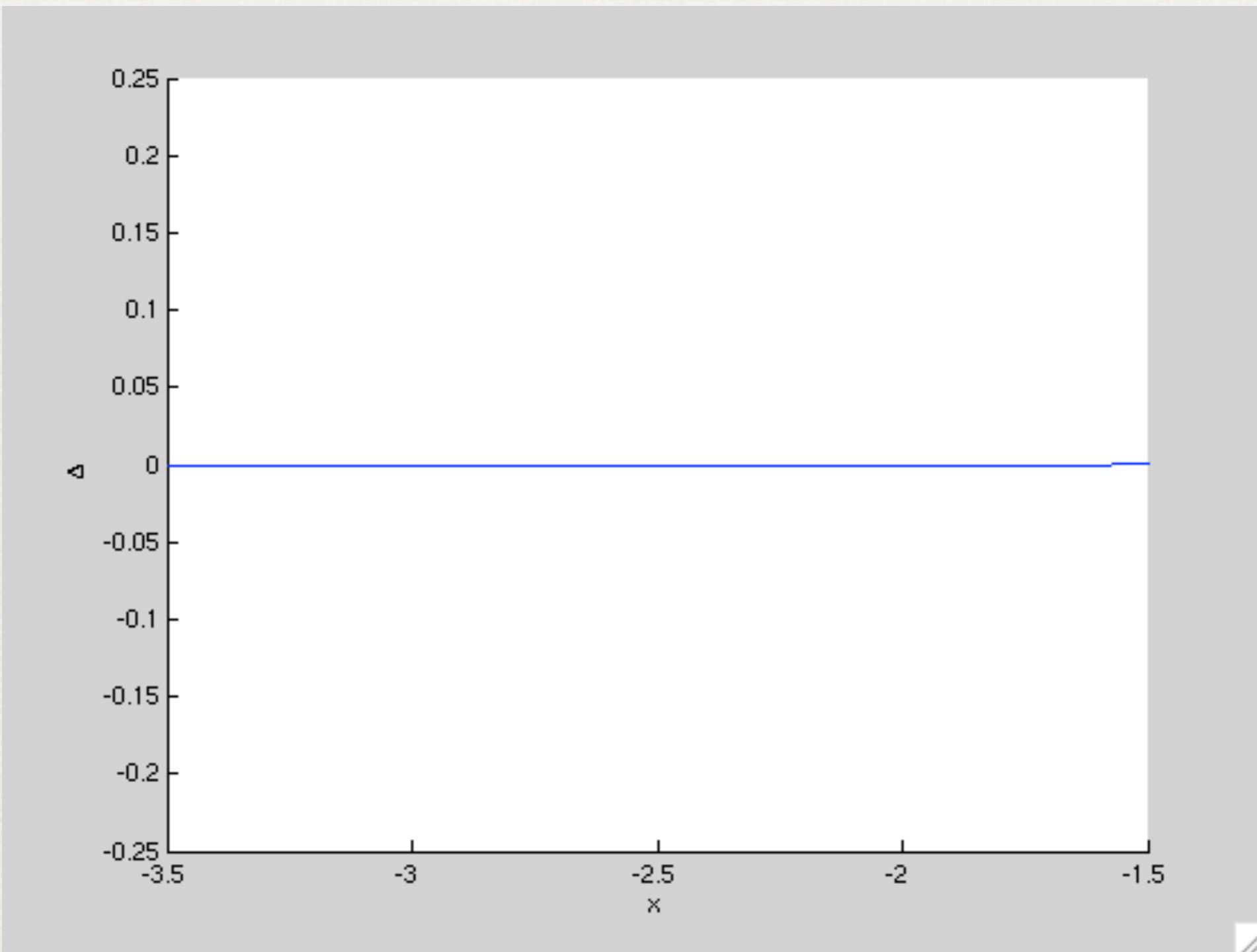
KdV and asymptotic solution



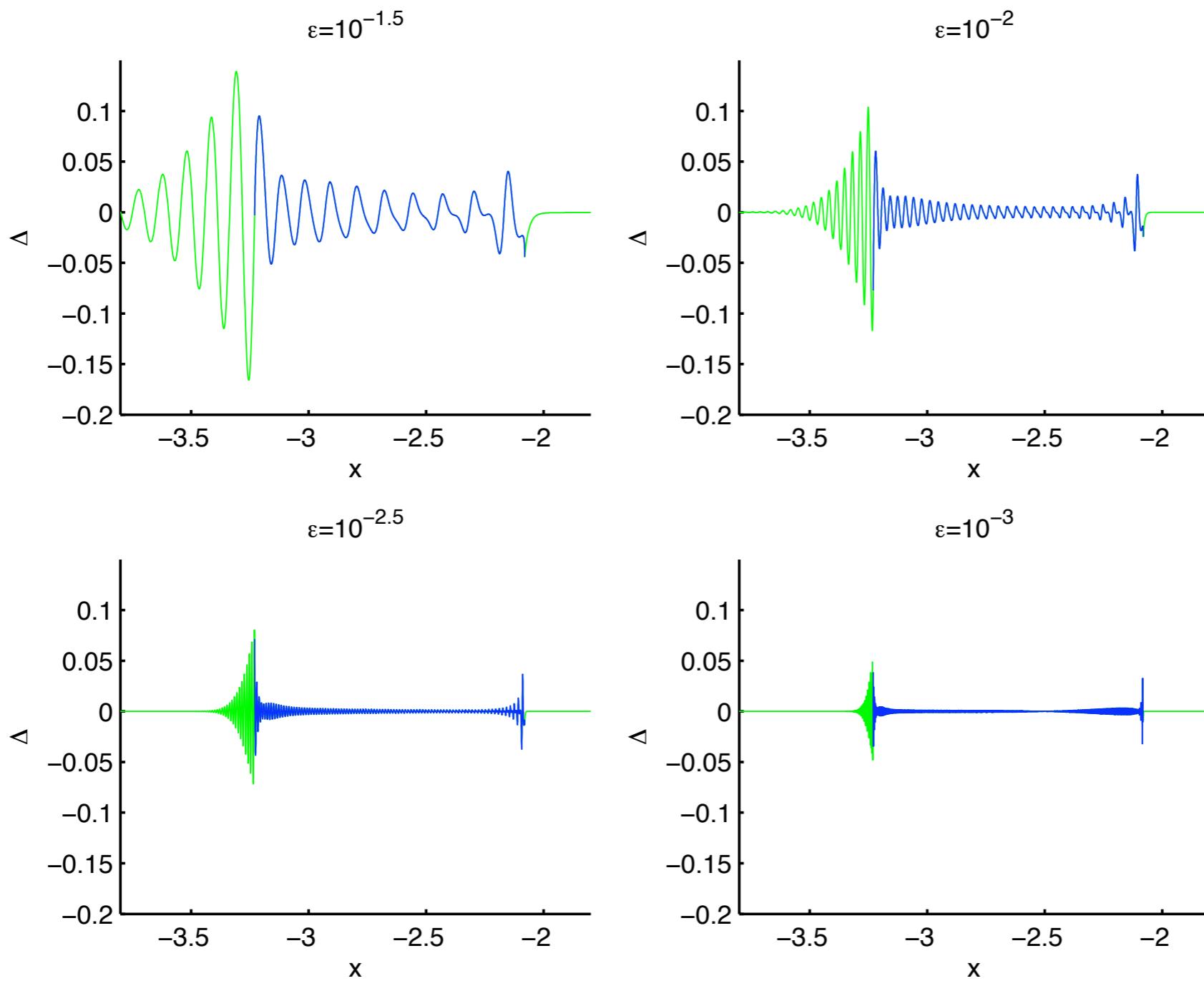
$$\epsilon = 0.01$$



Difference of KdV and asymptotic solution

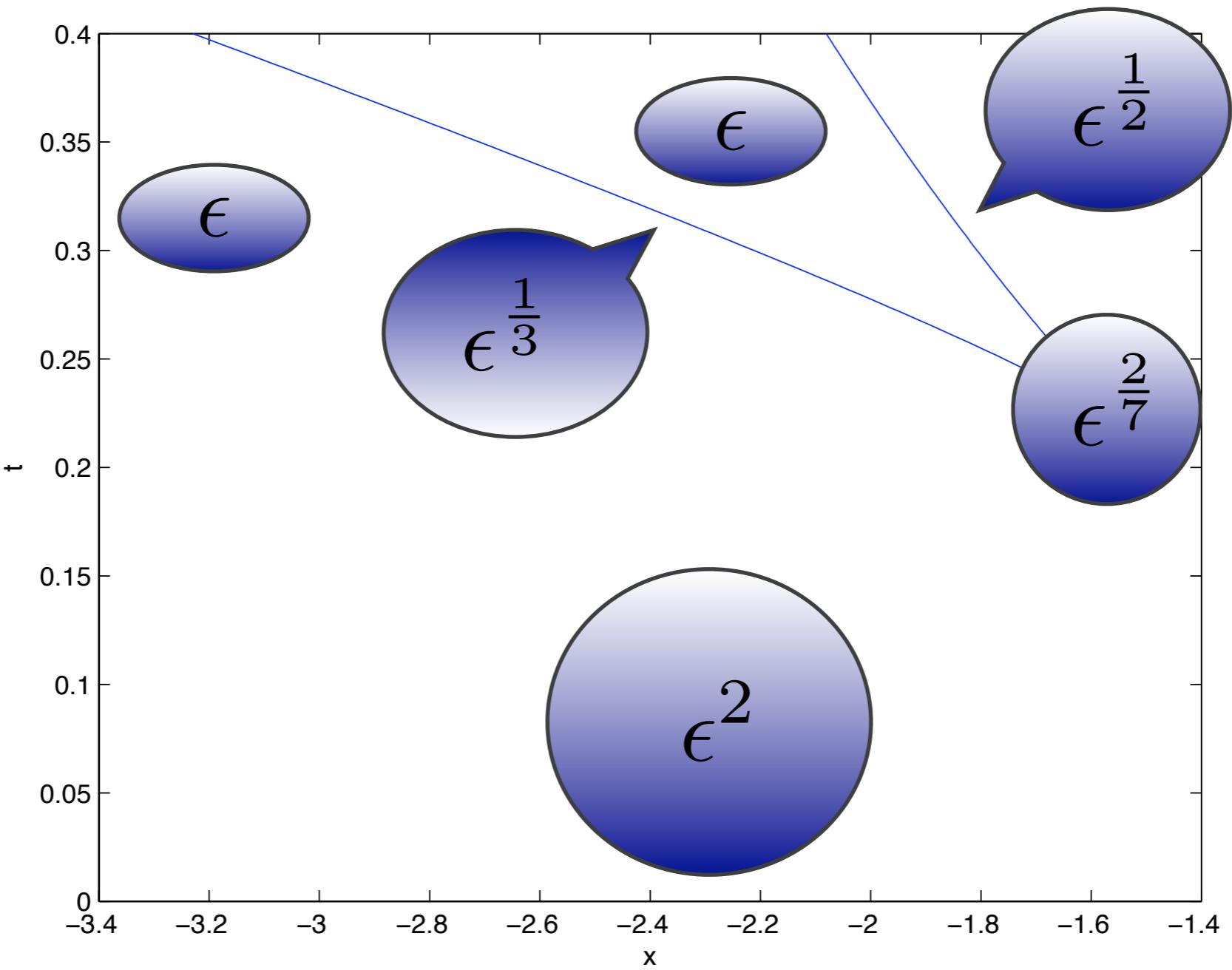


Difference of KdV and asymptotic solution

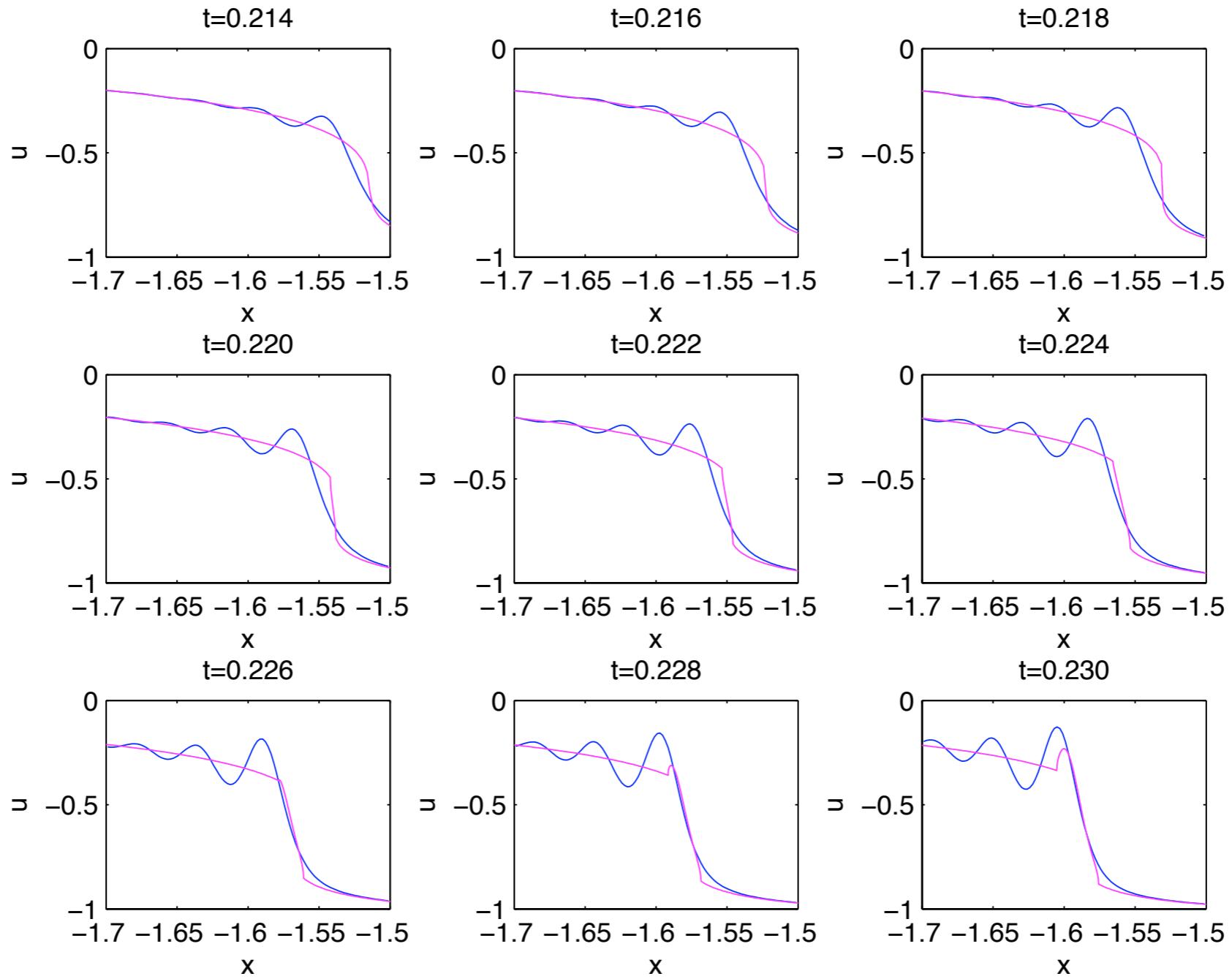


$t = 0.4$

x, t -plane



KdV and asymptotic solution at breakup $\epsilon=0.01$



Point of gradient catastrophe

- Dubrovin: universal behavior, solution near breakup characterized by cubic, $u \sim x^{1/3}$, multi-scale expansion
- solution given by fourth order equation, generalization of Painlevé I (x_c , t_c , u_c : quantities at breakup)

