

# Universality in Random Matrix Theory

## $F(\text{Tr}(V(M)))$ instead of $\text{Tr}(V(M))$

Projects with Peter Miller, Misha Stepanov, and Gernot Akemann

The basic example of random matrices:  $N \times N$  Hermitian matrices:

$M_{jj}$ : Gaussian random variable: Prob  $\{M_{jj} < x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}t^2} dt$

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$$\text{Prob } \left\{ M_{jk}^{(R)} < x \right\} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^x e^{-t^2} dt$$

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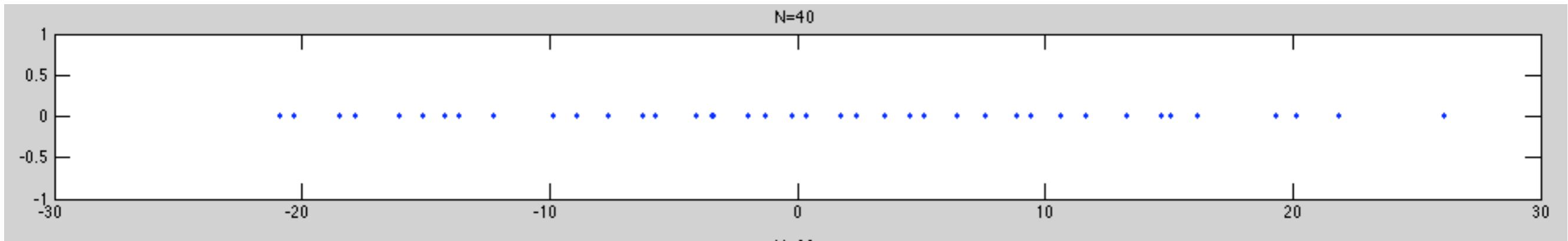
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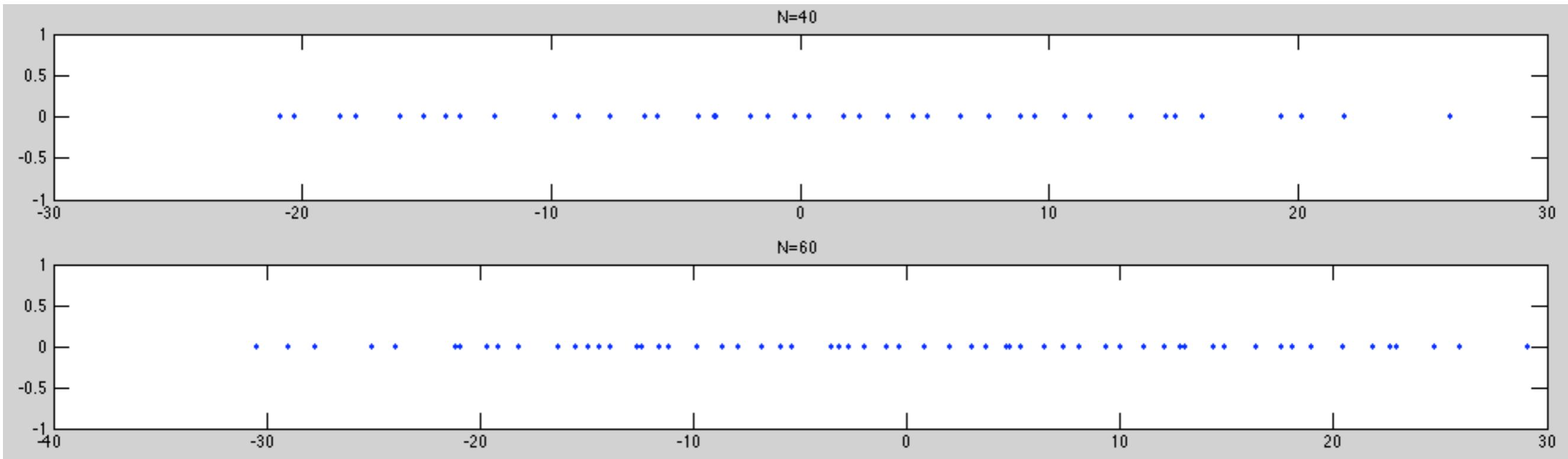
$$\begin{pmatrix} M_{11} & M_{12}^{(R)} + iM_{12}^{(I)} & \cdots & M_{1N}^{(R)} + iM_{1N}^{(I)} \\ M_{12}^{(R)} - iM_{12}^{(I)} & M_{22} & \cdots & M_{2N}^{(R)} + iM_{2N}^{(I)} \\ \vdots & \ddots & \ddots & \vdots \\ M_{1N}^{(R)} - iM_{1N}^{(I)} & M_{2N}^{(R)} + iM_{2N}^{(I)} & \cdots & M_{NN} \end{pmatrix}$$

This is referred to as the Gaussian Unitary Ensemble

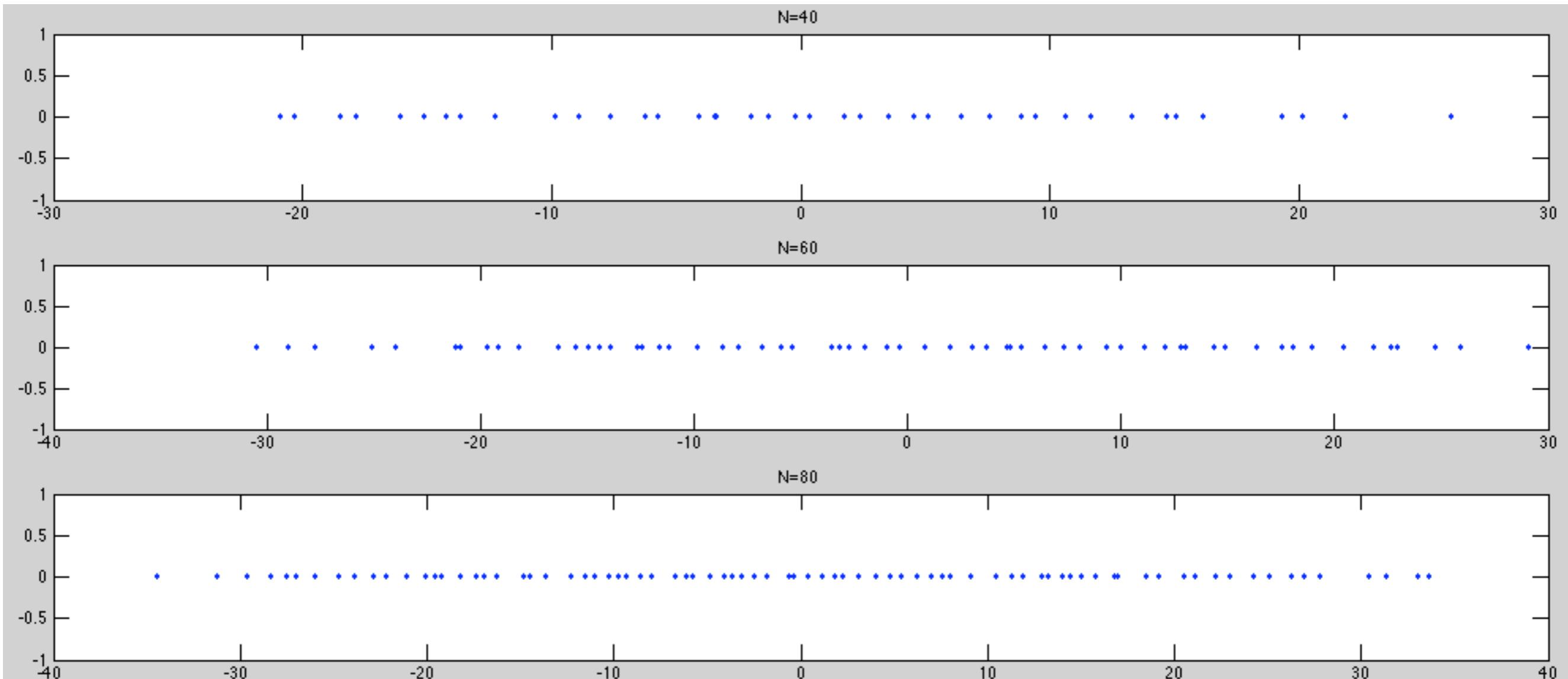
Using Matlab, we can generate random matrices and compute their eigenvalues