$$-6TF + F^3 + FF'' + \frac{1}{2}(F')^2 + \frac{1}{10}F'''' = X$$

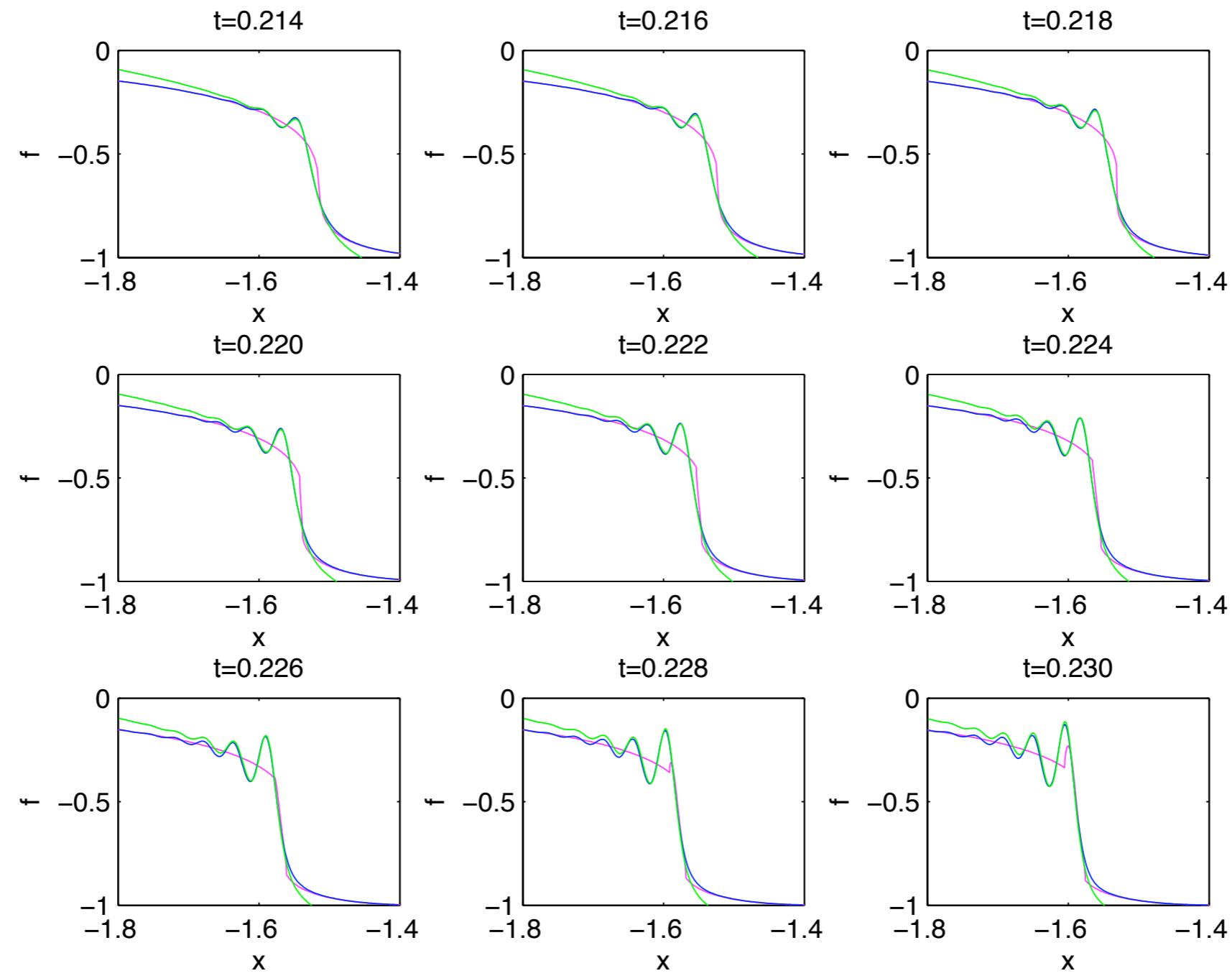
$$u(x, t, \epsilon) = u_c + \left(\frac{\epsilon}{k}\right)^{2/7} F(X)$$

where

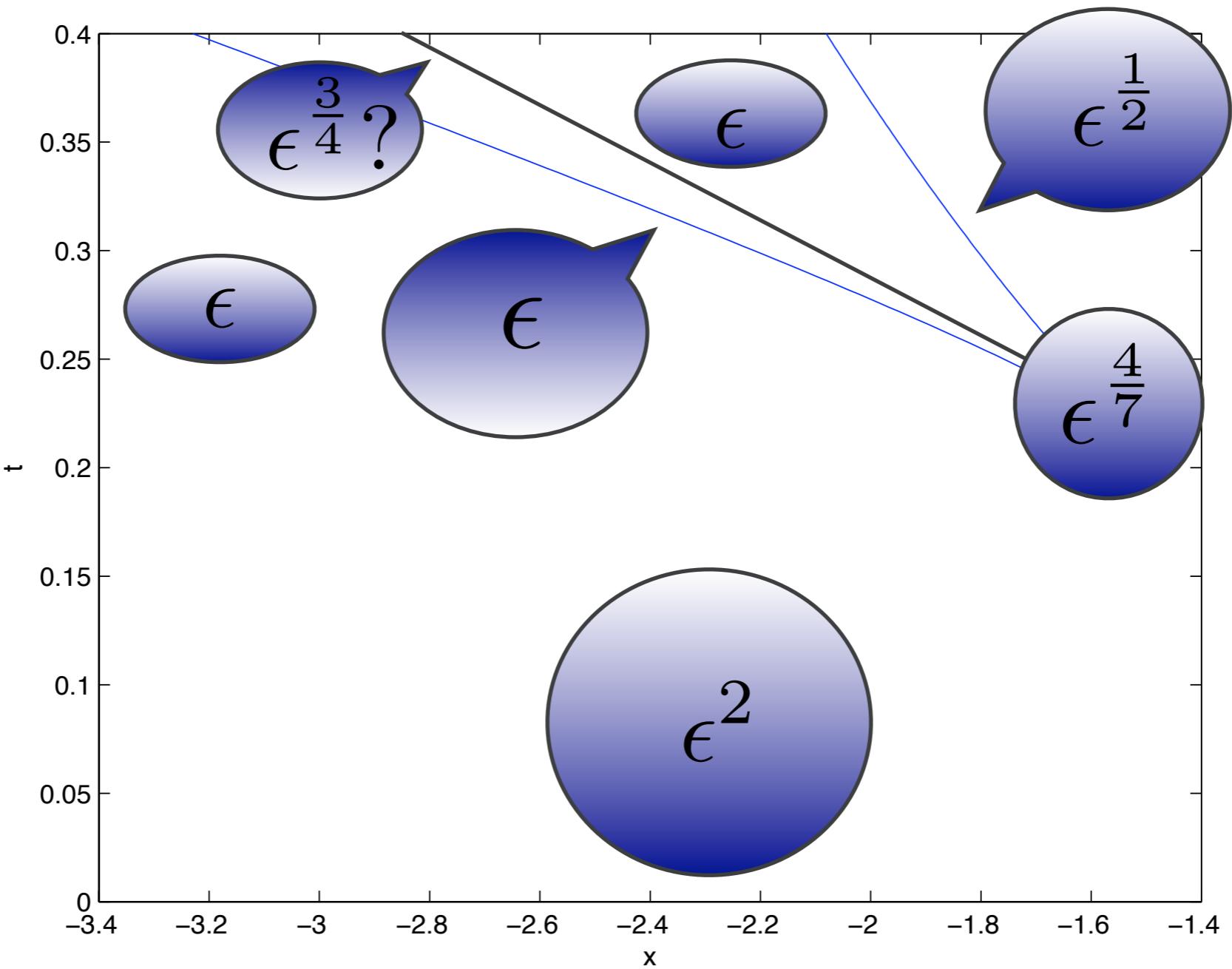
$$X = -\frac{1}{\epsilon} \left(\frac{\epsilon}{k}\right)^{1/7} (x - x_c - 6u_c(t - t_c))$$

$$T = \frac{1}{\epsilon} \left(\frac{\epsilon}{k}\right)^{3/7} (t - t_c)$$

KdV and PI2 near breakup



x, t -plane



Semiclassical Limit

- NLS equation: focusing ($\rho = -1$) and defocusing ($\rho = 1$)

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\psi_{xx} - \rho|\psi|^2\psi = 0$$

- slow variables: $u = |\psi|^2$, $v = \epsilon \Im((\ln \psi)_x)$

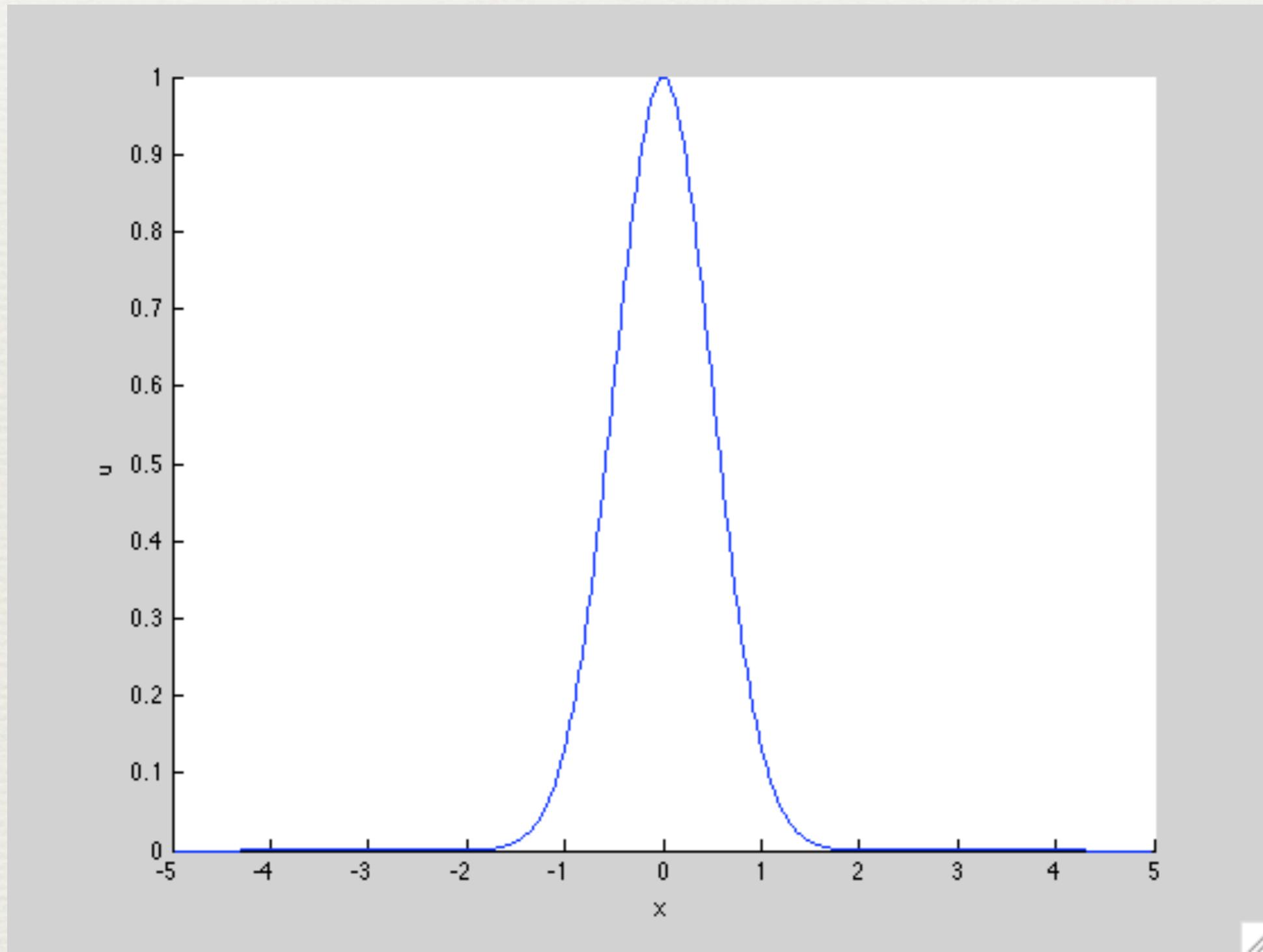
$$u_t + (uv)_x = 0$$

$$v_t + vv_x + \rho u_x + \frac{\epsilon^2}{4} \left(\frac{u_x^2}{2u^2} - \frac{u_{xx}}{u} \right)_x = 0$$

Defocusing NLS

$$\psi_0(x) = \exp(-x^2),$$

$$\begin{aligned}\epsilon &= 0.5, \\ 0 &\leq t \leq 1, \\ u &= |\psi|^2\end{aligned}$$



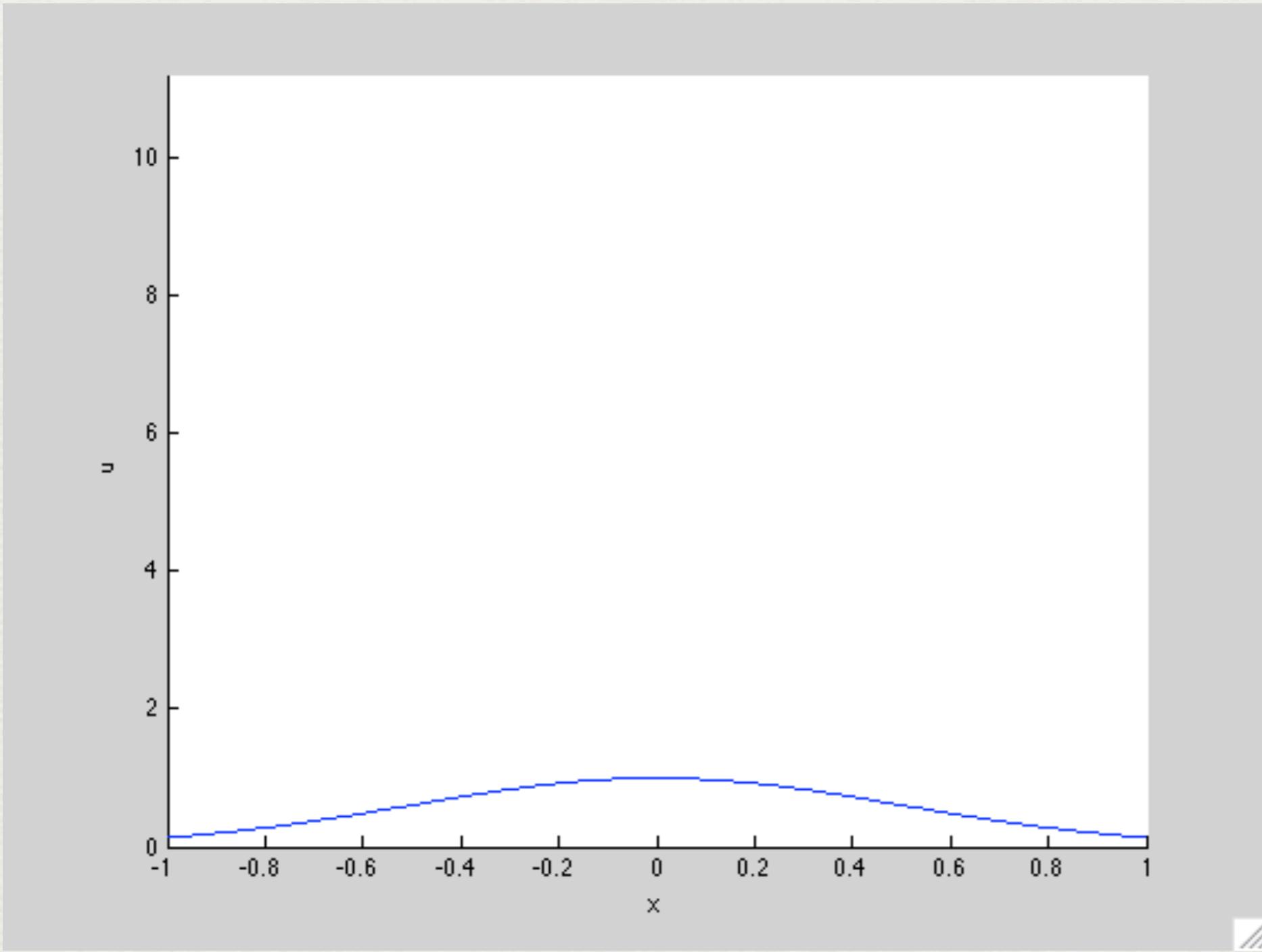
Focusing NLS

$$\psi_0(x) = \exp(-x^2),$$

$$\epsilon = 0.1,$$

$$0 \leq t \leq 0.8,$$

$$u = |\psi|^2$$



- semiclassical limit $\epsilon \rightarrow 0$: hydrodynamic system

$$u_t + (uv)_x = 0$$

$$v_t + vv_x + \rho u_x = 0$$

- coefficient matrix: $\begin{pmatrix} v & u \\ \rho & v \end{pmatrix}$, eigenvalues $\lambda = v \pm \sqrt{\rho u}$
- defocusing case similar to KdV, zeroth order and Whitham equations hyperbolic (Jin, Levermore, McLaughlin 1999)
- focusing case: zeroth order and Whitham equations elliptic, only special initial data studied:

Kamvissis, McLaughlin, Miller 2003: $\psi_0 = A \operatorname{sech} x$

Tovbis, Venakides, Zhou 2004: $\psi_0 = -\operatorname{sech} x \exp(-i\mu \ln \cosh x)$

Hadamard example

- Cauchy problem for the 2d Laplace equation ($k = 1, 2, 3, \dots$)

$$\begin{aligned}\Delta_2 u_k &= 0, \quad (x, y) \in \Omega = (0, 1) \times (0, 1) \\ u_k &= 0, \quad x \in (0, 1) \\ u_{k,y} &= \phi_k = (\sin \pi k x)/(\pi k), \quad x \in (0, 1)\end{aligned}$$

- unique solution

$$u_k = \frac{\sinh(\pi k y) \sin(\pi k x)}{\pi^2 k^2}$$

- $k \rightarrow \infty$: $\phi_k \rightarrow 0$, solution u_k trivial. But for small y , strongly oscillating solution with growing amplitude, no convergence.

Zeroth order solution

- Let $f(u, v)$ be a solution to the PDE

$$f_{vv} + u f_{uu} = 0$$

with $f_{uu}^2 + f_{vv}^2 \neq 0$ for u_0, v_0 with $f_v(u_0, v_0) = 0$.
Then the solution to

$$x = vt + f_u, \quad 0 = tu + f_v$$

solves the zeroth order system for sufficiently small t .