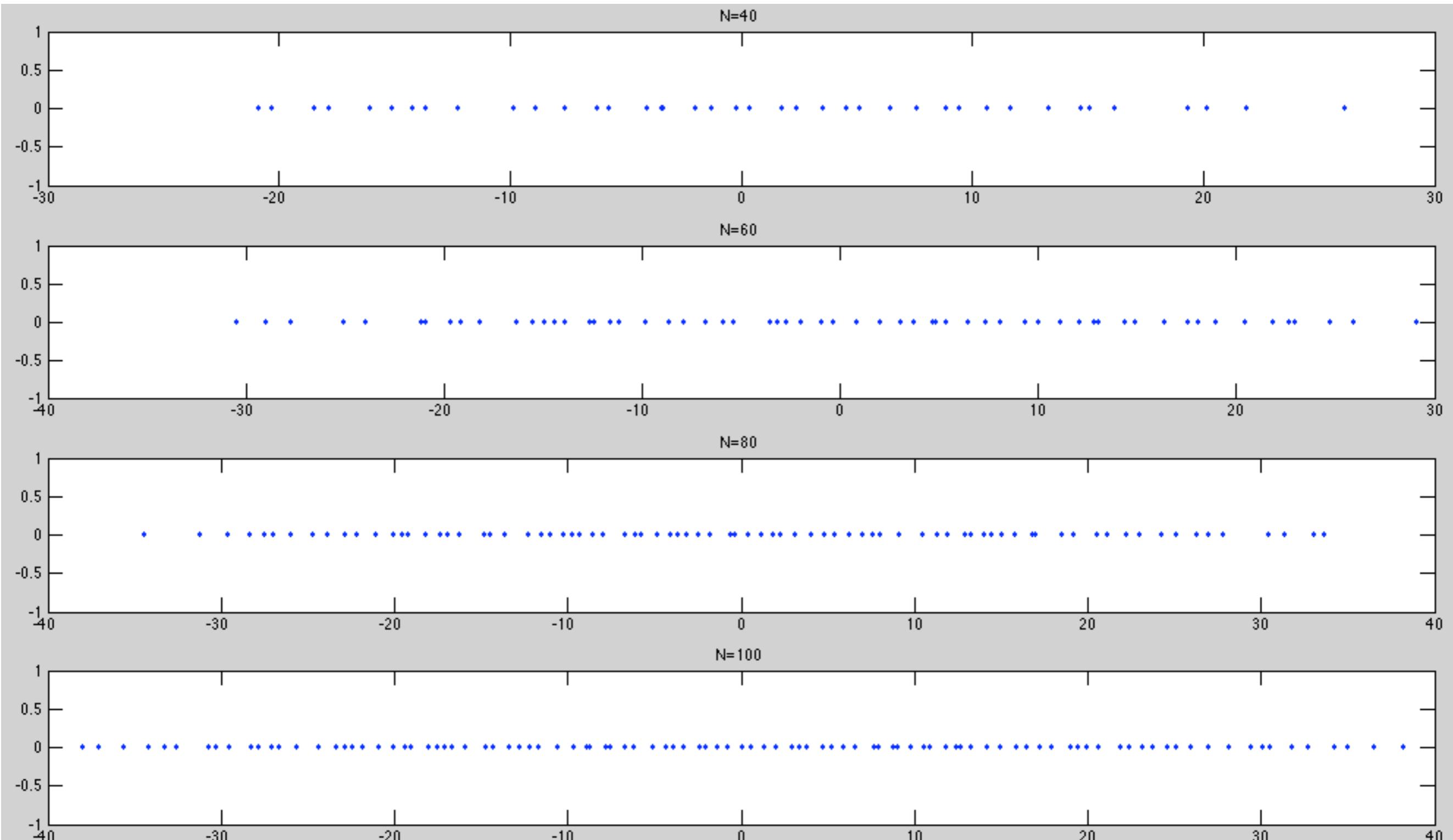
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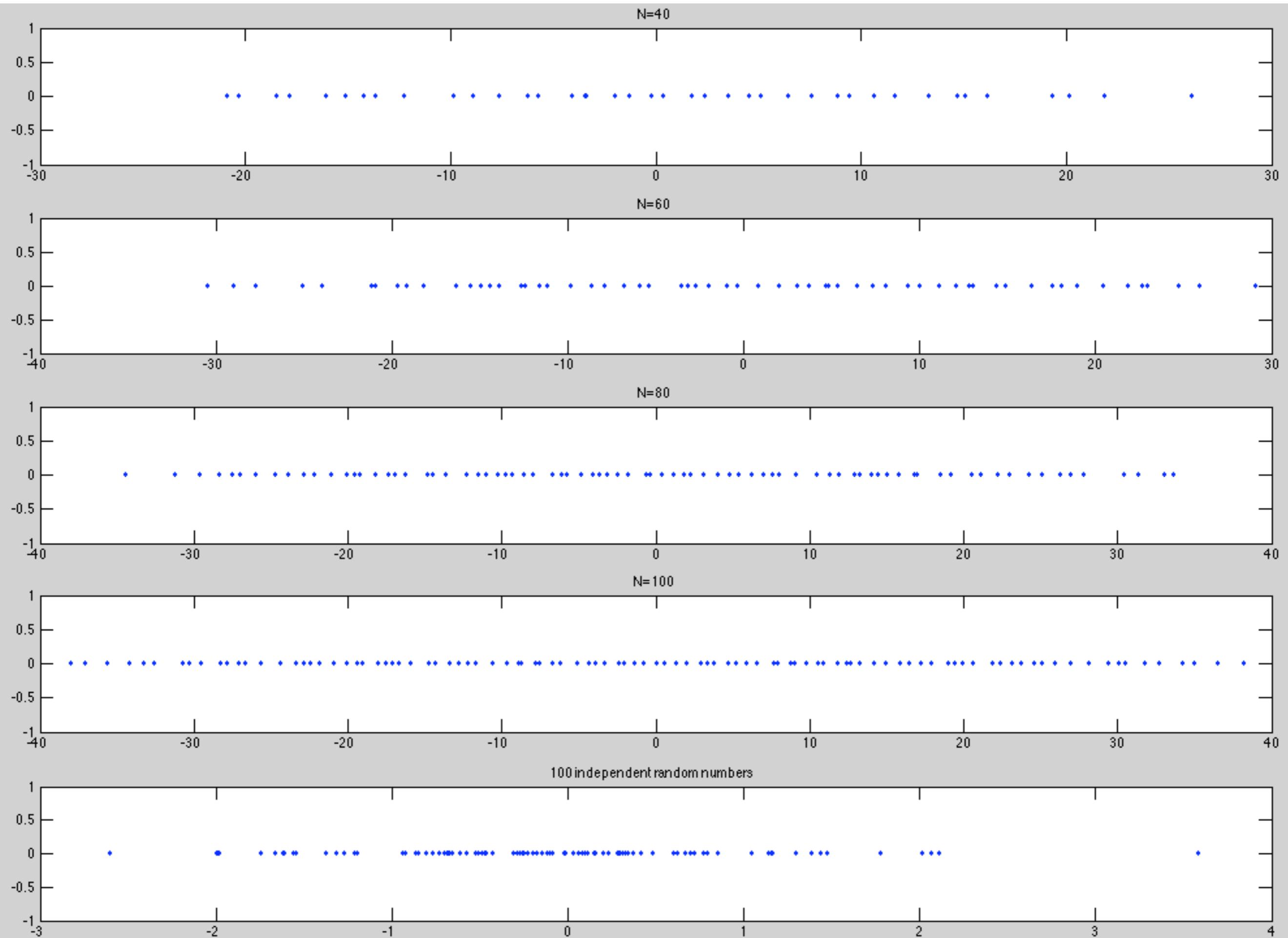
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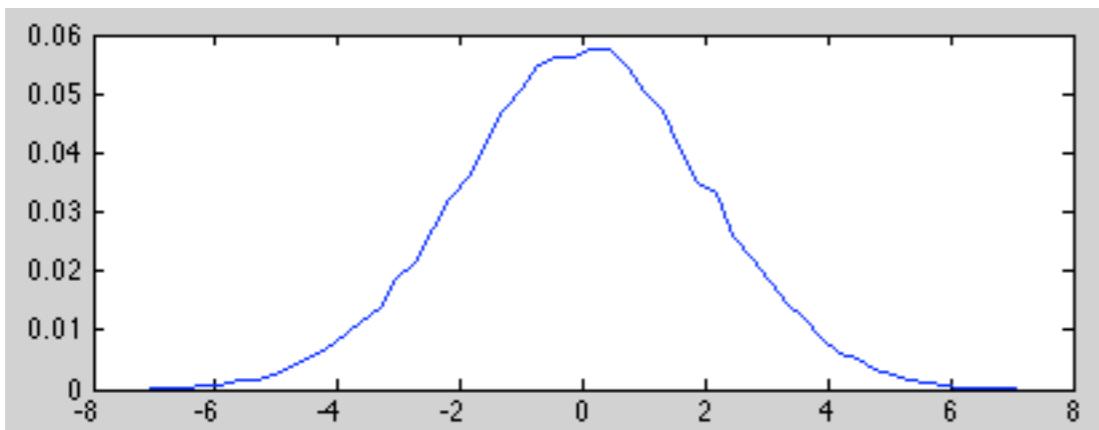
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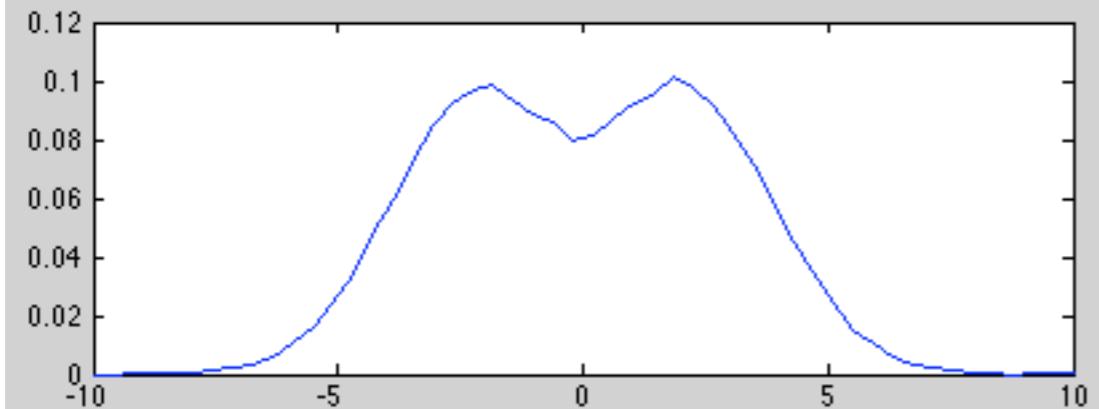
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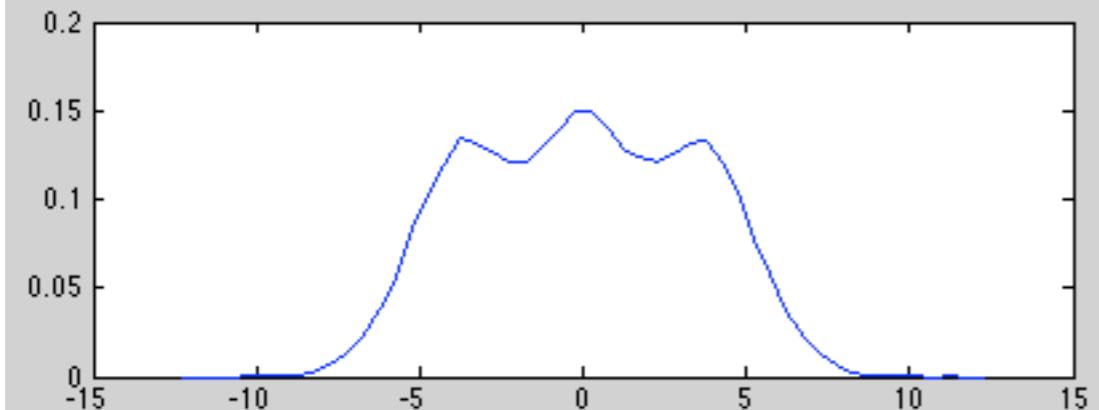
**N=1**



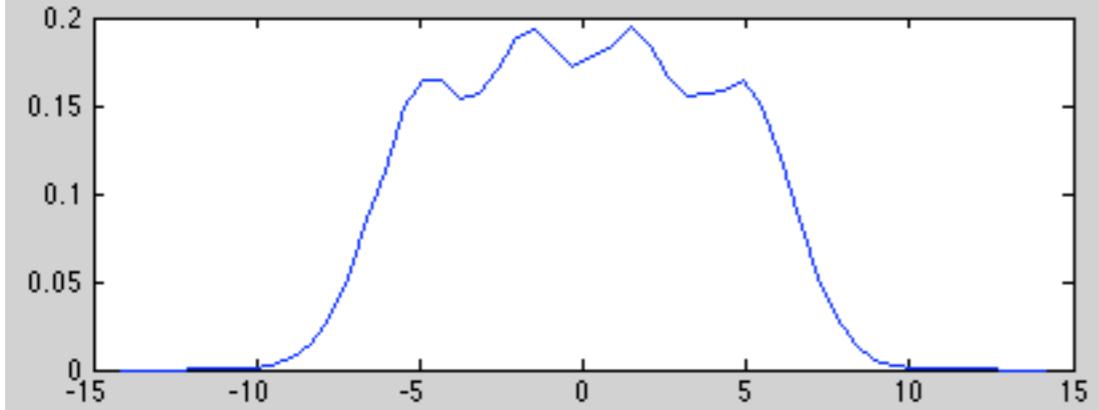
**N=2**



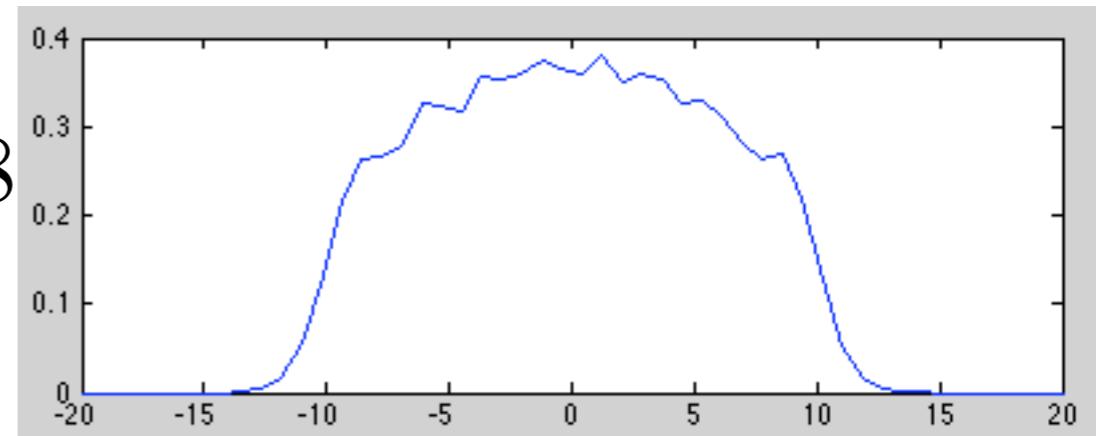
**N=3**



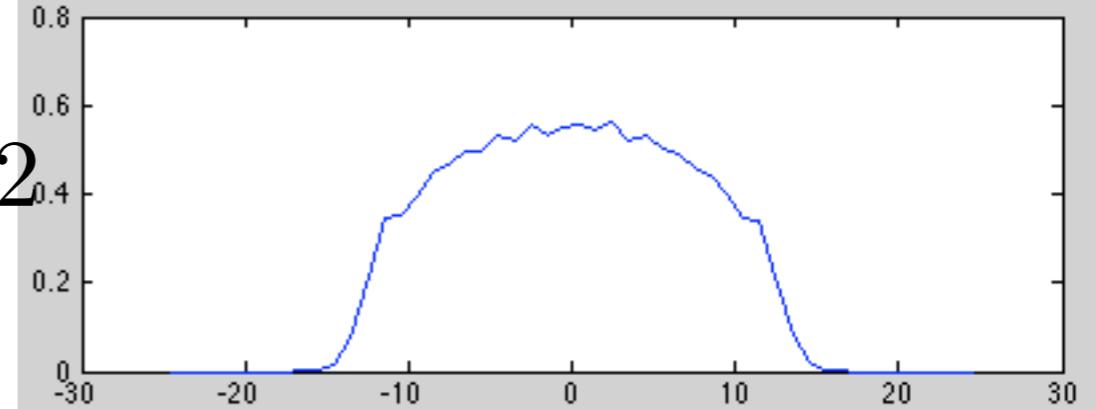
**N=4**



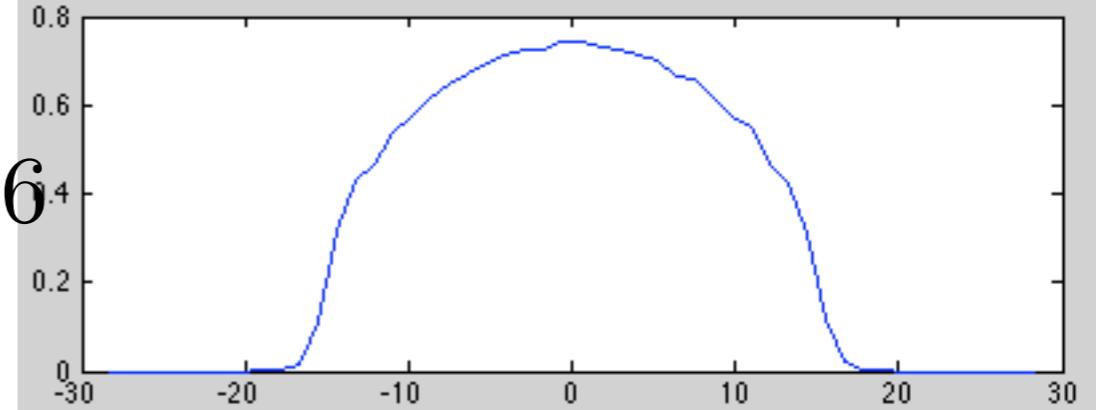
**N=8**



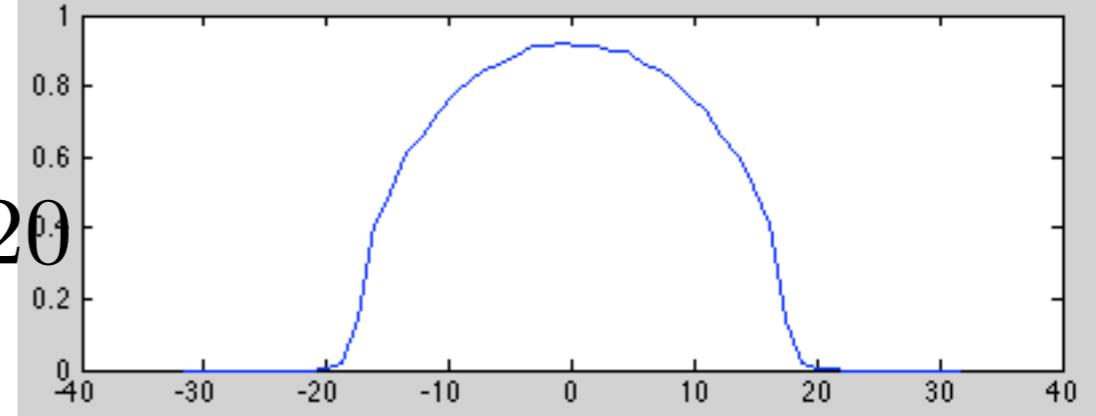
**N=12**



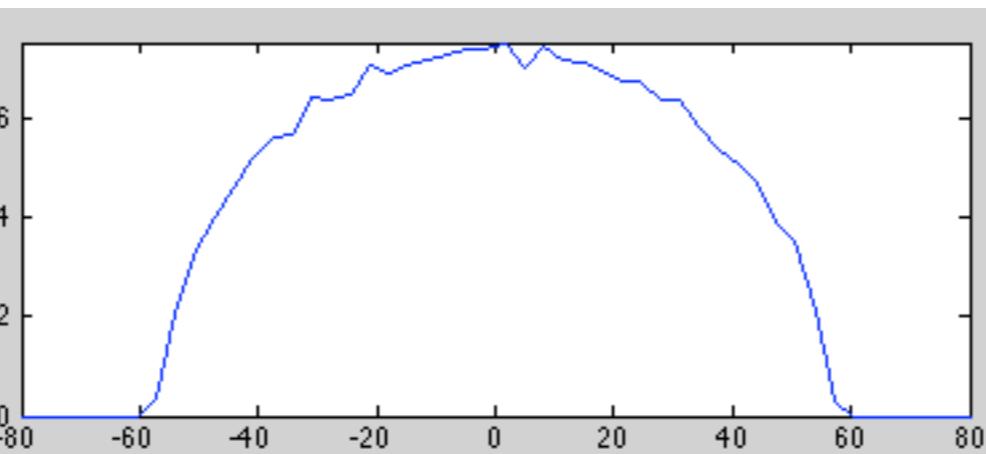
**N=16**



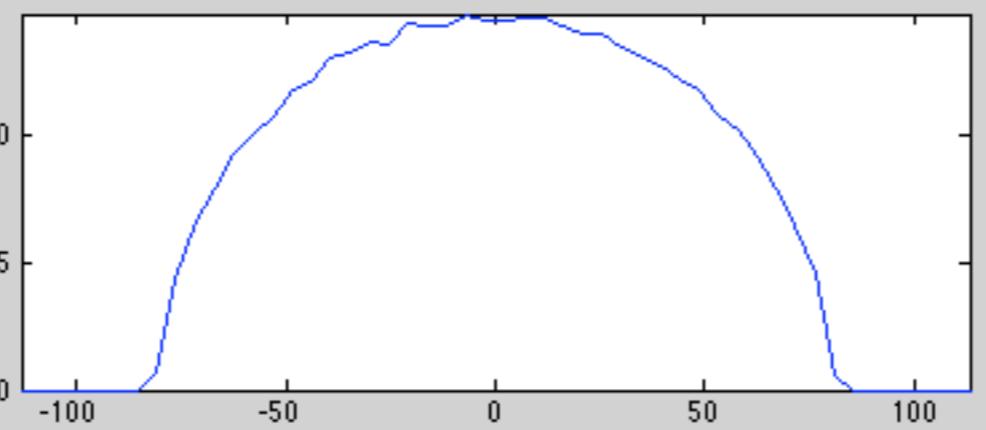
**N=20**



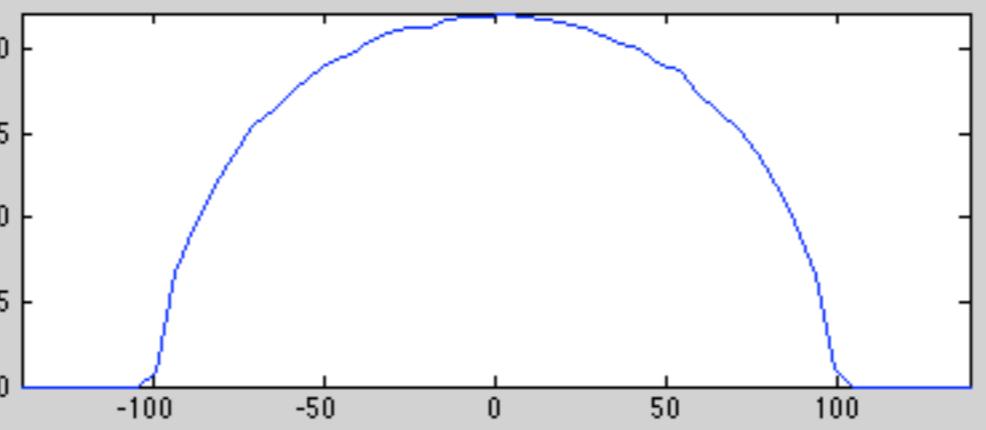
$N = 200$



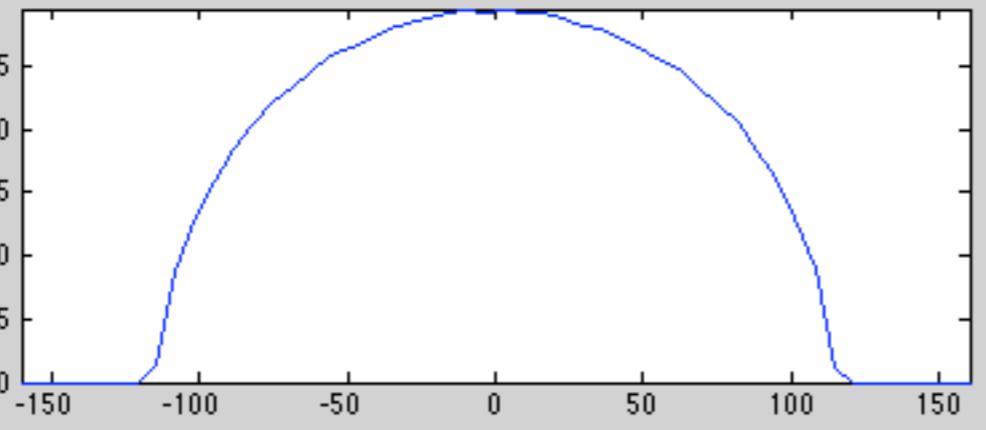
$N = 400$



$N = 600$



$N = 800$



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$M_{jk}$ : Complex Gaussian random variable,  $\frac{1}{\pi} e^{-|M_{jk}|^2} dM_{jk}^{(R)} dM_{jk}^{(I)}$