- point of gradient catastrophe:

$$f_{uu} = f_{vv} = 0, \quad f_{uv} = -t_c$$

Example

- initial data $\psi_0 = A \operatorname{sech} x$

$$f_u = \Re \left[\operatorname{arsinh} \left(\frac{-\frac{1}{2}v + iA}{\sqrt{u}} \right) \right], \quad f_v = -\Re \left[\sqrt{\left(-\frac{1}{2}v + iA \right)^2 + u} \right]$$

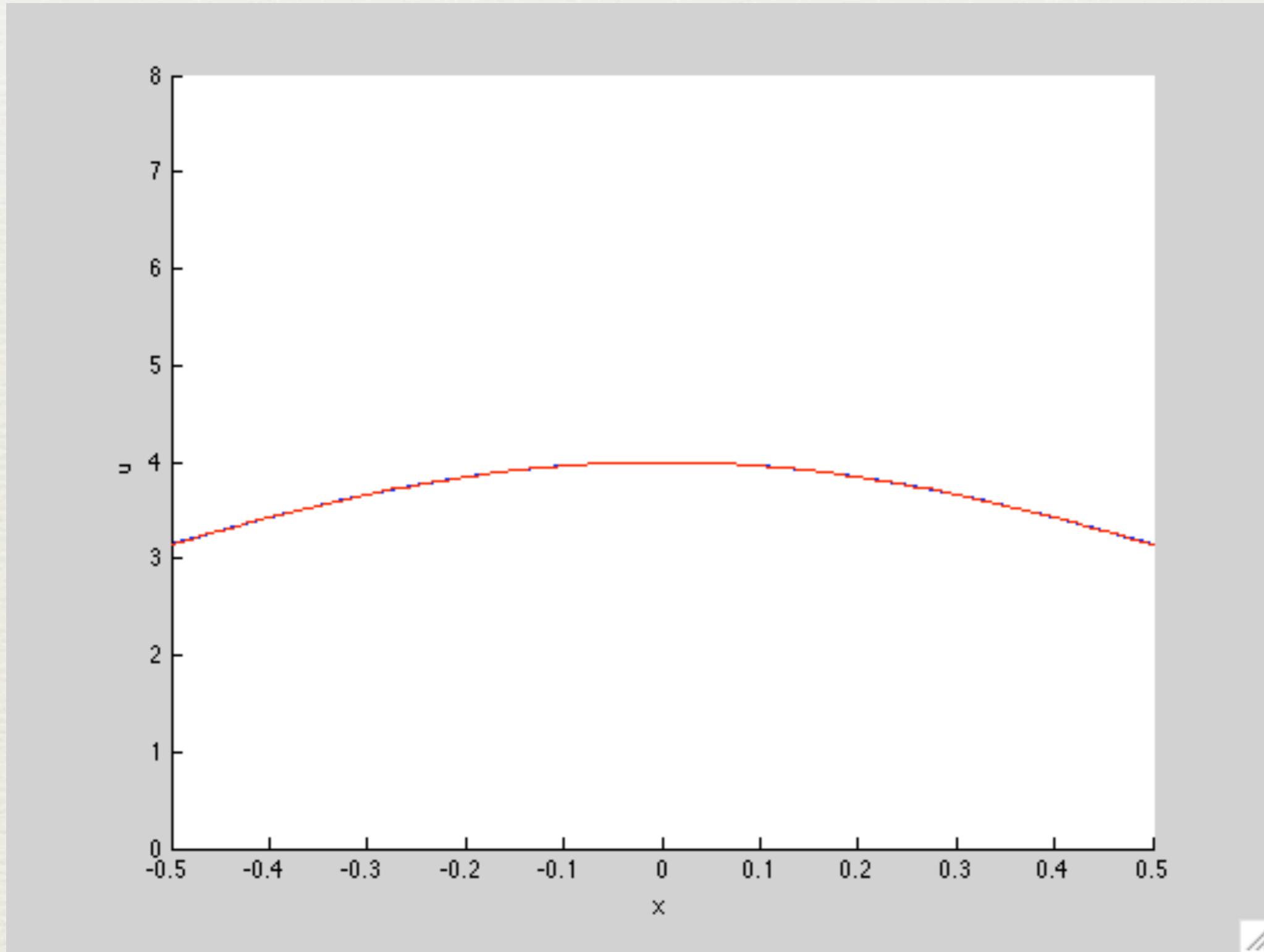
- critical point:

$$u_c = 2A^2, \quad v_c = 0, \quad x_c = 0, \quad t_c = \frac{1}{2A}$$

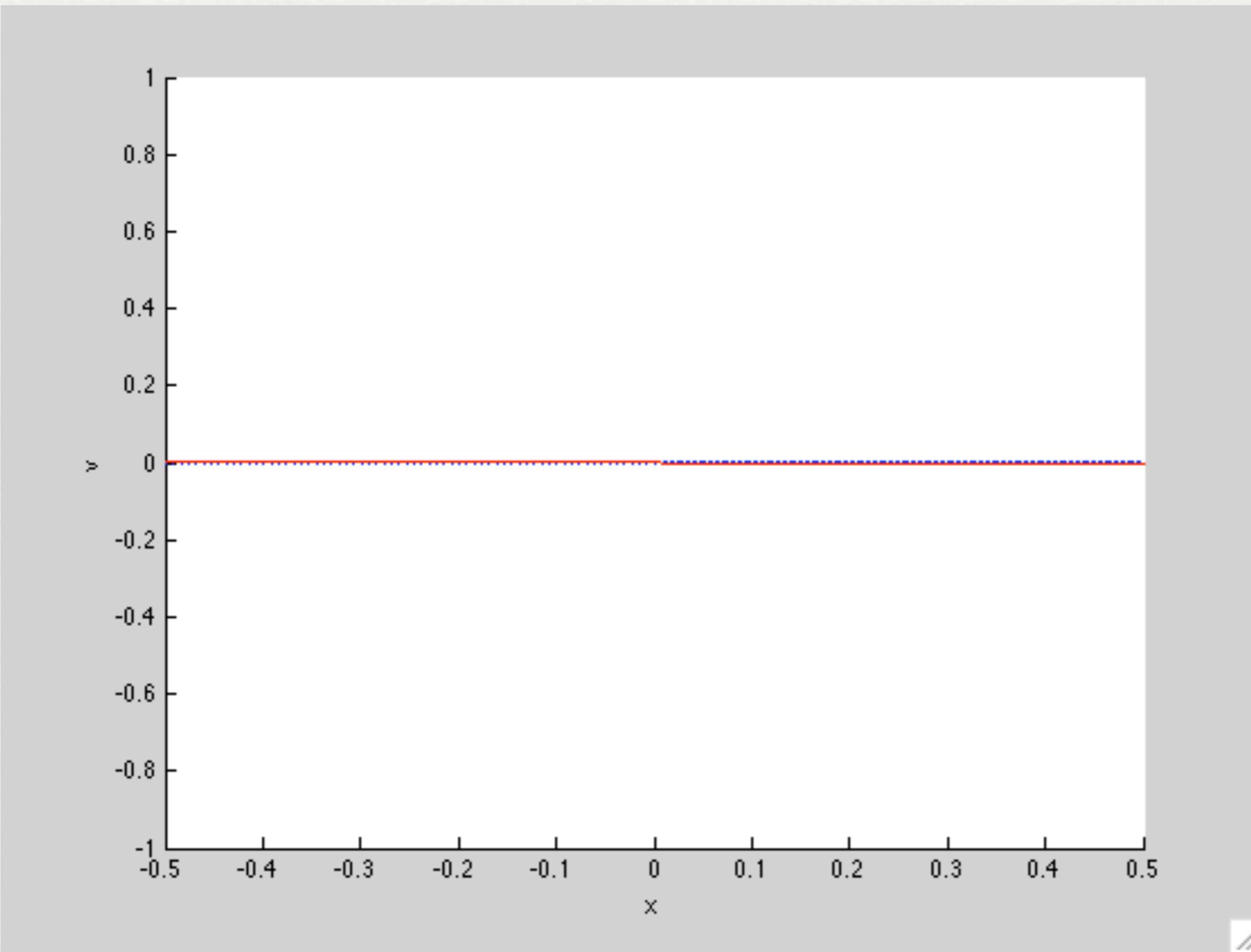
- cusp formation (α, β constant)

$$u = u_c - \alpha \sqrt{|x|} + \dots, \quad v = \beta \operatorname{sign}(x) \sqrt{|x|} + \dots$$

Absolute value



Phase ν



$$\psi_0 = 2 \operatorname{sech} x$$

$$\epsilon = 0.04$$



ϵ Scaling

- NlS and asymptotic solutions for several values of ϵ ($\epsilon = 0.1, 0.09, \dots, 0.03, 0.025$), linear regression analysis for the L_∞ norm of the difference between the solutions
- for $t \ll t_c$: $|u - u^0| = O(\epsilon^2)$
- for $t = t_c$: largest difference at the critical point
 $|u - u^0| = O(\epsilon^{2/5})$

Near breakup

- Conjecture: solution close to the critical point given in terms of a solution Ω_0 to the Painlevé I equation

$$\Omega_{\zeta\zeta} = 3\Omega^2 - \zeta$$

in the form

$$u + i\sqrt{u_c}v \approx u_c + i\sqrt{u_c}v_c - \tilde{t}re^{i\phi} + \epsilon^{2/5}(6r\sqrt{u_c})^{2/5}e^{2i\phi/5}\Omega_0(\zeta) + O(\epsilon^{4/5})$$

where

$$\zeta = \left(\frac{6r}{u_c^2}\right)^{1/5} \frac{1}{\epsilon^{4/5}} \left[-u_c \tilde{t} + i\sqrt{u_c} \tilde{x} + \frac{1}{2} r \tilde{t}^2 \right], \quad re^{i\phi} = \frac{1}{f_{uvv}^c + i\sqrt{u_c} f_{uuu}^c}$$

with $\tilde{t} = t - t_c$, $\tilde{x} = x - x_c$.

Painlevé I and *tritronquées* solutions

- Painlevé I:

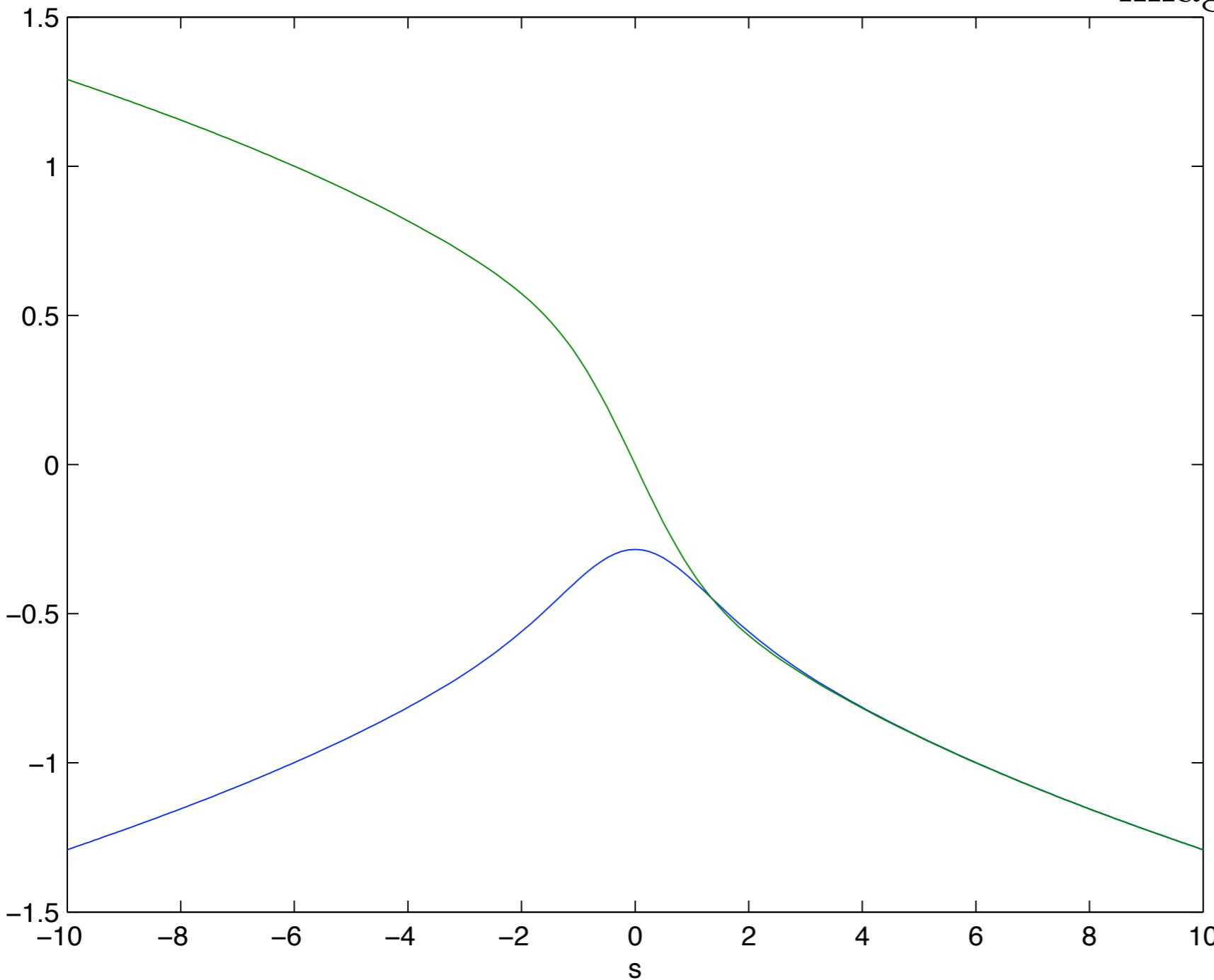
$$\Omega_{\zeta\zeta} = 3\Omega^2 - \zeta$$

- Boutroux 1913: asymptotic analysis, solutions asymptotically without poles for $|\arg\zeta| < 4\pi/5$.
- Joshi, Kitaev 2001: no pole for $\Re\zeta > -2.38\dots$
- numerical solution: asymptotic series (divergent)

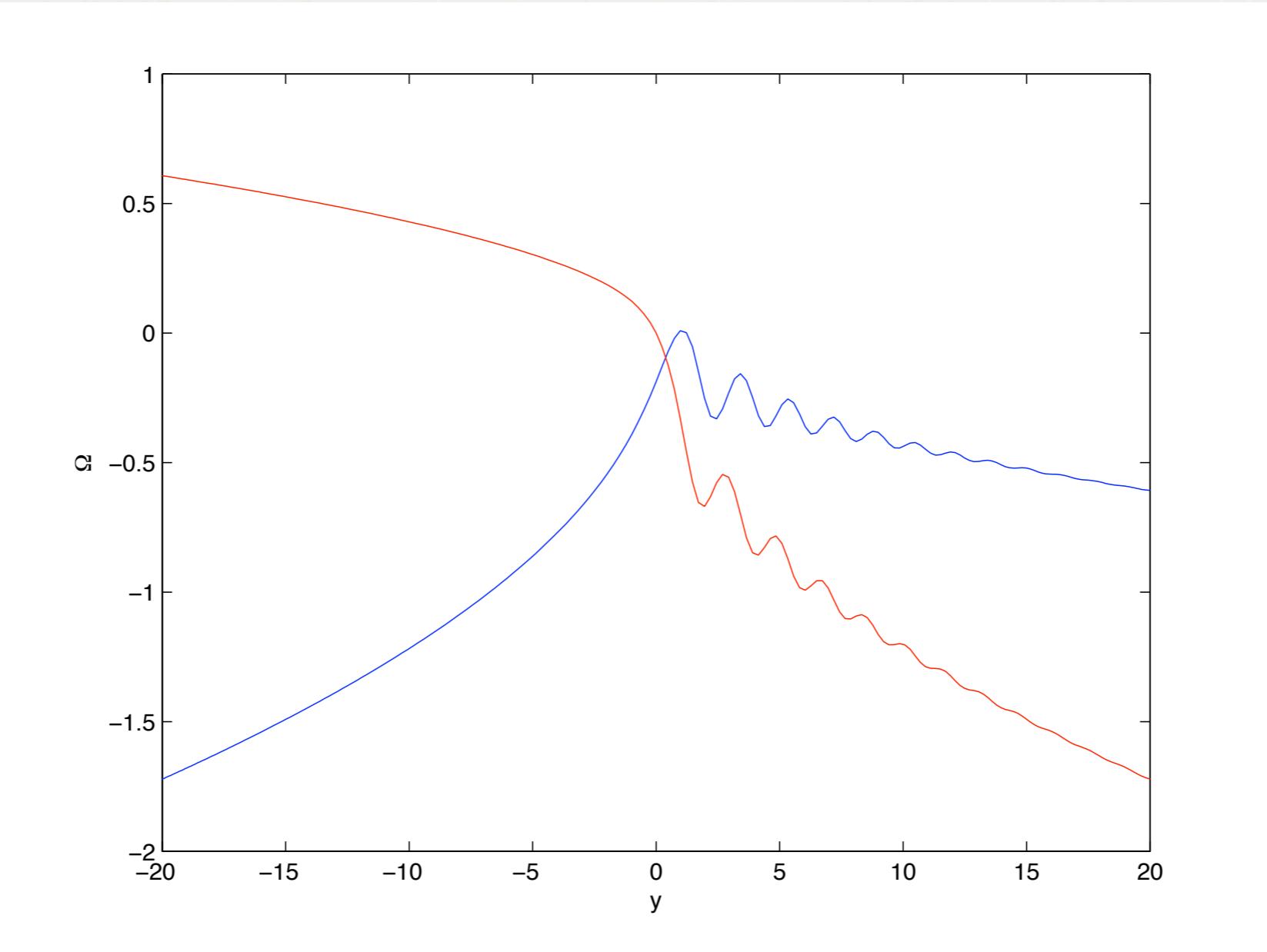
$$\Omega = \frac{\sigma}{\sqrt{3}}X + \sum_{n=0}^{\infty} \frac{\sigma a_n}{X^{5n-1}}, \quad X = \sqrt{\zeta}, \quad \sigma = \pm 1$$