Joint prob. measure:  $\frac{1}{\#_N} \exp \left\{ -\text{Tr} \left[ \frac{1}{2} M^2 \right] \right\} dM,$

$$dM = \prod_{j < k} dM_{jk}^{(R)} dM_{jk}^{(I)} \prod_{j=1}^N dM_{jj}$$

This is referred to as the Gaussian Unitary Ensemble

## Unitary Ensembles of Random Matrices

Consider the probability measure on  $N \times N$  Hermitean matrices given by

$$\frac{1}{\hat{Z}_N} \exp \left\{ -N \operatorname{Tr} \left[ \frac{1}{2} M^2 + \sum_{k=1}^{2v} t_k M^k \right] \right\} dM$$

$$dM = \prod_{j < k} dM_{jk}^R dM_{jk}^I \prod_{j=1}^N dM_{jj}$$

$$\hat{Z}_N = \int \exp \left\{ -N \operatorname{Tr} \left[ \frac{1}{2} M^2 + \sum_{k=1}^{2v} t_k M^k \right] \right\} dM$$

Historically, the interest has been in the probabilistic description of the eigenvalues, as  $N \rightarrow \infty$

Gaussian Unitary Ensemble: all the  $t_k$ 's =0.

## Unitary Ensembles of Random Matrices

Consider the probability measure on  $N \times N$  Hermitean matrices given by

$$\frac{1}{\hat{Z}_N} \exp \left\{ -N \operatorname{Tr} \begin{bmatrix} & V(M) \\ & \end{bmatrix} \right\} dM$$

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Gaussian Unitary Ensemble:  $V(x)=x^2/2$

# Mean density of eigenvalues

Consider the random variable  $\frac{1}{N} \# \{ \lambda_j < x \}$ .

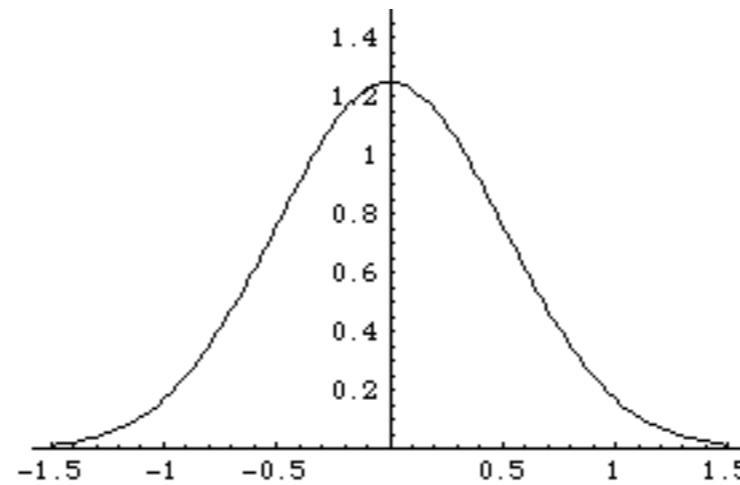
What is its average behavior?

$$\mathbb{E} \left[ \frac{1}{N} \# \{ \lambda_j < x \} \right] = \int_{-\infty}^x \rho_1^{(N)}(t) dt$$

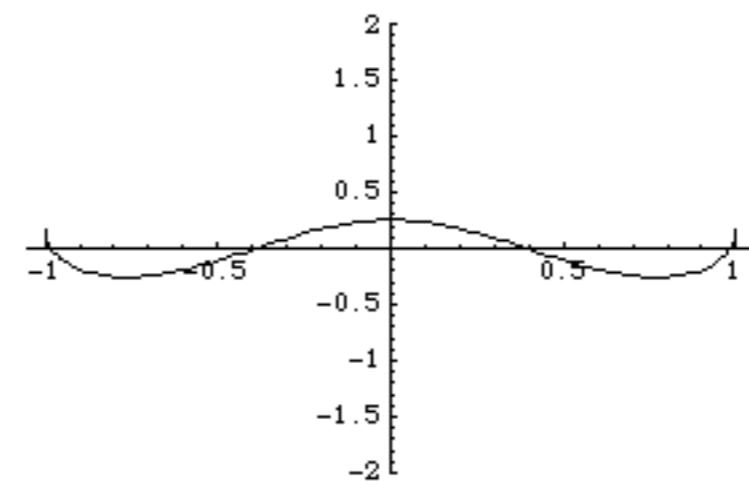
$\rho_1^{(N)}(t)$  is called the mean density of eigenvalues

Movie of  $\rho_1^{(N)}$  for  $N = 1$  through  $N = 50$ .

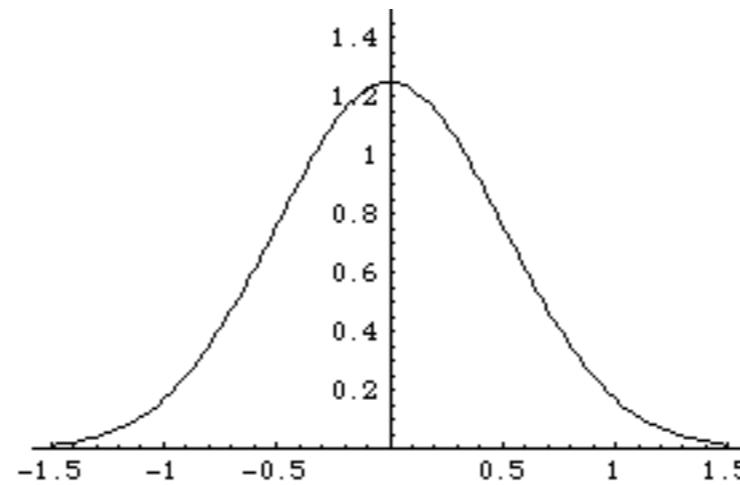
Plot of the error,  $\epsilon_1^{(N)}(x)$ , where  
 $\rho_1^{(N)}(x) = \psi(x) + N^{-1}\epsilon_1^{(N)}(x)$



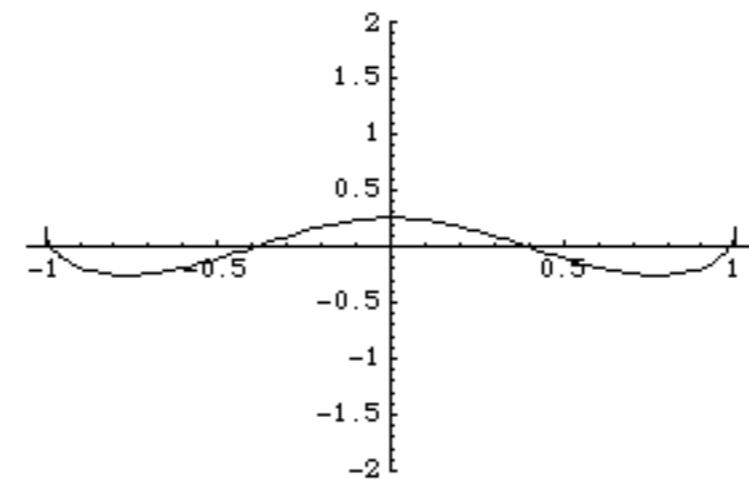
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Plot of the error,  $\epsilon_1^{(N)}(x)$ , where  
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more fundamental quantities: gap probabilities

$$\text{Prob } \{ \text{ no evals in } (a, b) \}$$

$$F_N(\mu) = \text{Prob } \{ \lambda_{max} < \mu \}$$

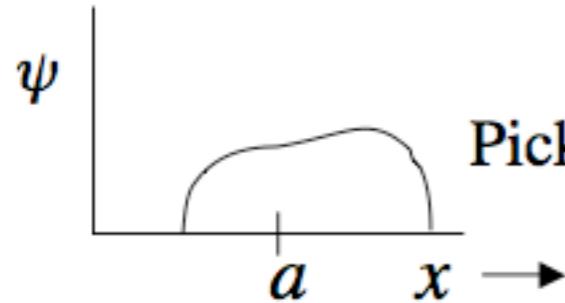
Basic asymptotic result: Under rather weak assumptions on  $V$ , the following limit exists.  
(Johansson, 1998)

$$\lim_{N \rightarrow \infty} \rho_1^{(N)}(x) = \psi(x), \quad \text{where } \psi \geq 0 \text{ solves a well-known variational problem.}$$

$$\sup_{\substack{0 \leq d\mu, \\ \int d\mu = 1}} \left[ - \int V d\mu + \iint \log|x-y| d\mu(x) d\mu(y) \right]$$

$V$  real analytic with suitable growth  $\Rightarrow$   $\psi$  is supported on finitely many intervals,  
and is analytic on the interior of each one.  
(Deift, Kriecherbauer, McL '98)

Much more detailed asymptotic result:



Pick  $a \in \text{supp}(\psi)$  and consider  $G_N(a,s) = \text{Prob}\left\{\text{no } \lambda_j \text{'s in } \left(a, a + \frac{s}{N\psi(a)}\right)\right\}$ .

One version of universality result: for any real analytic  $V$  with suitable growth, and any such  $a$ ,

$$\lim_{N \rightarrow \infty} G_N(a,s) = \det(1 - \mathbf{S})_{L^2(0,s)}$$

$$(\mathbf{S}h)(\sigma) = \int_0^s \frac{\sin(\pi(\sigma - \sigma'))}{\pi(\sigma - \sigma')} h(\sigma') d\sigma'$$

$V$  Quadratic: Gaudin&Mehta '69

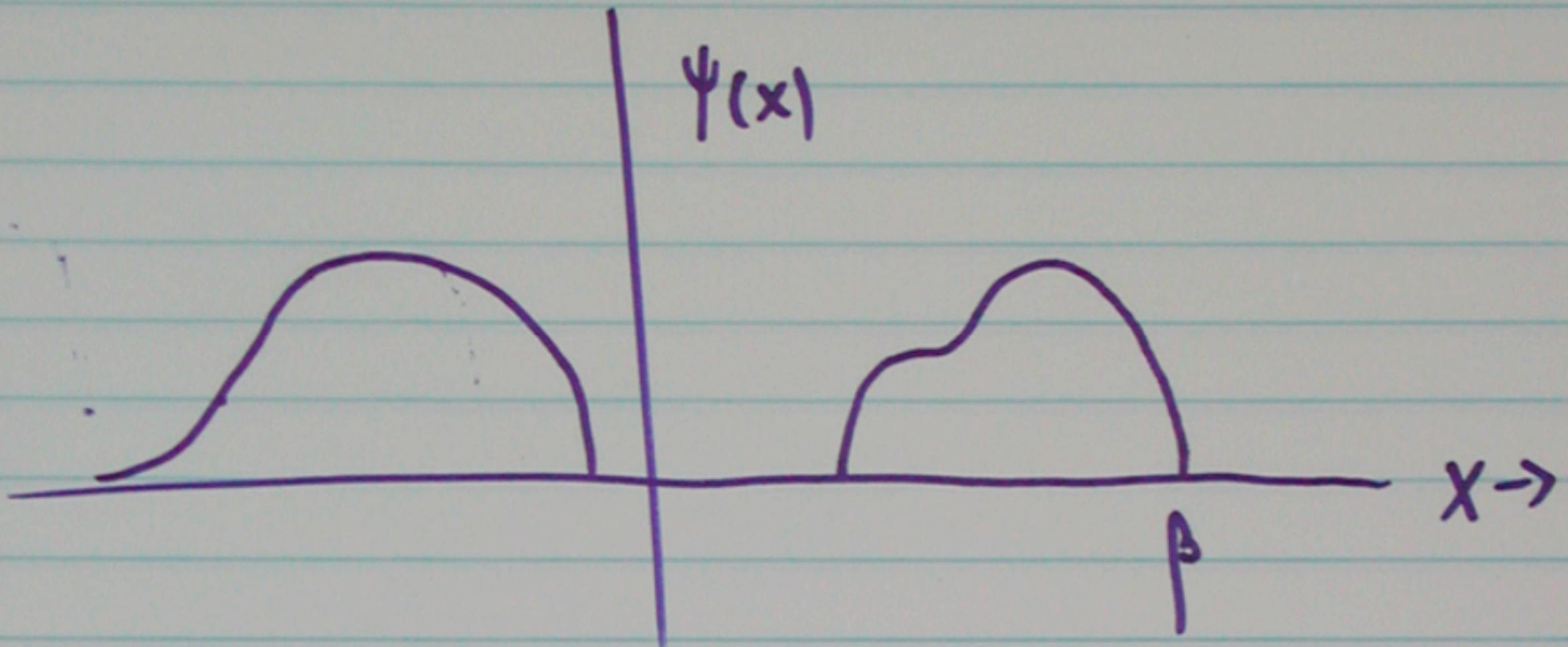
$V$  Quartic: Bleher&Its '99

$V$  real analytic: Deift, Kriecherbauer,

McL, Venakides, & Zhou '99

$V''$  Lipschitz continuous: Miller & McL '07

# Behavior of largest eval



$$\text{Prob}(|\lambda_N - \beta| > \epsilon) \xrightarrow{N \rightarrow \infty} 0$$

$$\text{Prob} \left\{ C (\lambda_N - \beta) N^{2/3} < s \right\} \rightarrow F_{TW}(s)$$

- $V(x) = x^2/2$ : Tracy-Widom

$$F_{TW}(s) = \exp \left\{ - \int_s^\infty (x-s) q^2(x) dx \right\},$$

$$q'' = xq + 2q^3$$

- Real analytic  $V$ : Bleher & Its (Quartic), DKMVZ
- $V''$  Lipschitz continuous: McL & Miller '07.

Typically interested in eigenvalues, whose induced probability density is:

$$\frac{1}{Z_N} \exp\left(-N \sum_{j=1}^N V(\lambda_j)\right) \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2,$$

$$V(\lambda) = \left[ \frac{1}{2} \lambda^2 + \sum_{k=1}^{2v} t_k \lambda^k \right]$$

Partition Function:  $Z_N = \int \exp\left(-N \sum_{j=1}^N V(\lambda_j)\right) \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2 d^N \lambda$

Closely related to polynomials  $\{p_j(x)\}_{j \geq 0}$  orthogonal wrt  $e^{-NV(x)} dx$

## Mean density of particles

Consider the random variable  $\frac{1}{N} \# \{\text{evals} < x\}$

The mean density is defined via

$$\rho_1(x) = \frac{d}{dx} \left\langle \frac{1}{N} \# \{\text{evals} < x\} \right\rangle$$

The connection to orthogonal polynomials:

$$\rho_1(x) = K_N(x, x)$$

$$K_N(x, y) = e^{-\frac{N(V(x)+V(y))}{2}} \sum_{\ell=0}^{N-1} p_\ell(x) p_\ell(y)$$

All statistical properties can be expressed in terms of the orthogonal polynomials!

E.G.

$$\text{Prob}\{\text{no } \lambda_j \text{'s in } (a,b)\} = \det(1 - \mathbf{K}_{\mathbf{N}})_{L^2(a,b)}$$

$$(\mathbf{K}_{\mathbf{N}} f)(x) = \int_a^b K_N(x, y) f(y) dy$$

In most applications, we are interested in the behavior for  $N \rightarrow \infty$

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For universality of “gap probabilities”,  $(a, b) = \left(a, a + \frac{s}{N\psi(a)}\right)$ .

# Universality is implied by

**Asymptotic Result 1:** *There is a constant  $c$  so that for every  $u, v \in \mathbb{R}$ , we have*

$$\lim_{N \rightarrow \infty} \frac{1}{(cN)^{2/3}} K_N \left( \beta + \frac{u}{(cN)^{2/3}}, \beta + \frac{v}{(cN)^{2/3}} \right) = \frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}(v)\text{Ai}'(u)}{u - v}. \quad (1)$$

**Asymptotic Result 2:** *For every  $a$  with  $\psi(a) > 0$ , and every  $u, v \in \mathbb{R}$ , we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N\psi(a)} K_N \left( a + \frac{u}{N\psi(a)}, a + \frac{v}{N\psi(a)} \right) = \frac{\sin \pi(u - v)}{\pi(u - v)} \quad (2)$$

$$K_N(x, y) = e^{-\frac{N}{2}(V(x)+V(y))} \sum_{\ell=0}^{N-1} p_\ell(x)p_\ell(y).$$

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Repeat Slide!

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# Unitary Ensembles of Random Matrices, revisited

$$\frac{1}{\hat{Z}_N} \exp \left\{ -\alpha N \text{Tr}(M^2) - \tau \left( \text{Tr} (M^2) \right)^2 \right\} dM$$

Induced Measure on eigenvalues:

$$\frac{1}{Z_N} \exp \left\{ -\alpha N \sum_{j=1}^N \lambda_j^2 - \tau \left( \sum_{j=1}^N \lambda_j^2 \right)^2 \right\} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2 d^N \lambda$$

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This fouls up our connection  
to Orthogonal Polynomials

Wishart-Laguerre Ensembles,  
following Akemann&Vivo, arXiv:0806.1861

$$\frac{1}{\hat{Z}_N} \left( 1 + \frac{n\beta}{\gamma} V(\mathbf{X}^\dagger \mathbf{X}) \right)^{-\gamma} d\mathbf{X} ,$$

$X: (N + \nu) \times N$ ,  $\nu \geq 0$ , complex entries

$$\gamma > cN(N + \nu)$$

$dX$ : Lebesgue measure over independent matrix entries  
Induced Measure on eigenvalues of  $X^\dagger X$ :

$$\frac{1}{Z_N} \left( 1 + \frac{2n}{\gamma} \sum_{i=1}^N V(\lambda_i) \right)^{-\gamma} \prod_{i=1}^N \lambda_i^\nu \prod_{j>k}^N |\lambda_j - \lambda_k|^2 d^N \lambda$$

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Three measures on eigenvalues, (simplest choice of  $V$ )

$$\frac{1}{Z_N} \exp \left\{ -\frac{N}{2} \sum_{j=1}^N \lambda_j^2 \right\} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2 d^N \lambda$$

$$\frac{1}{Z_N} \exp \left\{ -\alpha N \sum_{j=1}^N \lambda_j^2 - \tau \left( \sum_{j=1}^N \lambda_j^2 \right)^{\textcolor{red}{2}} \right\} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2 d^N \lambda$$

$$\frac{1}{Z_N} \left( 1 + \frac{2n}{\gamma} \sum_{i=1}^N \lambda_i^2 \right)^{-\gamma} \prod_{i=1}^N \lambda_i^\nu \prod_{j>k}^N |\lambda_j - \lambda_k|^2 d^N \lambda$$

Note: Probability measures on **unordered** eigenvalues.

# Why use Orthogonal Polynomials?

$$\frac{1}{Z_N} \exp \left\{ -\frac{N}{2} \sum_{j=1}^N \lambda_j^2 \right\} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2 d^N \lambda$$

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Orthogonal Polynomials (Hermite!)

$$p_j(x) = \kappa_j x^j + \text{l.o.t.}, \quad \kappa_j > 0$$

$$\int_{\mathbb{R}} p_j(x) p_k(x) e^{-\frac{N}{2}x^2} dx = \delta_{jk}.$$

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Reproducing Kernel

$$K_N(x, y) = e^{-\frac{N}{4}(x^2 + y^2)} \sum_{\ell=0}^{N-1} p_\ell(x) p_\ell(y).$$

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Reproducing Kernel

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$$\mathbb{P}_N(\lambda_1, \dots, \lambda_N) = \frac{1}{N!} \det \begin{pmatrix} K_N(\lambda_1, \lambda_1) & \cdots & K_N(\lambda_1, \lambda_N) \\ \vdots & \ddots & \vdots \\ K_N(\lambda_N, \lambda_1) & \cdots & K_N(\lambda_N, \lambda_N) \end{pmatrix}.$$

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Must explain the formula!

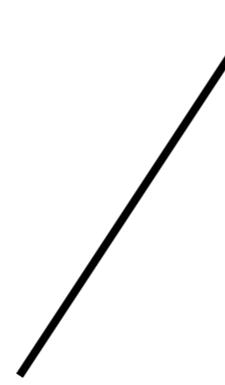
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**Useful:** Orthogonality yields

Mean density of particles

Consider the random variable  $\frac{1}{N} \# \{\text{evals} < x\}$

The mean density is defined via

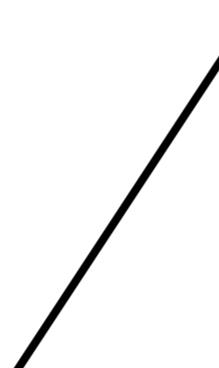
$$\rho_1(x) = \frac{d}{dx} \left\langle \frac{1}{N} \# \{\text{evals} < x\} \right\rangle$$

The connection to orthogonal polynomials:

$$\rho_1(x) = K_N(x, x)$$

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All statistical properties can be expressed in terms of the orthogonal polynomials!

Consider the random variable  $\frac{1}{N} \# \{\text{evals} < x\}$

E.G.