tritronquées solution

solution on the imaginary axis
real part blue,
imaginary part green

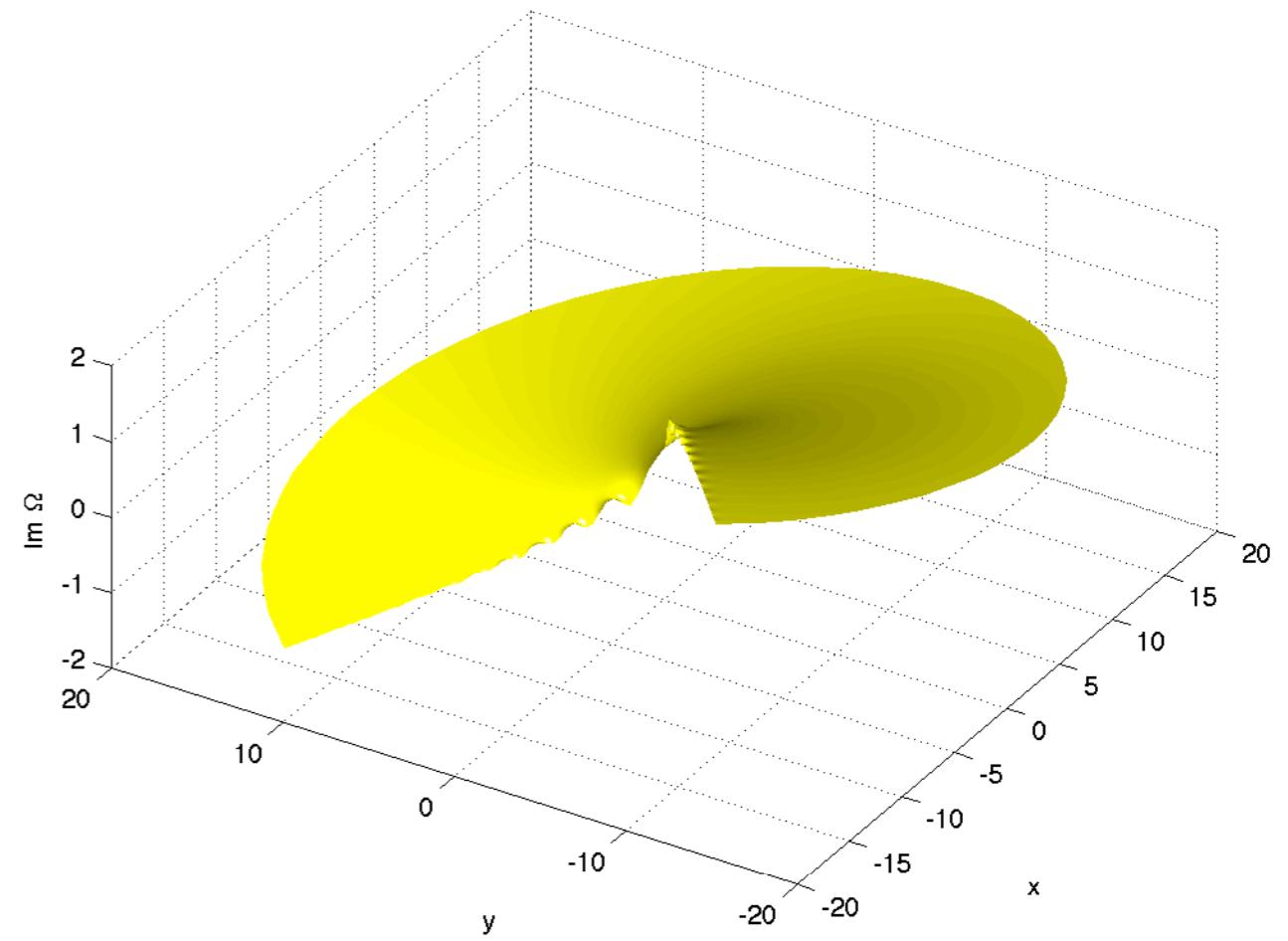
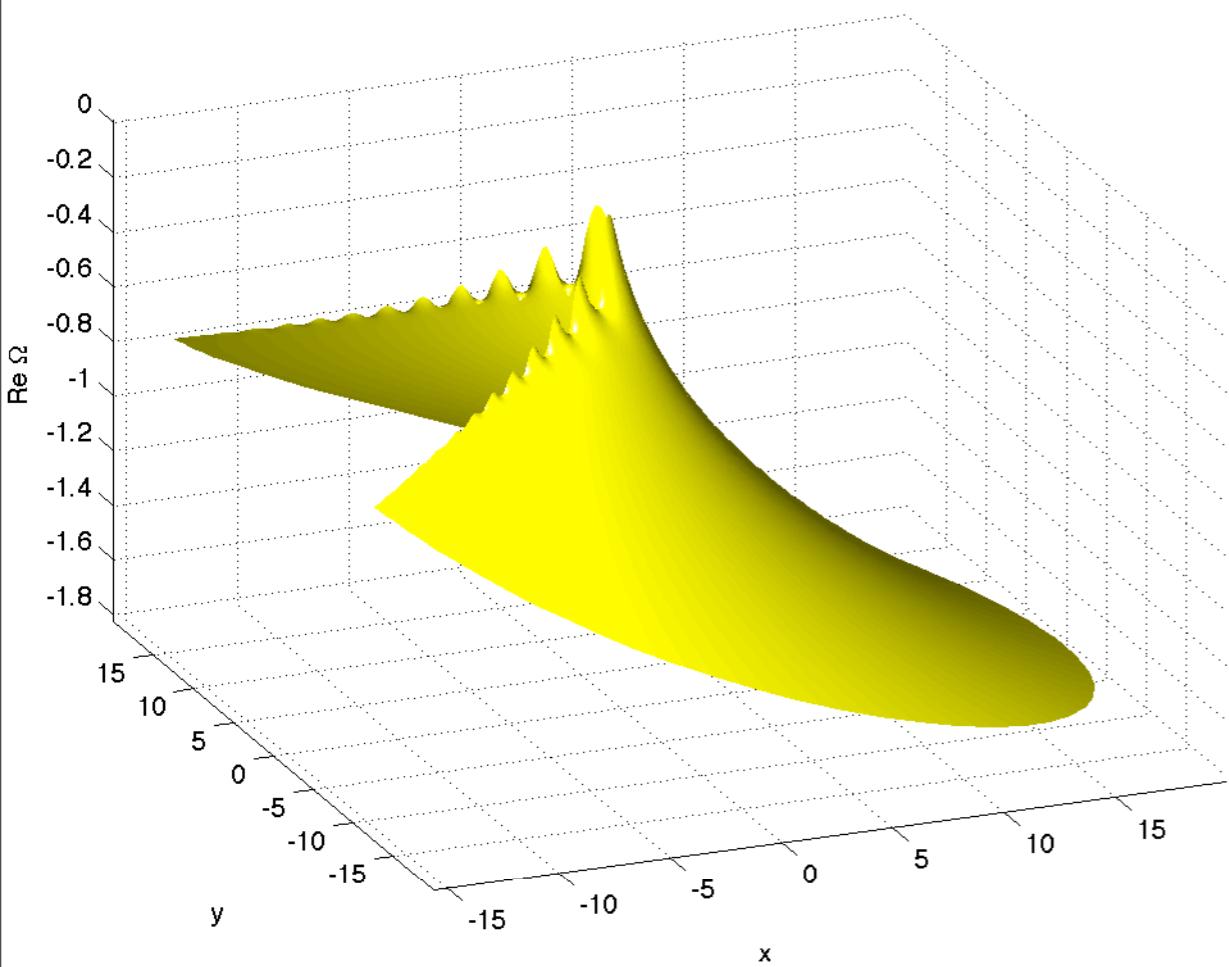


tritronquées solution



- Real (blue) and imaginary part (red) of the *tritronquée* solution close to the critical line (for $\zeta = \exp(i(4\pi/5 - .05))y$) with oscillations of slowly decreasing amplitude

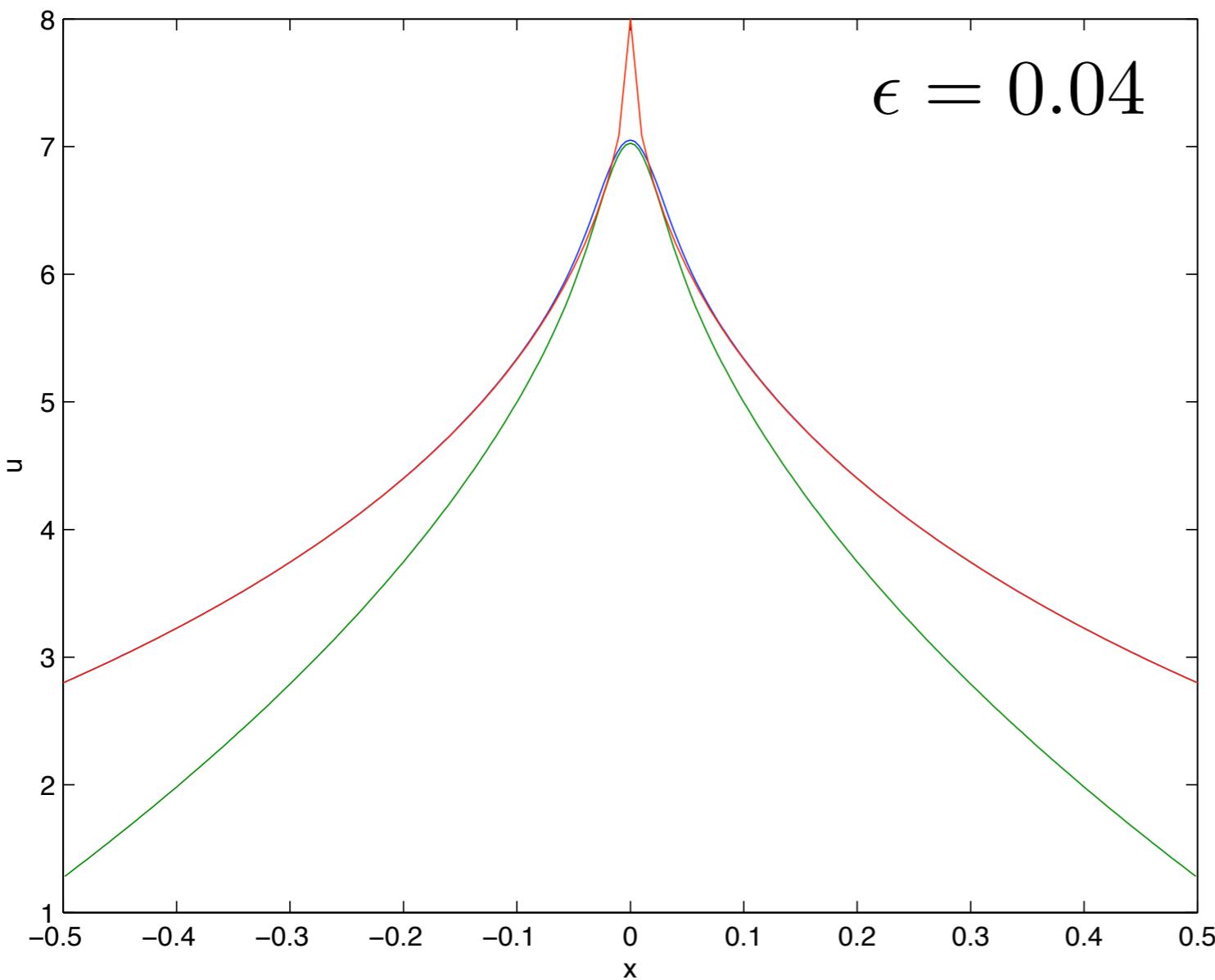
Conjecture: no poles in the sector $|arg(z)| < 4\pi/5$



- harmonic function with *tritronquée* boundary data

Critical point

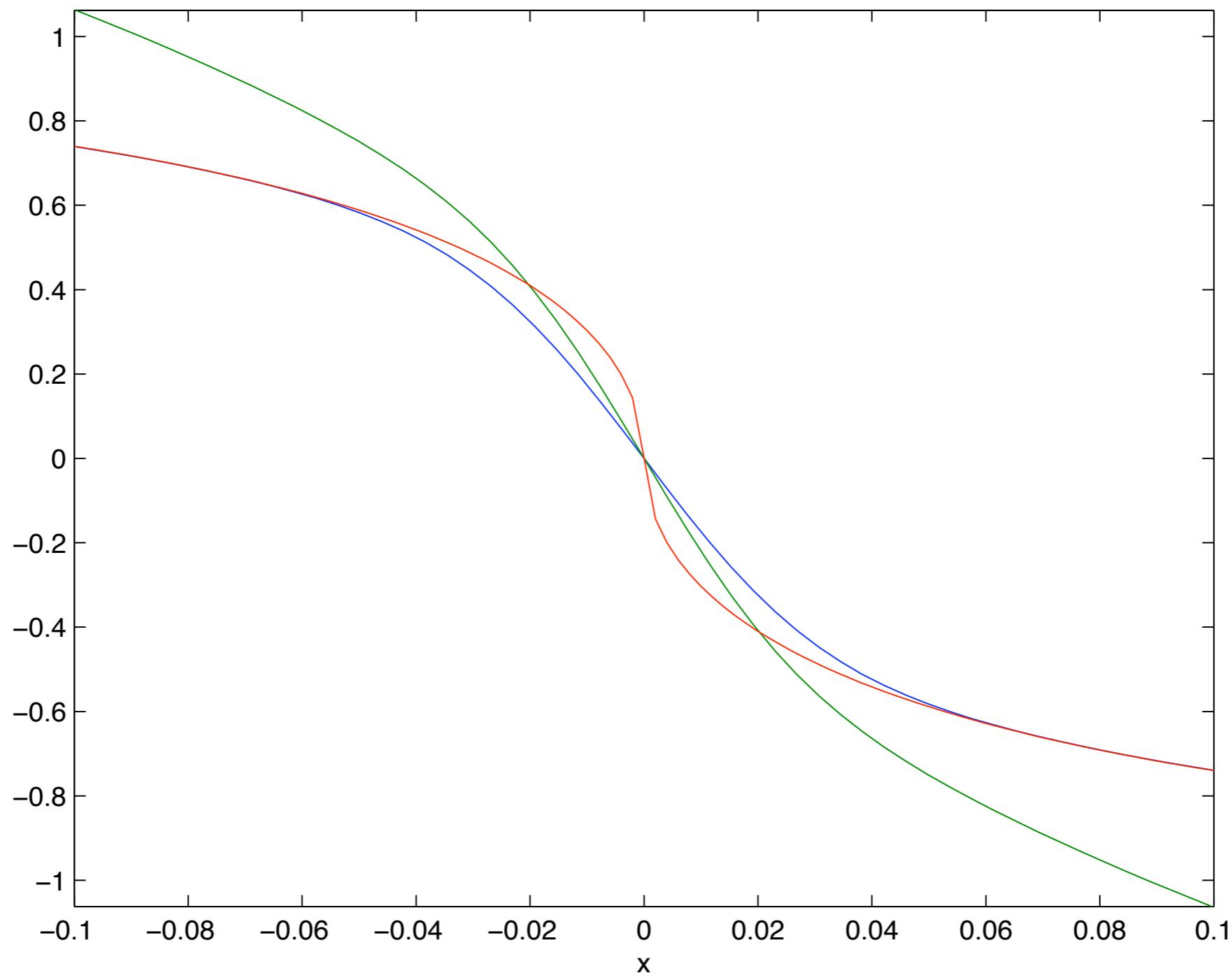
NLS (blue), zeroth order (red), PI (green)



singularity classification (Thom):