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$$\text{Prob}\{\text{no } \lambda_j \text{'s in } (a,b)\} = \det(1 - \mathbf{K}_N)_{L^2(a,b)}$$

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In most applications, we are interested in the behavior for  $N \rightarrow \infty$

## Why use Orthogonal Polynomials?

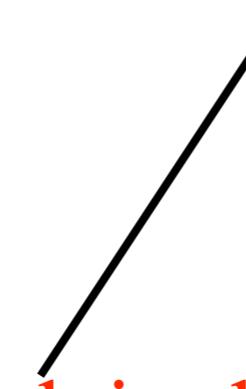
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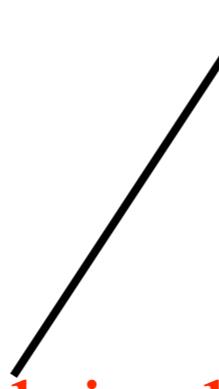


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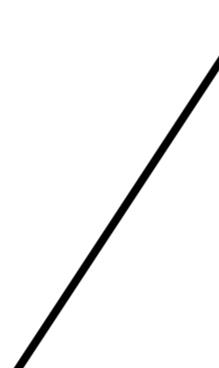
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Now orthogonality lets us evaluate the integrals:  $\frac{1}{Z_N \prod_{j=0}^{N-1} \kappa_j^2} = \frac{1}{N!}$

Three measures on eigenvalues, (simplest choice of  $V$ )

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$$= \frac{\prod_{j=0}^{N-1} \kappa_j^{-2}}{Z_N} \exp \left\{ -\tau \left( \sum_{j=1}^N \lambda_j^2 \right)^2 \right\} \begin{pmatrix} e^{-\frac{\alpha N}{2} \lambda_1^2} p_0(\lambda_1) & \cdots & e^{-\frac{\alpha N}{2} \lambda_N^2} p_0(\lambda_N) \\ e^{-\frac{\alpha N}{2} \lambda_1^2} p_1(\lambda_1) & \cdots & e^{-\frac{\alpha N}{2} \lambda_N^2} p_1(\lambda_N) \\ \vdots & \ddots & \vdots \\ e^{-\frac{\alpha N}{2} \lambda_1^2} p_{N-1}(\lambda_1) & \cdots & e^{-\frac{\alpha N}{2} \lambda_N^2} p_{N-1}(\lambda_N) \end{pmatrix}$$

A clever trick!

$$\exp \left\{ -\alpha N s - \tau (s)^2 \right\} = \frac{N}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} d\beta \quad e^{-\frac{\beta^2 N^2}{4\tau}} \quad \left( e^{-N(\alpha+i\beta)s} \right).$$

Which yields

$$\begin{aligned} \mathbb{P}_N(\lambda_1, \dots, \lambda_N) &= \frac{1}{Z_N} \exp \left\{ -\alpha N \sum_{j=1}^N \lambda_j^2 - \tau \left( \sum_{j=1}^N \lambda_j^2 \right)^{\textcolor{red}{2}} \right\} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2 \\ &= \frac{1}{Z_N} \int_{\mathbb{R}} d\beta \quad e^{-\frac{\beta^2 N^2}{4\tau}} \quad \left( e^{-N(\alpha+i\beta) \sum_{j=1}^N \lambda_j^2} \right) \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^2 \end{aligned}$$

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where

$$\int_{\mathbb{R}} p_n(x) p_j(x) e^{-N(\alpha+i\beta)x^2} dx = \delta_{nj},$$

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Mean Density of Eigenvalues:

$$\rho_1^{(N)}(\lambda) = \left[ N \int_{\mathbb{R}} d\beta \ e^{-\frac{\beta^2 N^2}{4\tau}} (\alpha + i\beta)^{-\frac{N^2}{2}} \right]^{-1} \int_{\mathbb{R}} d\beta \ e^{-\frac{\beta^2 N^2}{4\tau}} \left\{ (\alpha + i\beta)^{-\frac{N^2}{2}} K_N(\lambda, \lambda; \alpha, \beta) \right\}$$

$$\text{Prob } \{ \text{ no eigenvalues in } S \} = \frac{\int_{\mathbb{R}} d\beta \ e^{-\frac{\beta^2 N^2}{4\tau}} (\alpha + i\beta)^{-\frac{N^2}{2}} \ \det(1 - \mathcal{K}_N)}{\int_{\mathbb{R}} d\beta \ e^{-\frac{\beta^2 N^2}{4\tau}} (\alpha + i\beta)^{-\frac{N^2}{2}}} .$$

Where  $\mathcal{K}_N$  is the integral operator on  $L^2(S)$ :

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Take  $S = (\lambda, \infty)$ :

$$\text{Prob } \{\lambda_{\max} < \lambda\} = \frac{\int_{\mathbb{R}} d\beta \ e^{-\frac{\beta^2 N^2}{4\tau}} (\alpha + i\beta)^{-\frac{N^2}{2}} \det(1 - \mathcal{K}_N)}{\int_{\mathbb{R}} d\beta \ e^{-\frac{\beta^2 N^2}{4\tau}} (\alpha + i\beta)^{-\frac{N^2}{2}}} ,$$

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**Probing the extent of universality**

# Universality in Random Matrices:

$$\psi(x) = \lim_{N \rightarrow \infty} \rho_1^{(N)}(x) \quad \text{Is supported on a single interval } (-\beta, \beta)$$

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$$\lim_{N \rightarrow \infty} \frac{1}{(cN)^{2/3}} K_N \left( \beta + \frac{u}{(cN)^{2/3}}, \beta + \frac{v}{(cN)^{2/3}} \right) = \frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}(v)\text{Ai}'(u)}{u - v}. \quad (1)$$

**Asymptotic Result 2:** *For every  $a$  with  $\psi(a) > 0$ , and every  $u, v \in \mathbb{R}$ , we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N\psi(a)} K_N \left( a + \frac{u}{N\psi(a)}, a + \frac{v}{N\psi(a)} \right) = \frac{\sin \pi(u - v)}{\pi(u - v)} \quad (2)$$

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**Asymptotic Result 1:** *There is a constant  $c$  so that for every  $u, v \in \mathbb{R}$ , we have*

$$\lim_{N \rightarrow \infty} \frac{1}{(cN)^{2/3}} K_N \left( \beta + \frac{u}{(cN)^{2/3}}, \beta + \frac{v}{(cN)^{2/3}} \right) = \frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}(v)\text{Ai}'(u)}{u - v}. \quad (1)$$

**Asymptotic Result 2:** *For every  $a$  with  $\psi(a) > 0$ , and every  $u, v \in \mathbb{R}$ , we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N\psi(a)} K_N \left( a + \frac{u}{N\psi(a)}, a + \frac{v}{N\psi(a)} \right) = \frac{\sin \pi(u - v)}{\pi(u - v)} \quad (2)$$

What assumptions on  $V$  would yield  $a$  so that

- $\psi(x) > 0$  in a neighborhood of  $x = a$ ,
- **Asymptotic Result 2** fails to hold true?

What about the Laguerre-Wishart case?

$$\frac{1}{Z_N} \left( 1 + \frac{2n}{\gamma} \sum_{i=1}^N \lambda_i^2 \right)^{-\gamma} \prod_{i=1}^N \lambda_i^\nu \prod_{j>k}^N |\lambda_j - \lambda_k|^2 d^N \lambda$$

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A clever trick!

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A few asymptotic ( $N \rightarrow \infty$ ) results (mostly due to Akemann and Vivo)

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$$\nu = 0$$

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$$\int_0^\infty p_k p_j e^{-\frac{2\xi n N}{\alpha}x} dx = \delta_{jk}$$

$$K_N(x, y; \xi, \alpha) = e^{-\frac{\xi n N}{\alpha}(x+y)} \sum_{\ell=0}^{N-1} p_\ell(x) p_\ell(y)$$

$$\rho_1(x) = \frac{1}{\int_0^\infty d\xi \xi^\alpha e^{-\xi}} \int_0^\infty d\xi \xi^\alpha e^{-\xi} \frac{1}{N} K_N(x, x; \xi, \alpha)$$

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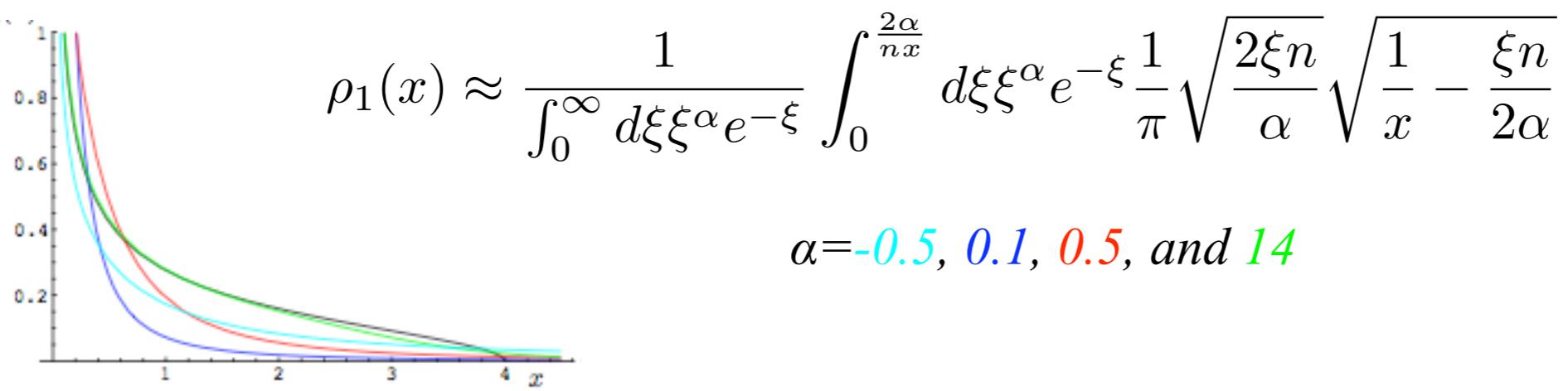
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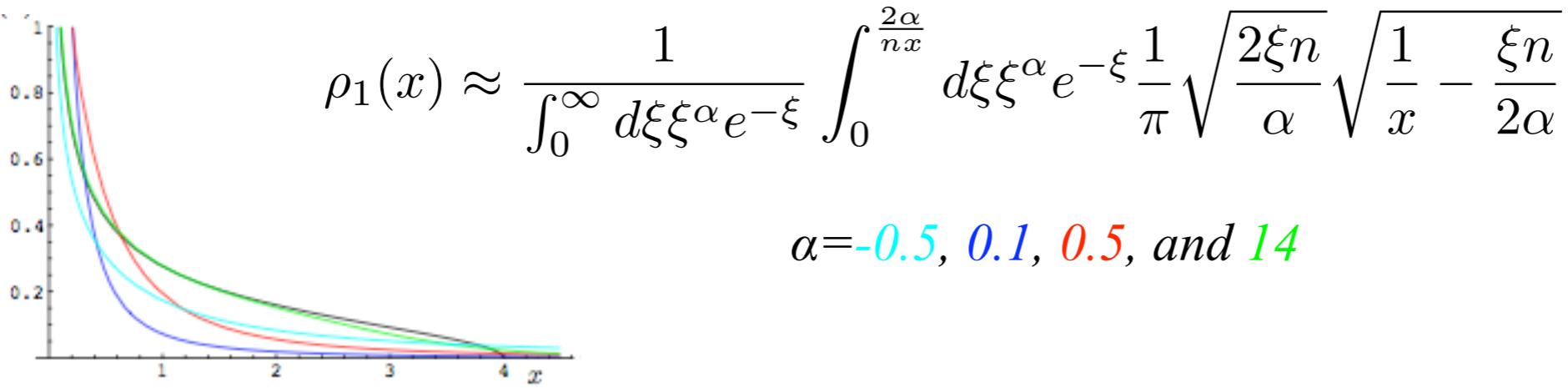
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$$K_N(x, x; \xi, \alpha) \rightarrow \frac{1}{\pi} \sqrt{\frac{2\xi n}{\alpha}} \sqrt{\frac{1}{x} - \frac{\xi n}{2\alpha}}, \quad x \in \left(0, \frac{2\alpha}{\xi n}\right)$$

$$\Rightarrow \rho_1(x) \approx \frac{1}{\int_0^\infty d\xi \xi^\alpha e^{-\xi}} \int_0^{\frac{2\alpha}{nx}} d\xi \xi^\alpha e^{-\xi} \frac{1}{\pi} \sqrt{\frac{2\xi n}{\alpha}} \sqrt{\frac{1}{x} - \frac{\xi n}{2\alpha}}$$