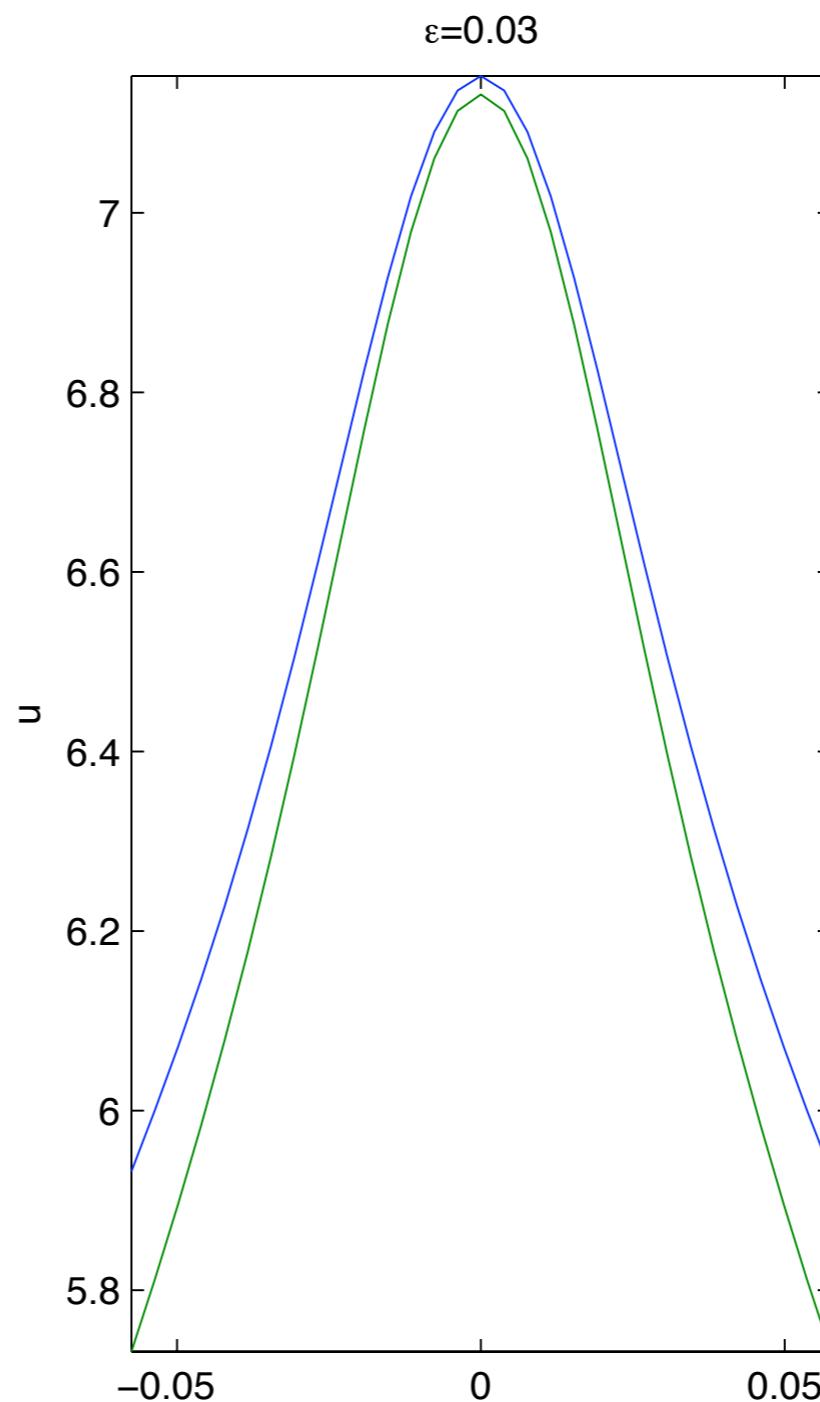
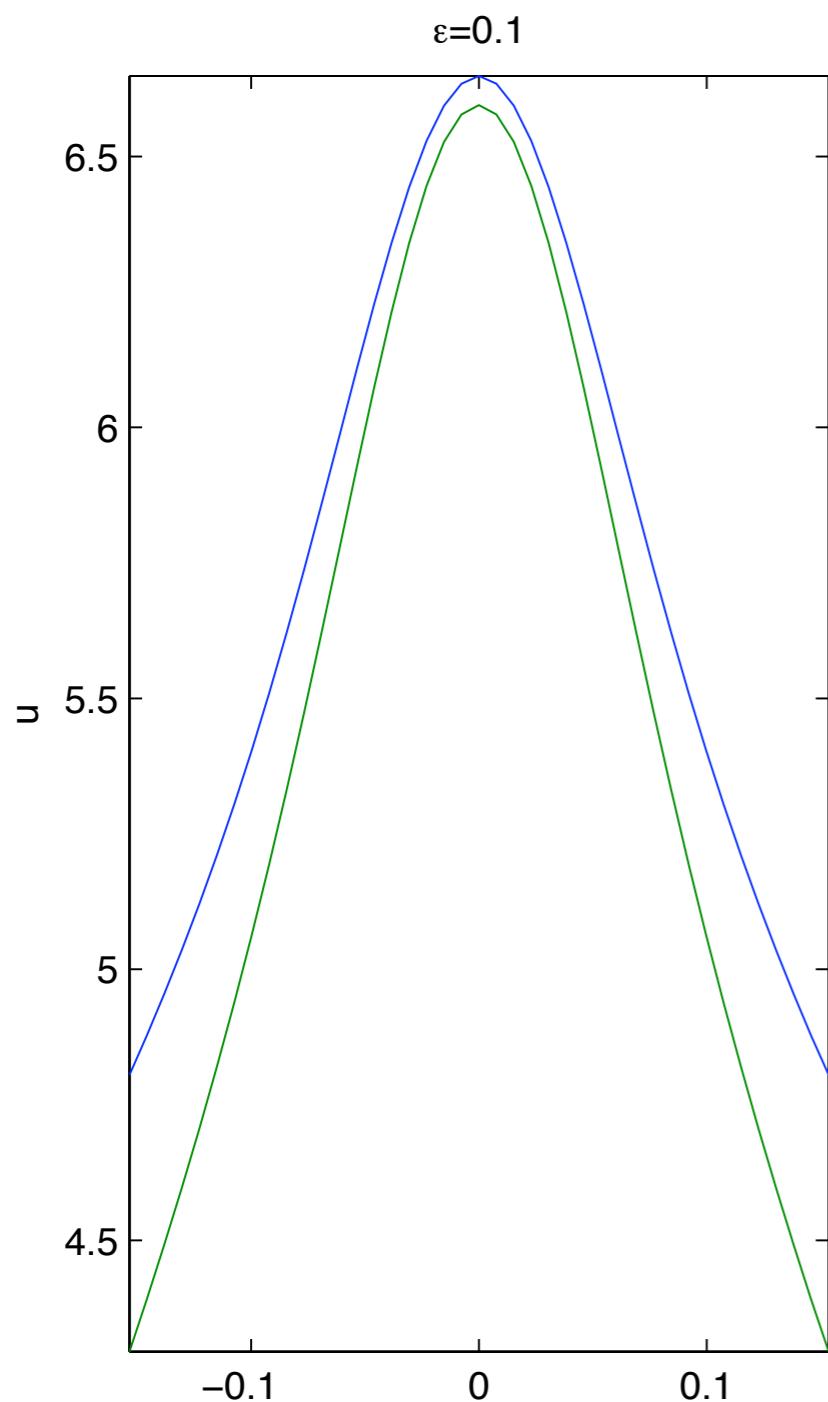
- ♦ focusing NLS: D4, local solution PI
- ♦ KdV: A2, local solution PI2 (B. Dubrovin CMP 2006, T. Grava, C.K., math-ph/0702038, 2007)

Critical point



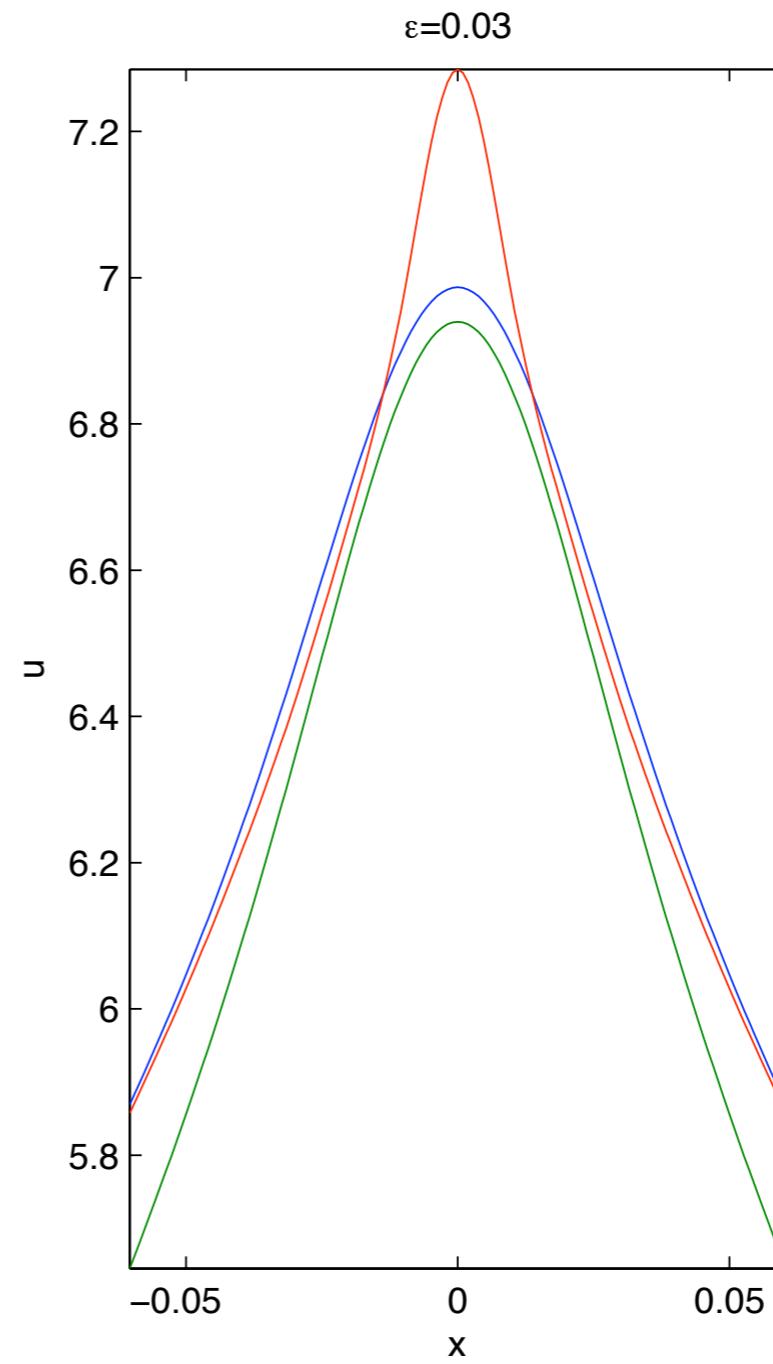
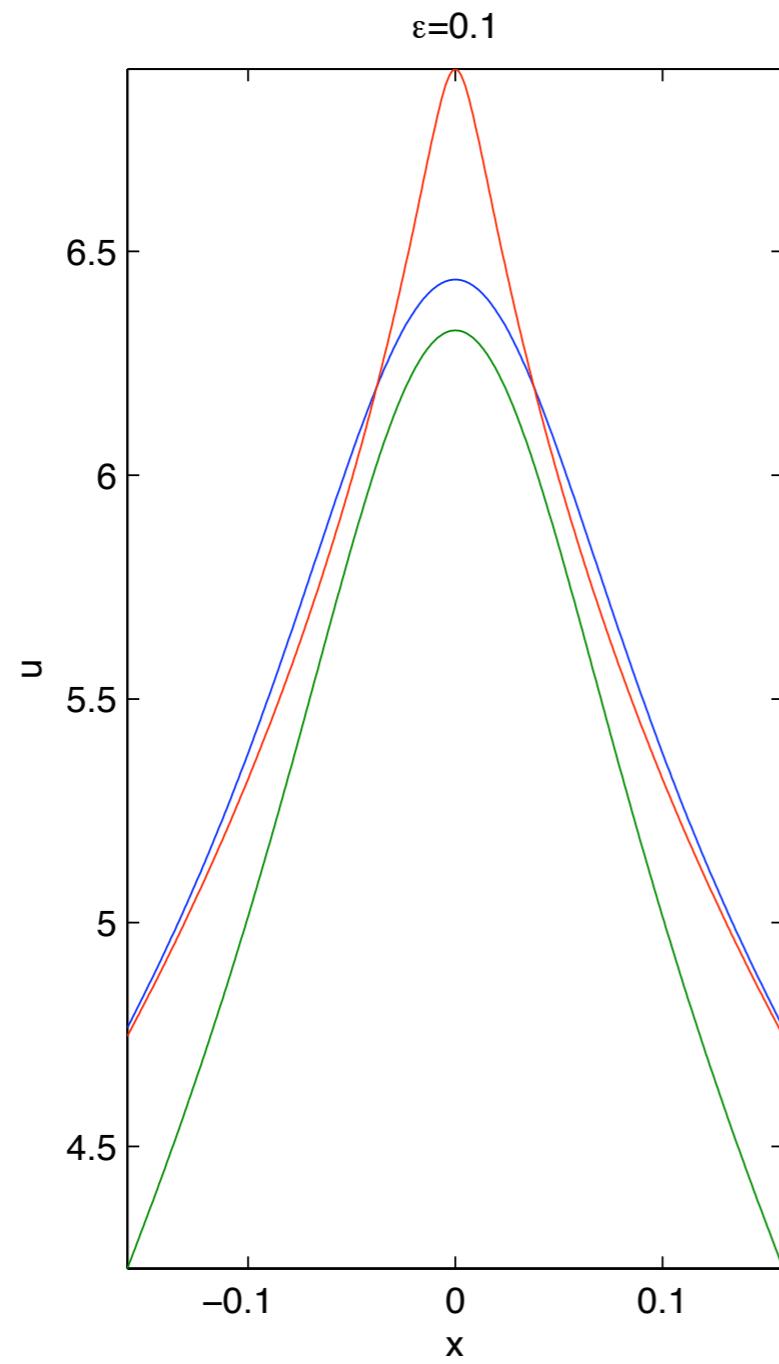
ϵ Scaling at breakup

$$|u - u_{tri}| \propto \epsilon^{4/5}$$



Before breakup

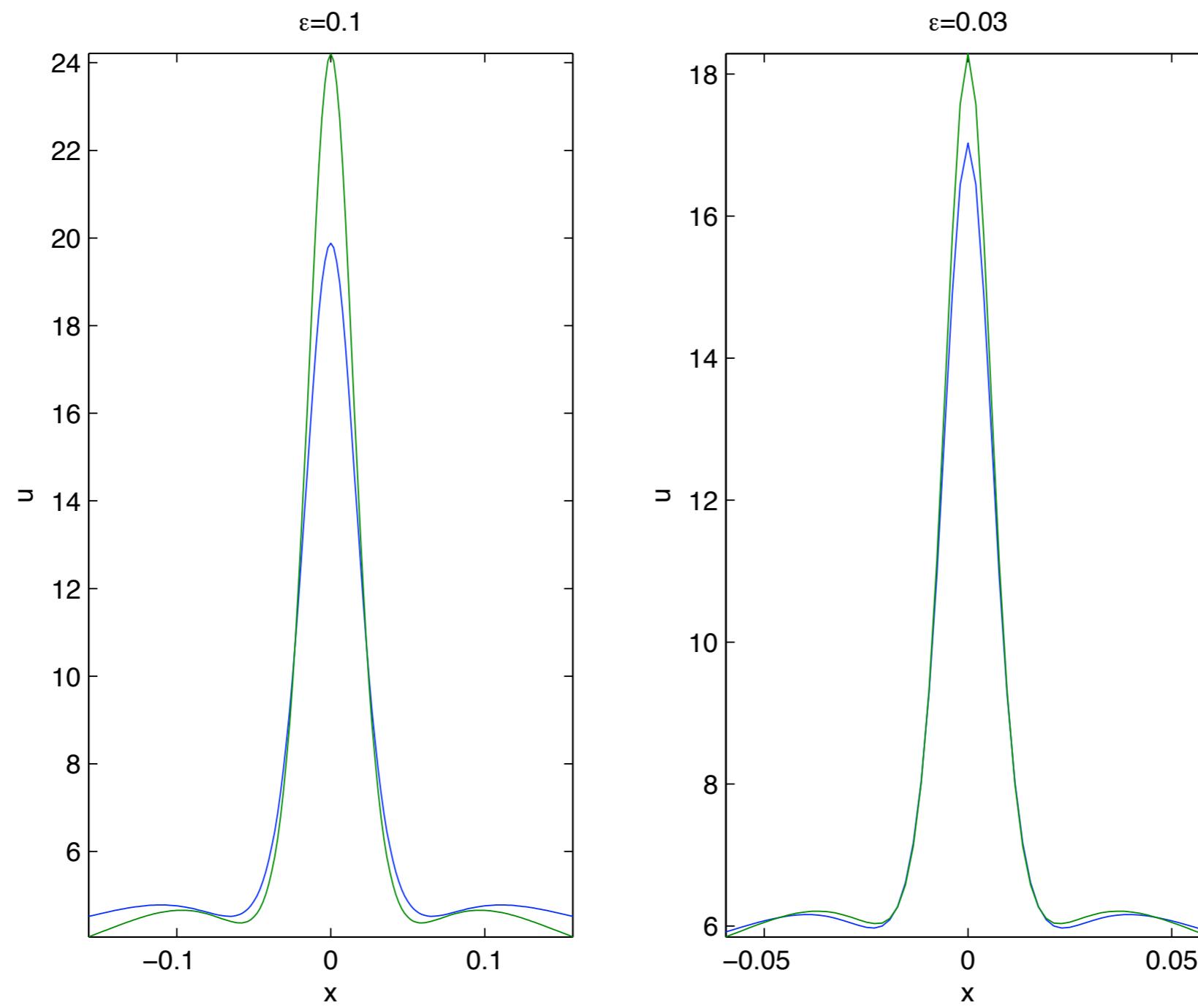
$$t_-(\epsilon) = t_c + u_0/r - \sqrt{(u_0/r)^2 - \epsilon^{4/5} \beta}$$