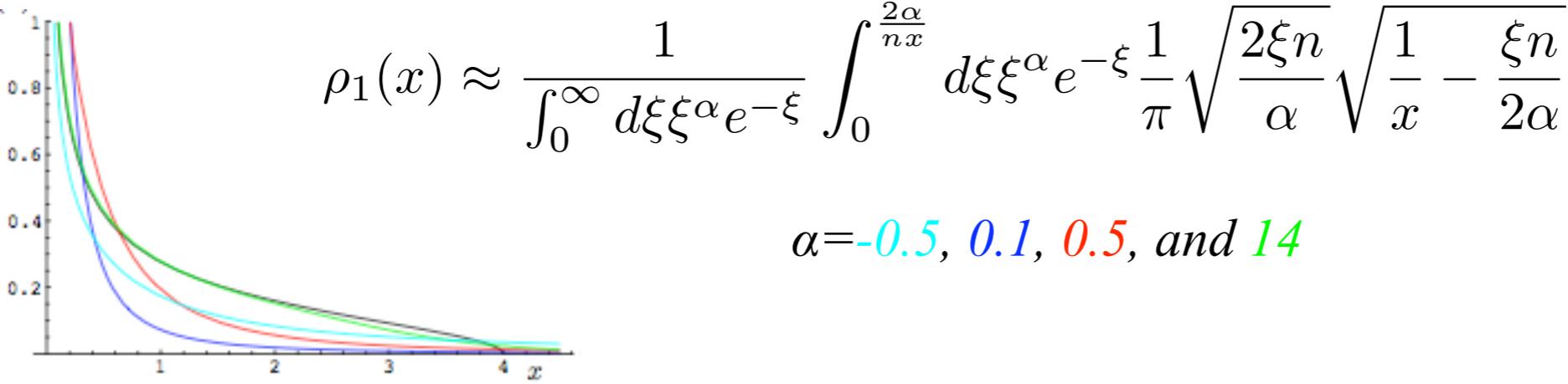




Largest eigenvalue:

$$\text{Prob}\{\lambda_{\max} < \mu\} = \frac{1}{\int_0^\infty d\xi \xi^\alpha e^{-\xi}} \int_0^\infty d\xi \xi^\alpha e^{-\xi} \tilde{F}_N(\mu, \xi, \alpha)$$

$F_N(\mu, \xi, \alpha)$ : largest eigenvalue distribution associated to  $e^{-\frac{2\xi n N}{\alpha} \text{Tr}(X^\dagger X)}$ .



$$\alpha = -0.5, 0.1, 0.5, \text{ and } 14$$

Largest eigenvalue:

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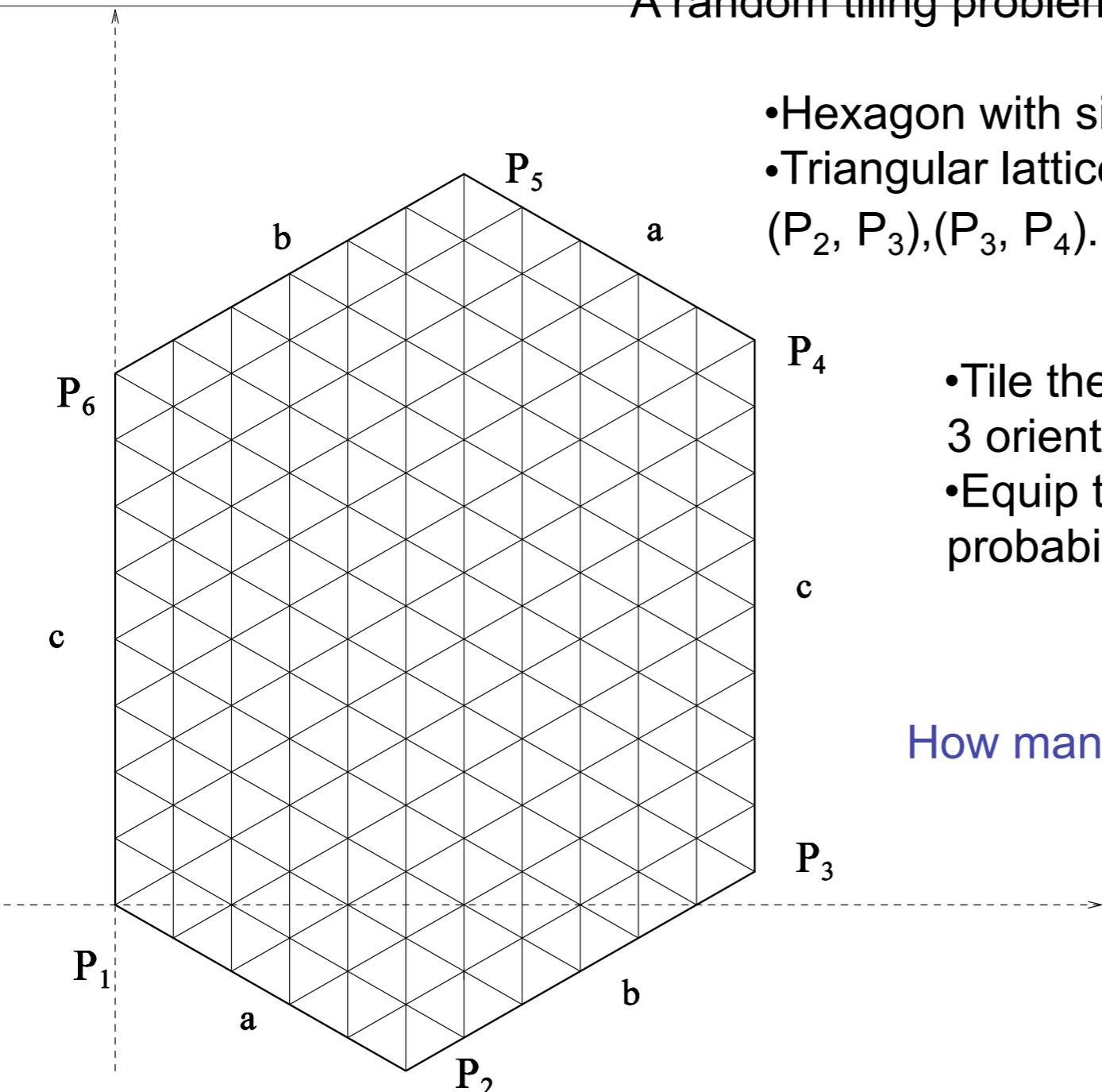
$$F_N(\mu, \xi, \alpha) \rightarrow \begin{cases} 0 & \xi < \frac{2\alpha}{n\mu} \\ 1 & \xi > \frac{2\alpha}{n\mu} \end{cases} \quad \text{as } N \rightarrow \infty$$

$$\implies \text{Prob} \{ \lambda_{\max} < \mu \} \approx \frac{1}{\int_0^\infty d\xi \xi^\alpha e^{-\xi}} \int_{\frac{2\alpha}{n\mu}}^\infty d\xi \xi^\alpha e^{-\xi}$$

Random Tiling Problems:

Emergence of curves  
From the mists of randomness.

## A random tiling problem.

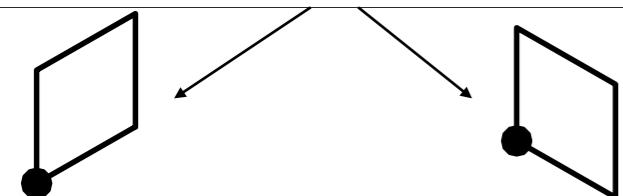


- Hexagon with sides of length  $a, b, c, a, b, c$ . ( $a, b, c$  integers).
- Triangular lattice  $\mathcal{L}$ , which includes  $(P_6, P_1), (P_1, P_2)$ ,  $(P_2, P_3), (P_3, P_4)$ .

- Tile the  $abc$ -hexagon with rhombi, which come in 3 orientations.
- Equip the set of all rhombus tilings with uniform probability.

How many tilings are there?