$\beta = 0.01$

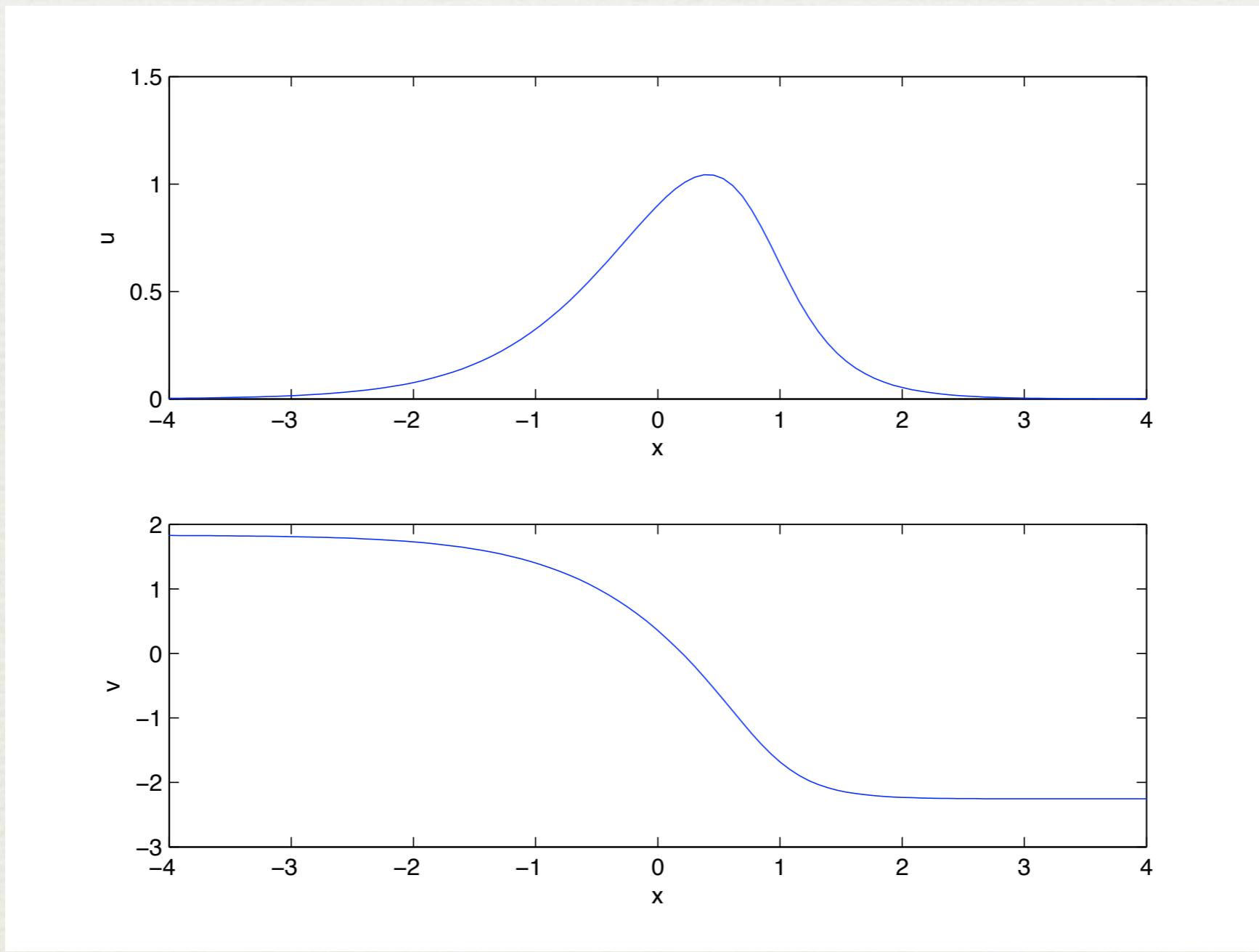
After breakup

$$t_+(\epsilon) = t_c + u_0/r - \sqrt{(u_0/r)^2 + \epsilon^{4/5} \beta}$$

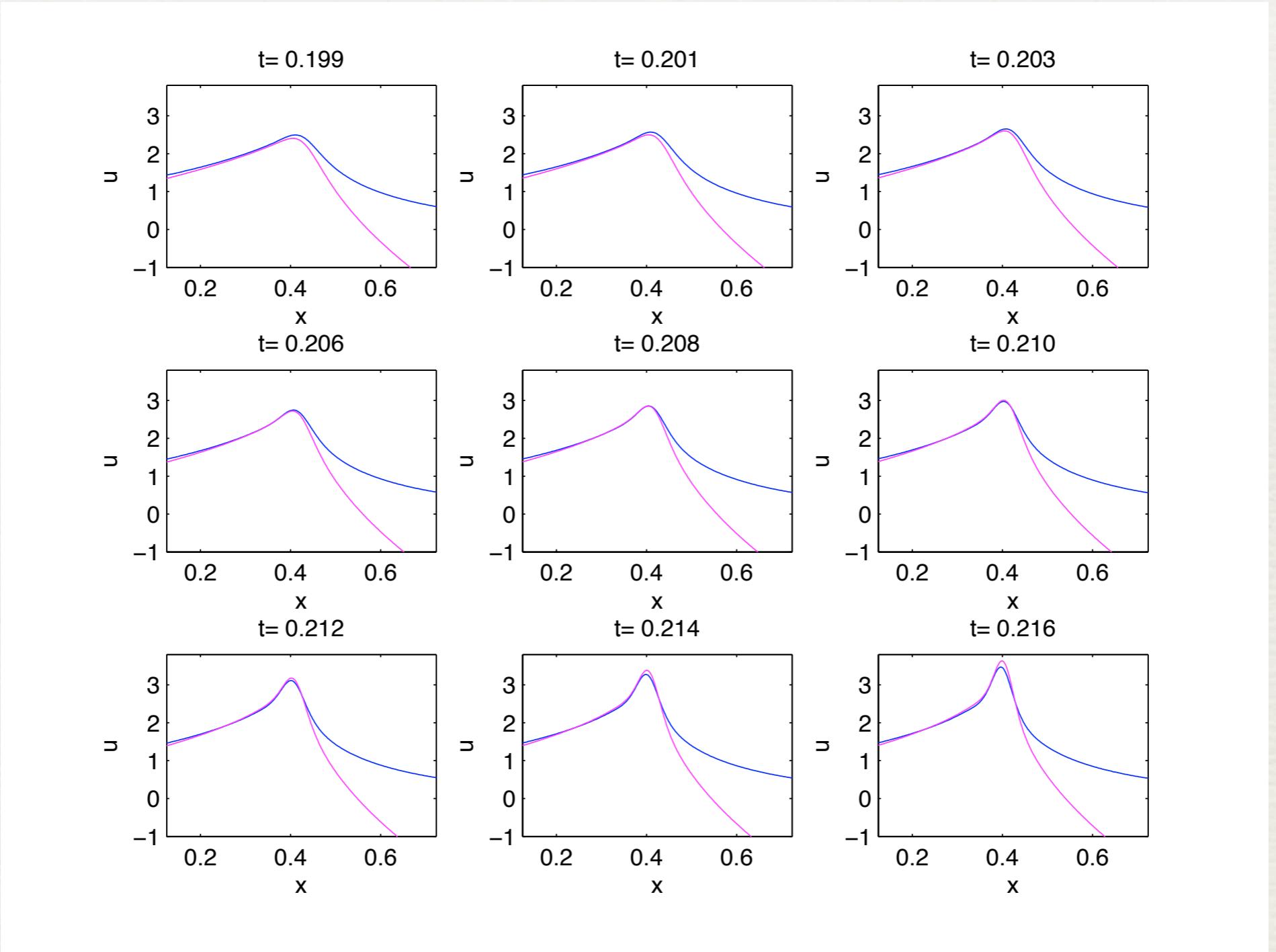


$\beta = 0.1$

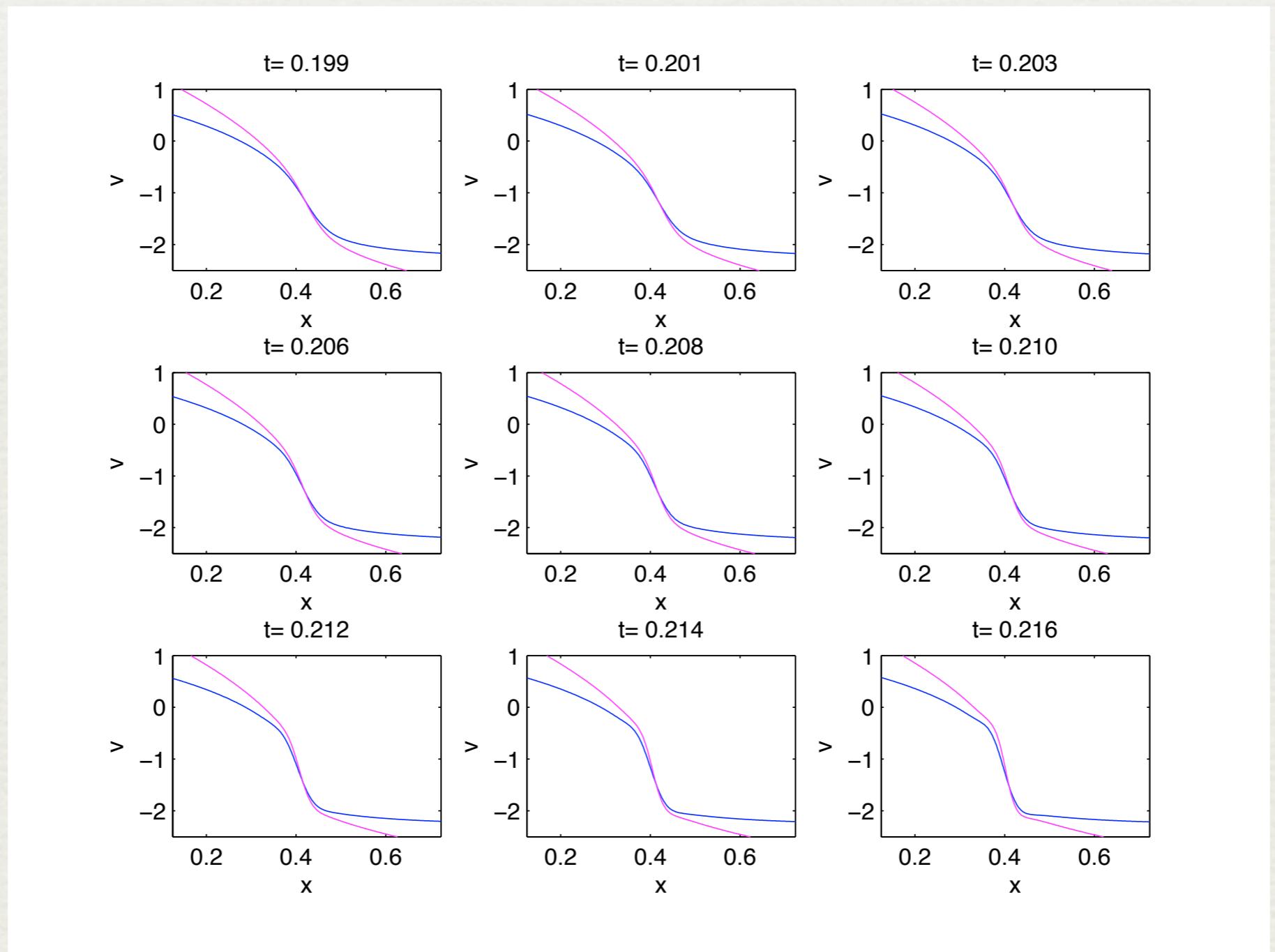
Non-symmetric initial data



Non-symmetric initial data



Non-symmetric initial data



Numerical methods

- task: resolve steep gradients in rapid oscillations
- Fourier series for spatial coordinates, method of lines
- fourth-order time stepping to avoid aliasing, integrating factor method (fourth-order Runge-Kutta), exponential time differencing, sliders (Driscoll), time splitting
- Krasny filtering (modulational instability) or more than double precision ($\varepsilon > 0.025$)

- Fourier space: equation of the form

$$U_t = cU + N[U]$$

here: U vector (1+1) or matrix (2+1), c array, $N[U]$ convolution,
steep gradients: high frequency terms in c lead to large absolute values
despite small ϵ

- exponential time differencing: time discretization and integration with integrating factor

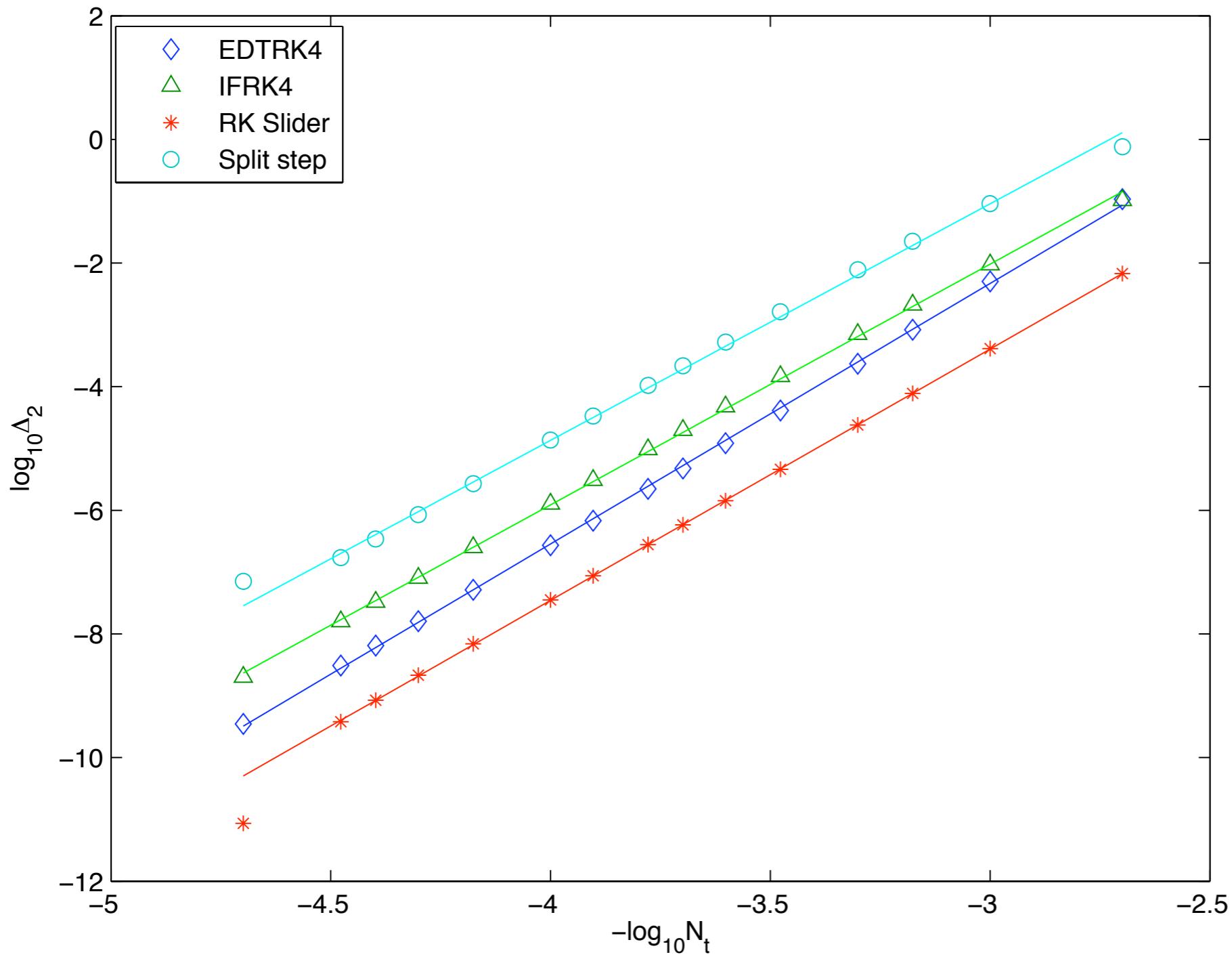
$$U(t_n + h) = e^{ch} U(t_n) + \int_0^h d\tau e^{c(h-\tau)} N[U(t_n + \tau)]$$

fourth-order Runge-Kutta scheme (Cox-Matthews), coefficients via contour integrals (Kassam-Trefethen)

- integrating factor, fourth-order Runge-Kutta (e.g. Trefethen):

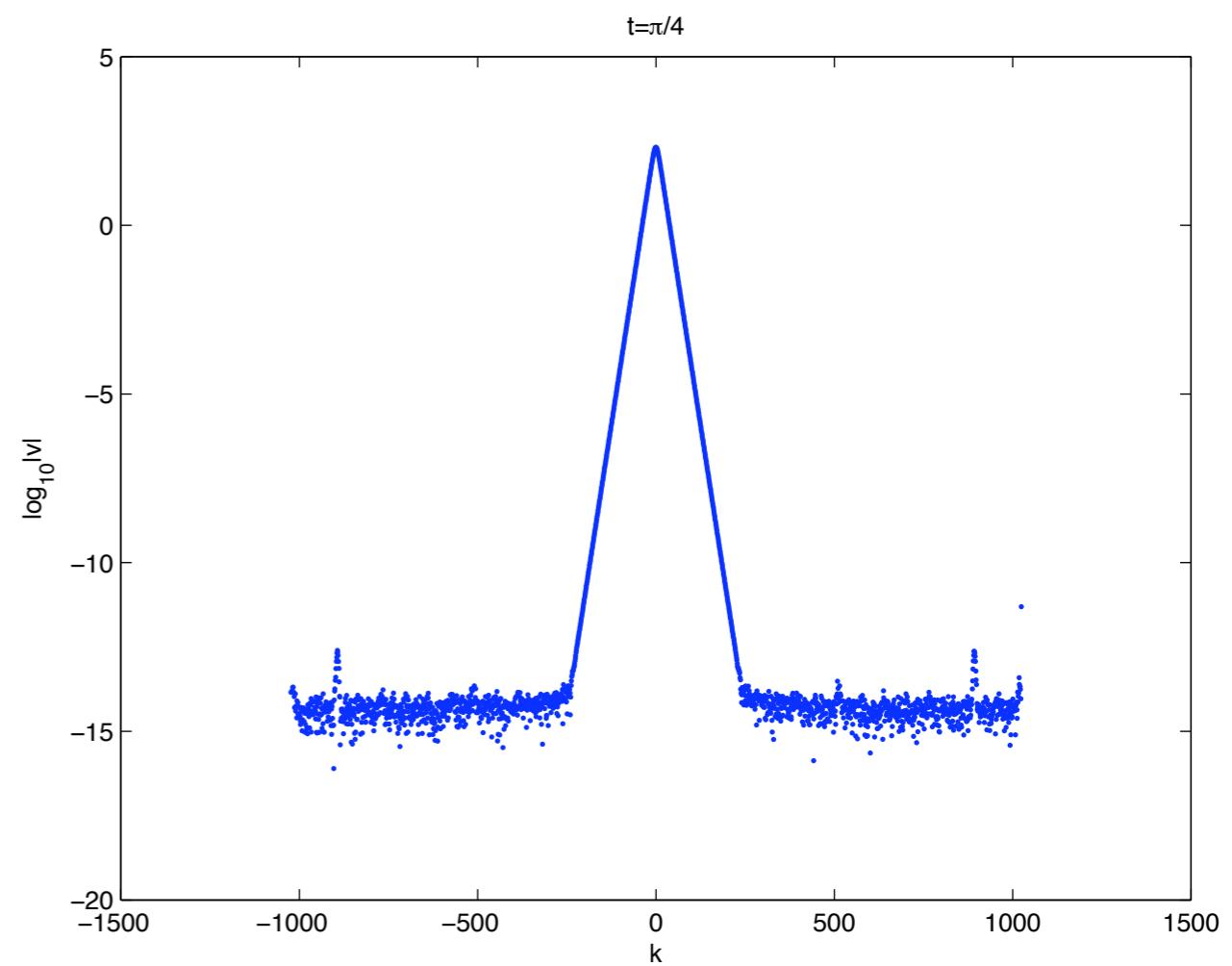
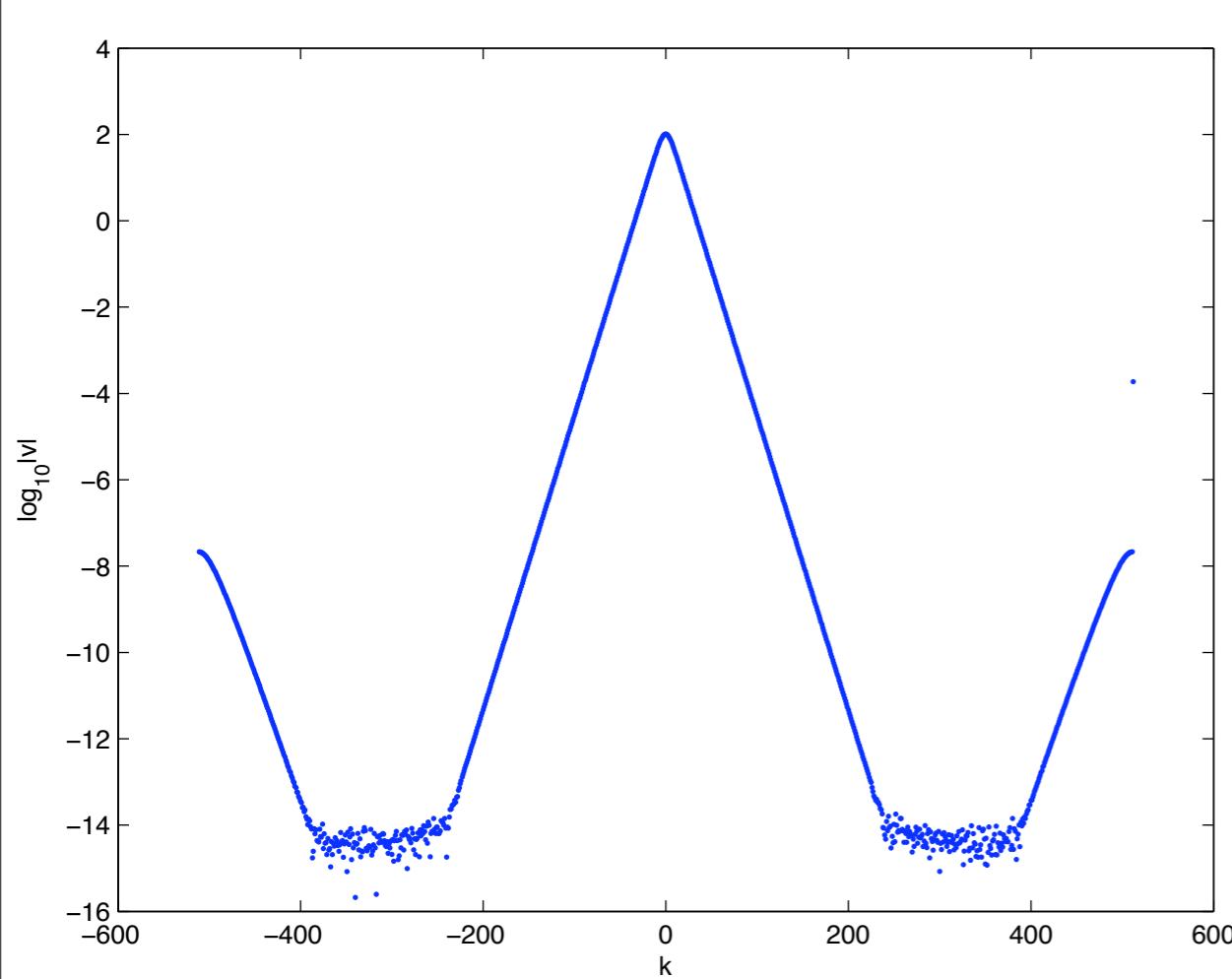
$$(e^{-ct} U)_t = e^{-ct} N[U]$$

Performance

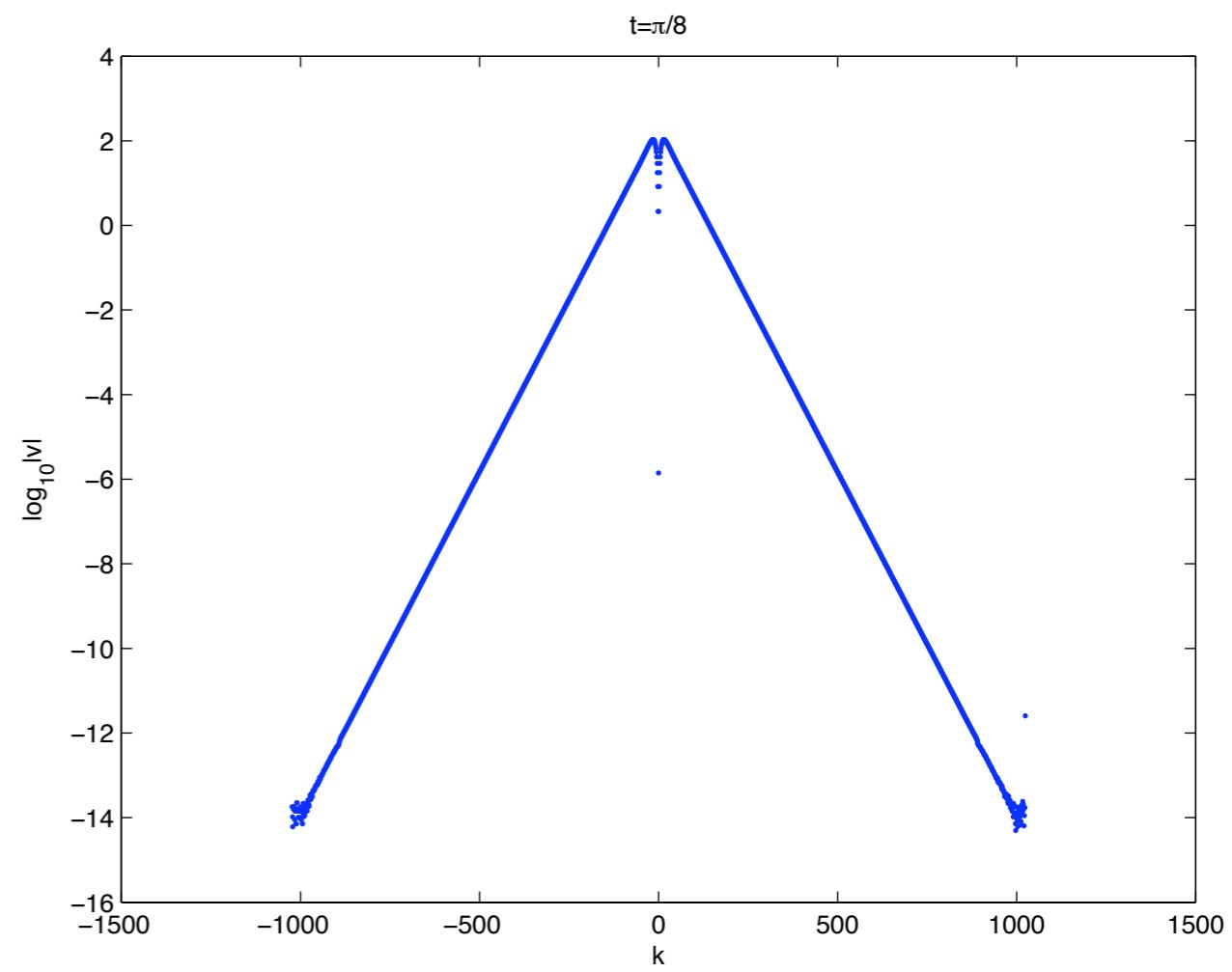
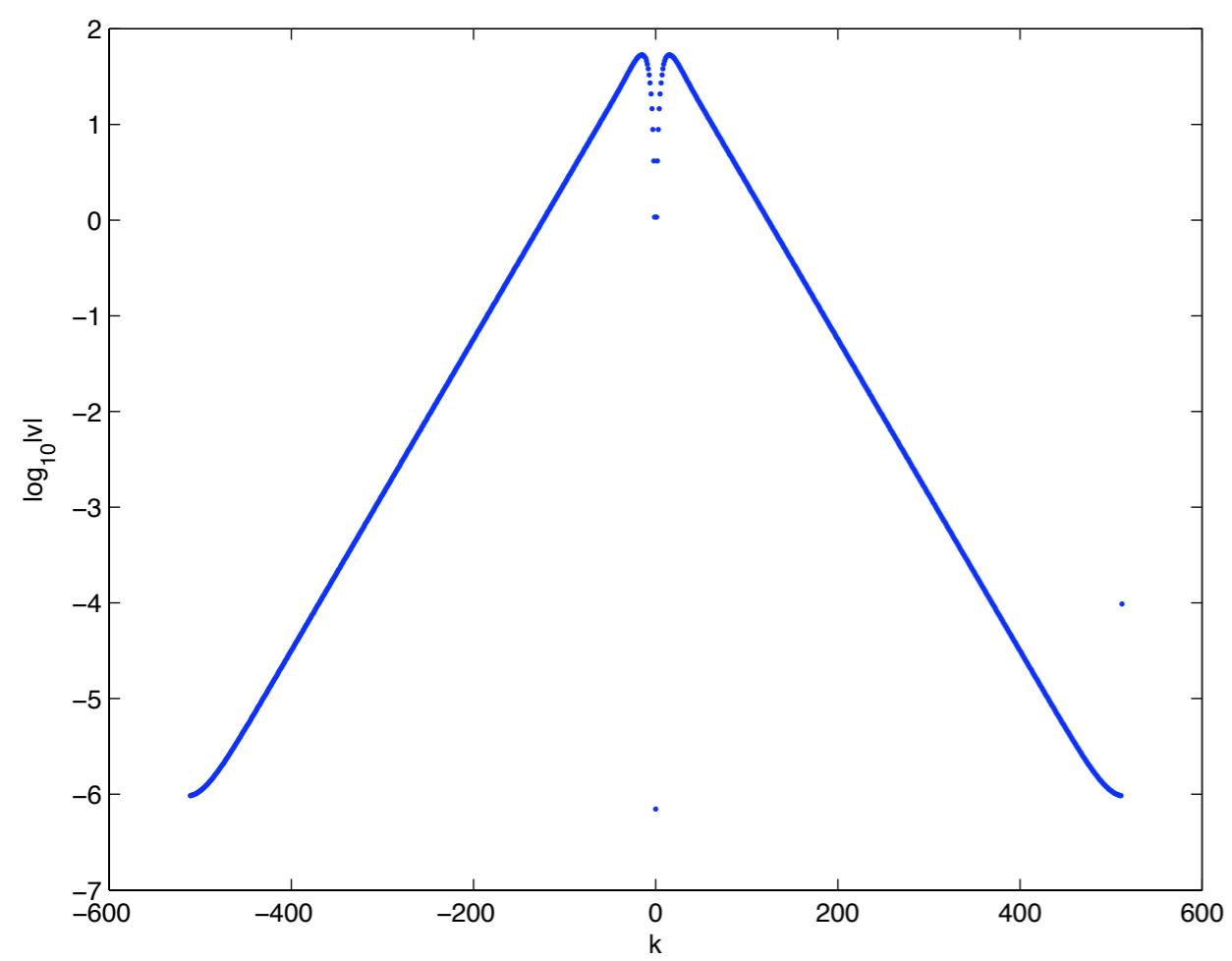


$\psi_0 = \exp(-x^2)$
 $\epsilon = 0.1$
 $t \leq 0.8$

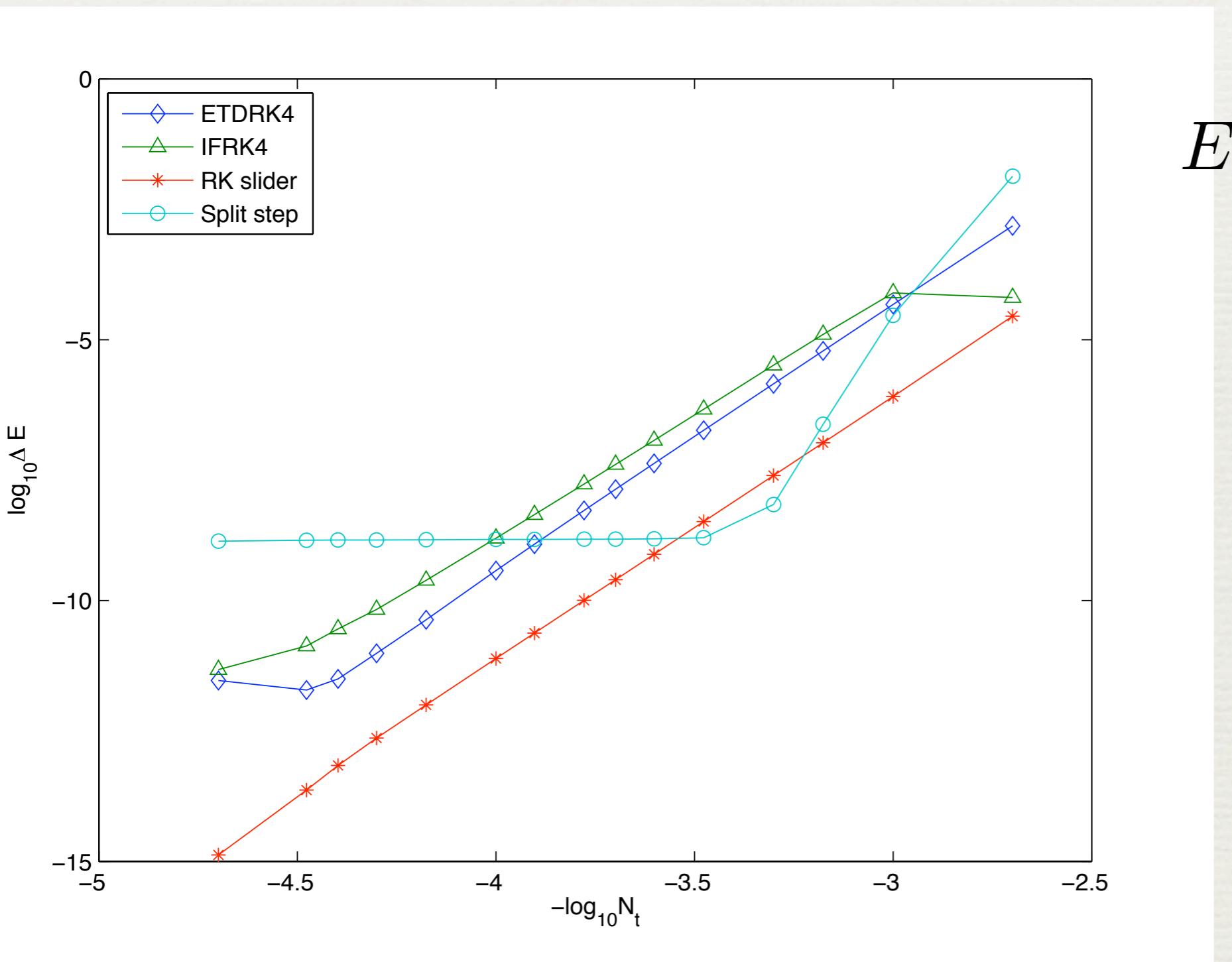
Modulational instability: breather, $t = \pi/4$



breather, $t = \pi/8$



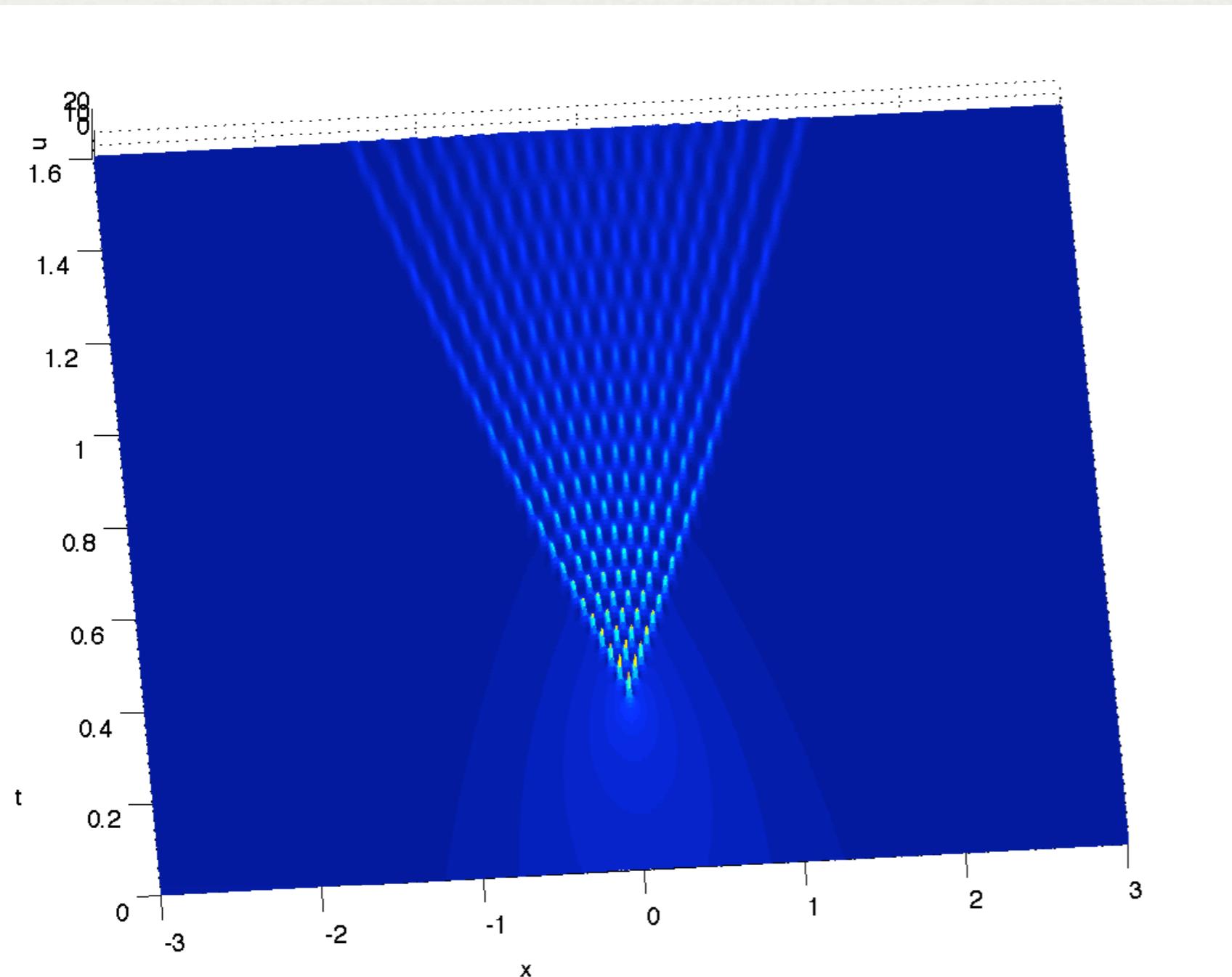
„Energy conservation“



$$E = \epsilon^2 |\psi_x|^2 + \rho |\psi|^4$$

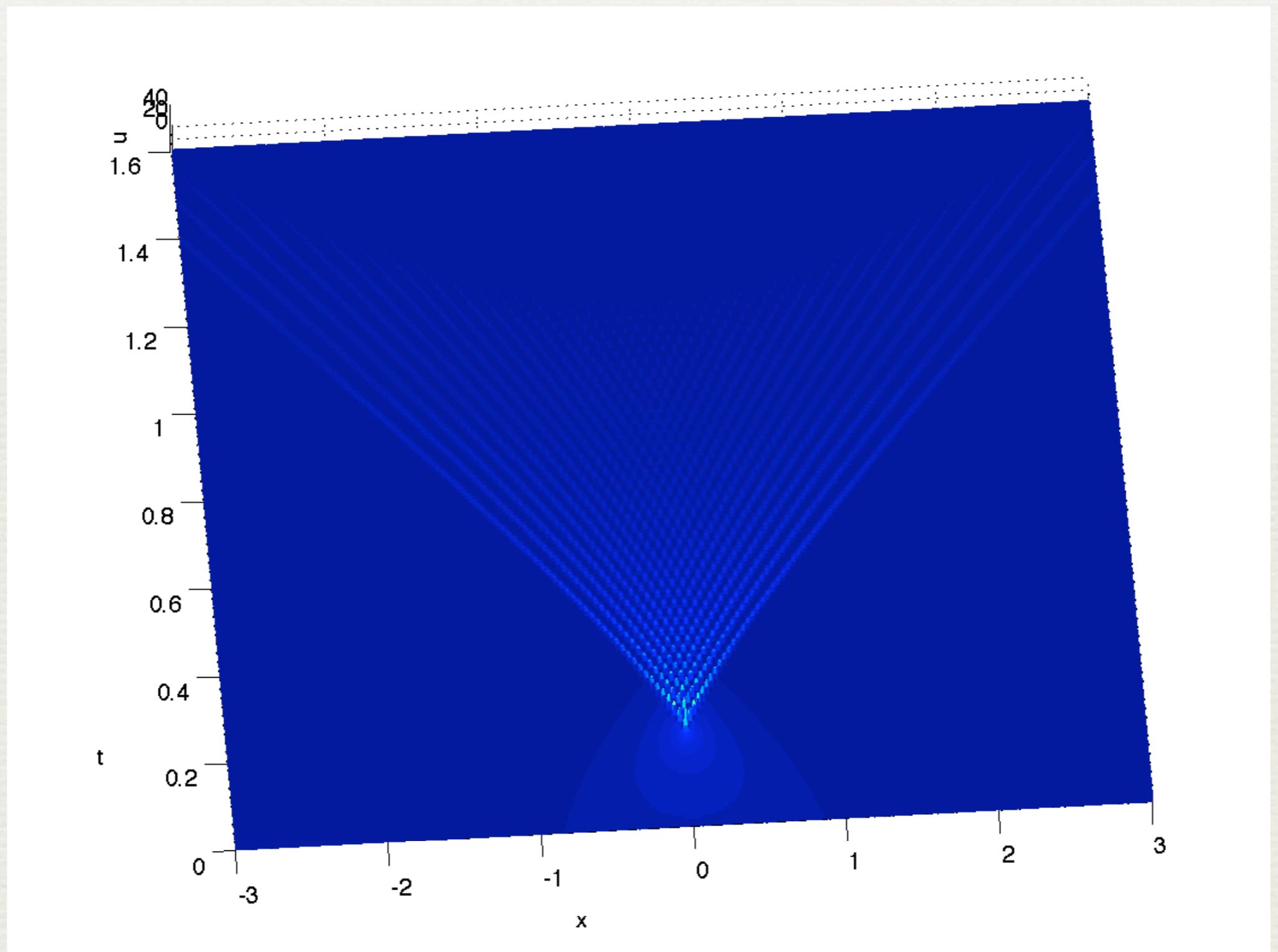
Large Times

$$\psi_0 = -\operatorname{sech} x \exp(-i\mu \ln \cosh x), \quad t_c = 1/(2 + \mu)$$

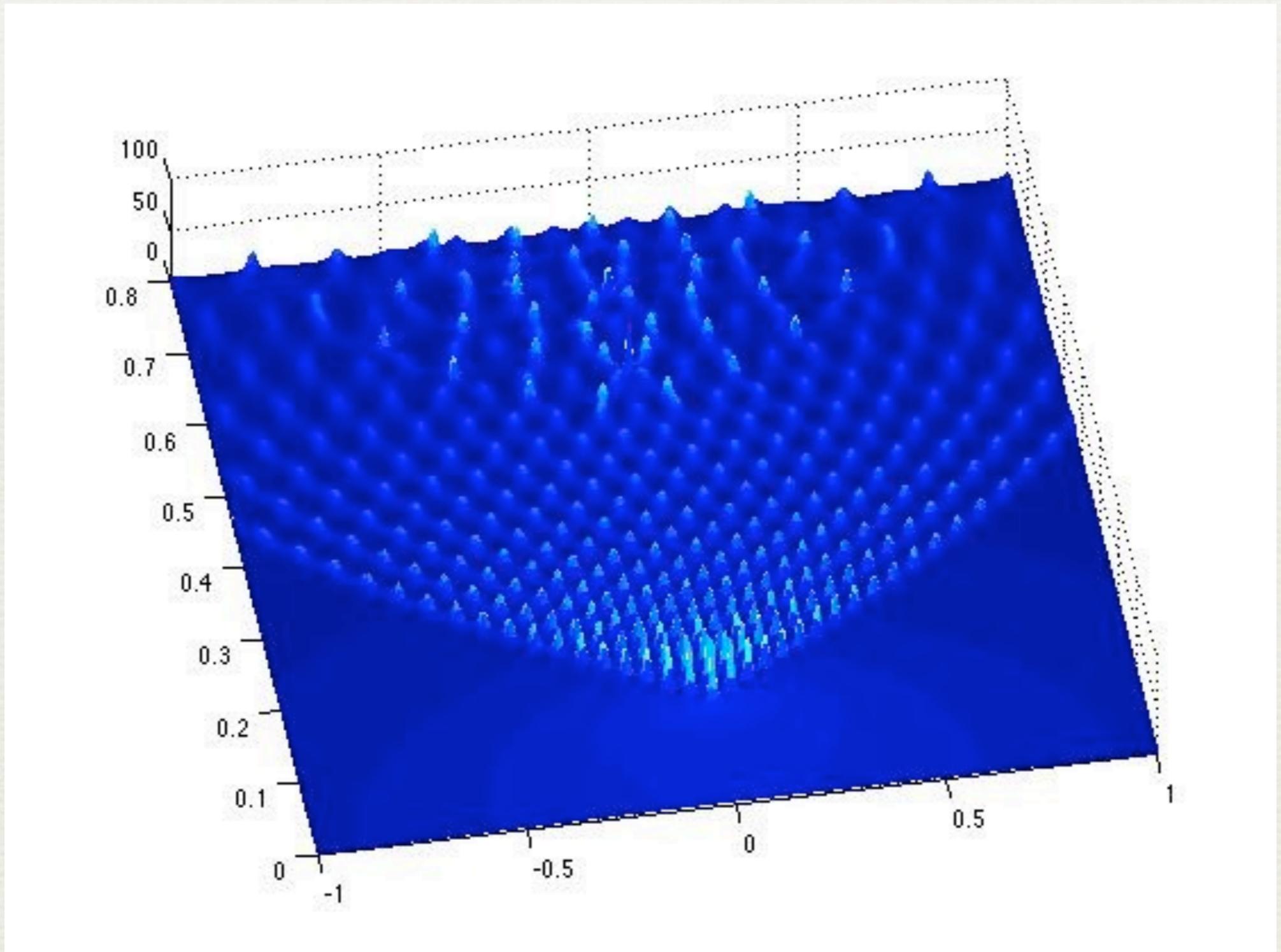


$\mu = 1$, solitonic
 $\epsilon = 0.04$

$\mu=3$, pure radiation



$\mu=0$, solitonic, second caustic



Davey-Stewartson equation

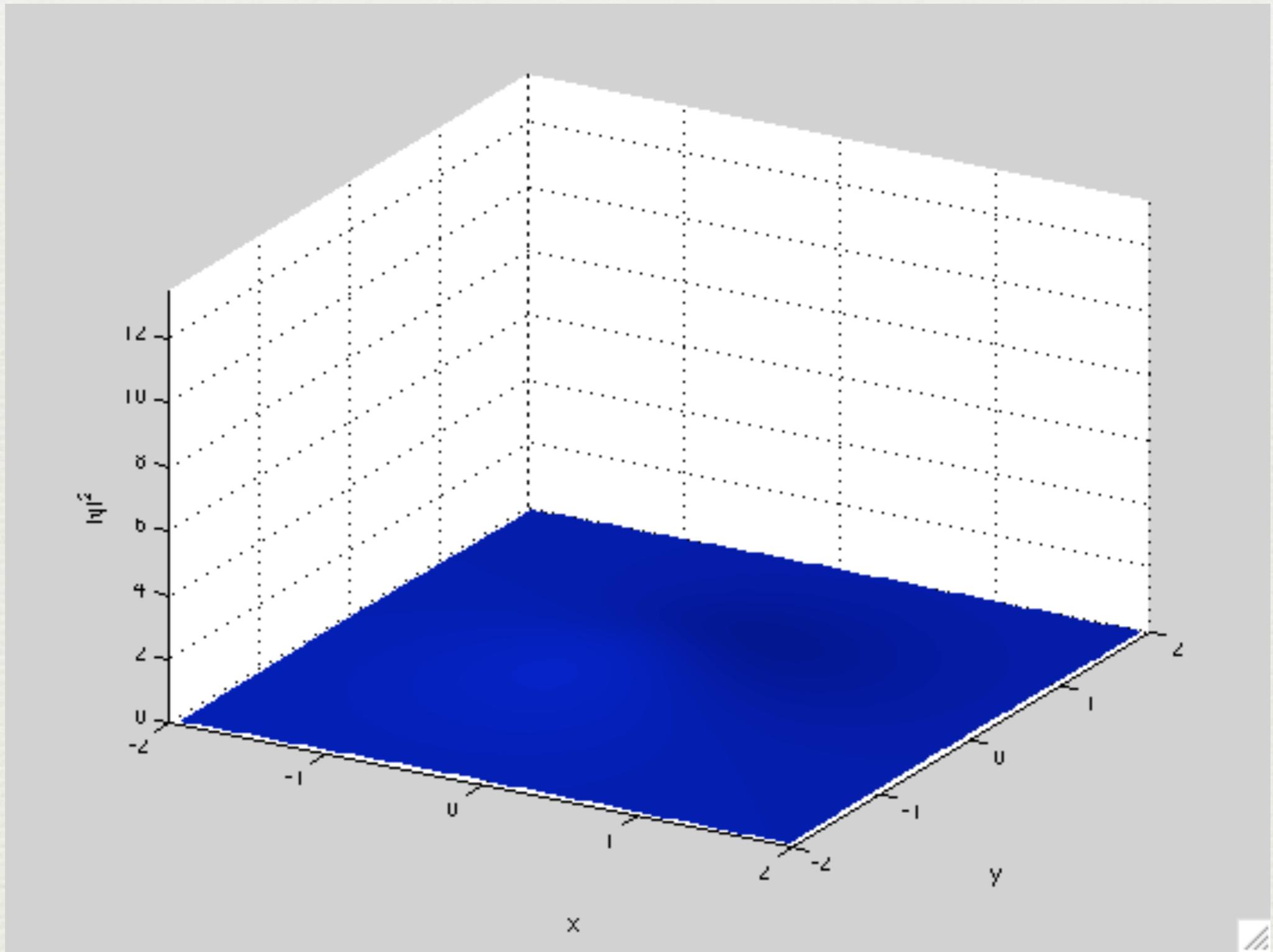
- equation for the amplitude

$$\begin{cases} i \partial_\tau \psi_0 - 3\eta \partial_{\xi\xi} \psi_0 + \frac{\lambda}{\eta} \partial_{yy} \psi_0 - \left(\frac{1}{6\eta} |\psi_0|^2 + \eta \phi \right) \psi_0 = 0, \\ 3\eta^2 \partial_{\xi\xi} \phi + \lambda \partial_{yy} \phi + \partial_{\xi\xi} |\psi_0|^2 = 0. \end{cases}$$

- y -independent case: non-linear Schrödinger equation (NLS)

Focusing DS

$$u_0 = \operatorname{sech} R, \quad R^2 = x^2 + y^2$$



$$\epsilon = 0.1$$