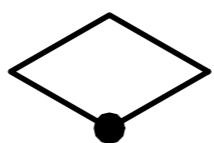
Horizontal rhombi



Type I

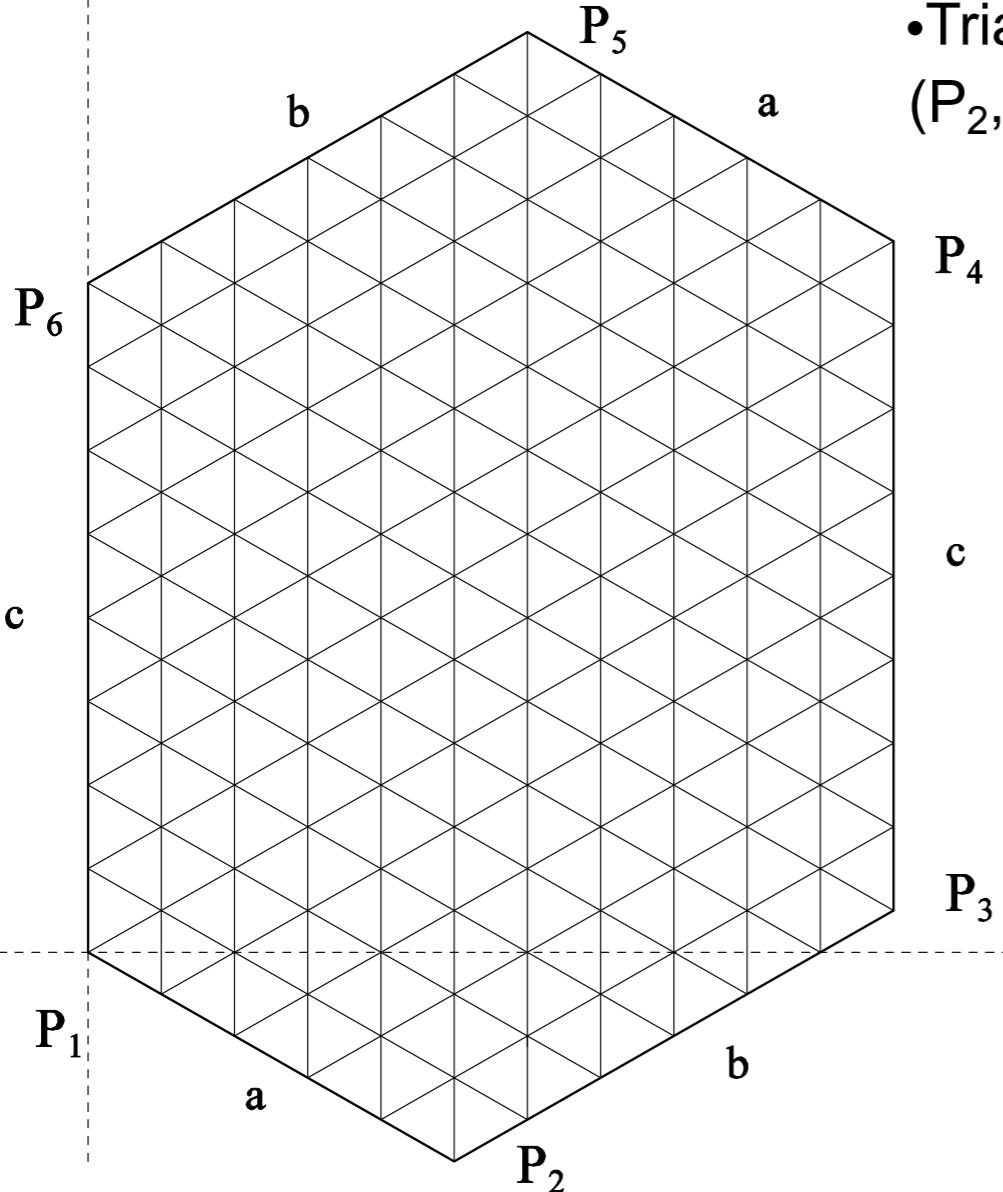
Type II

Vertical rhombi



Type III

## A random tiling problem.



- Hexagon with sides of length  $a, b, c, a, b, c$ . ( $a, b, c$  integers).
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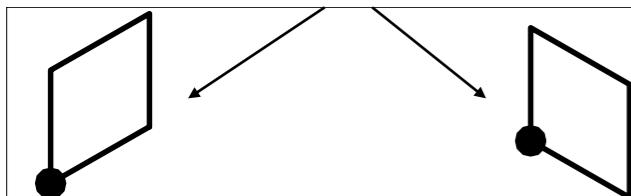
- Tile the  $abc$ -hexagon with rhombi, which come in 3 orientations.
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How many tilings are there?

$$\prod_{i=1}^a \prod_{j=1}^b \prod_{k=1}^c \frac{i+j+k-1}{i+j+k-2}.$$

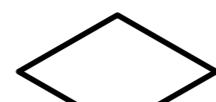
Interest: limiting statistics of tilings, when size of hexagon goes to infinity.

Horizontal rhombi



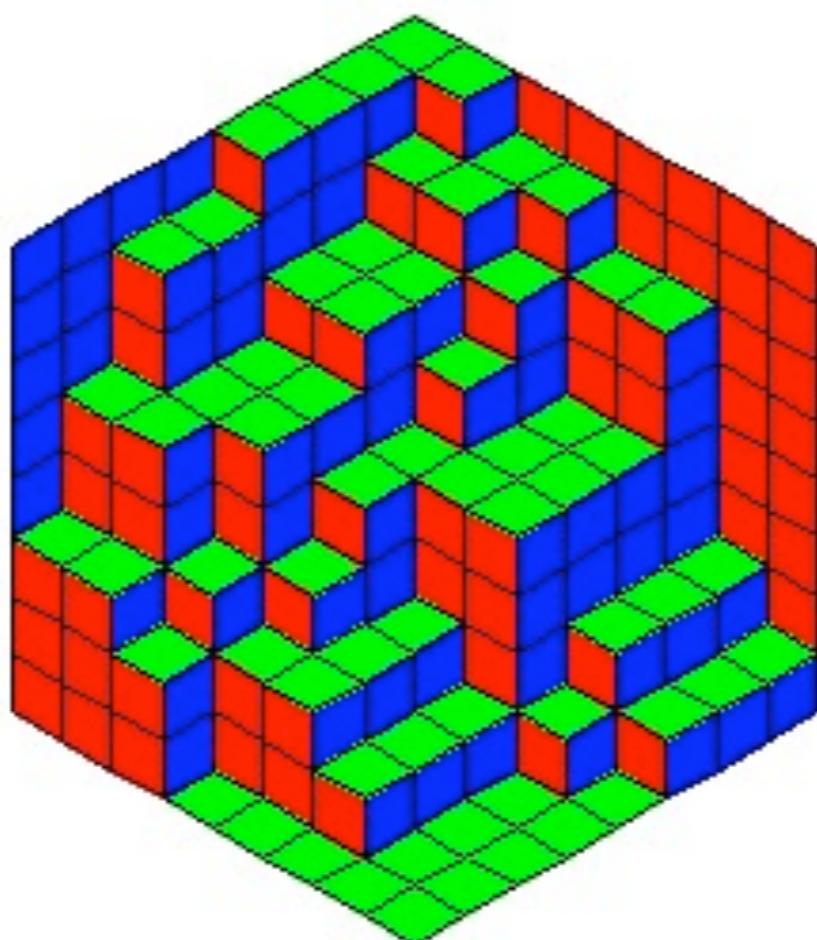
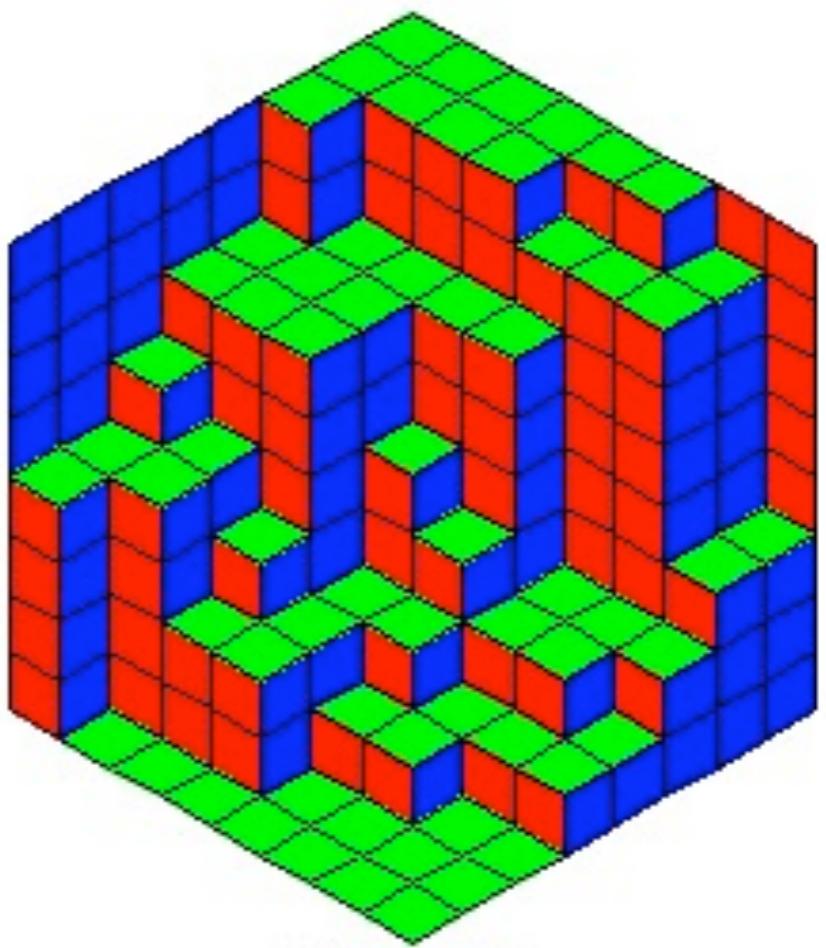
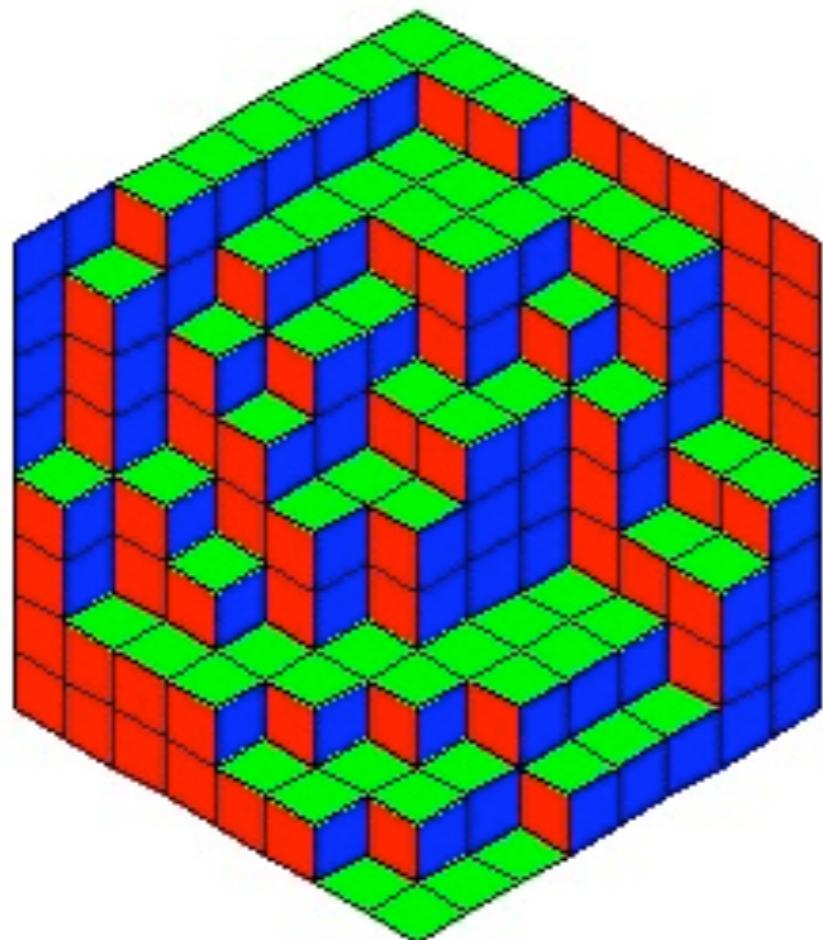
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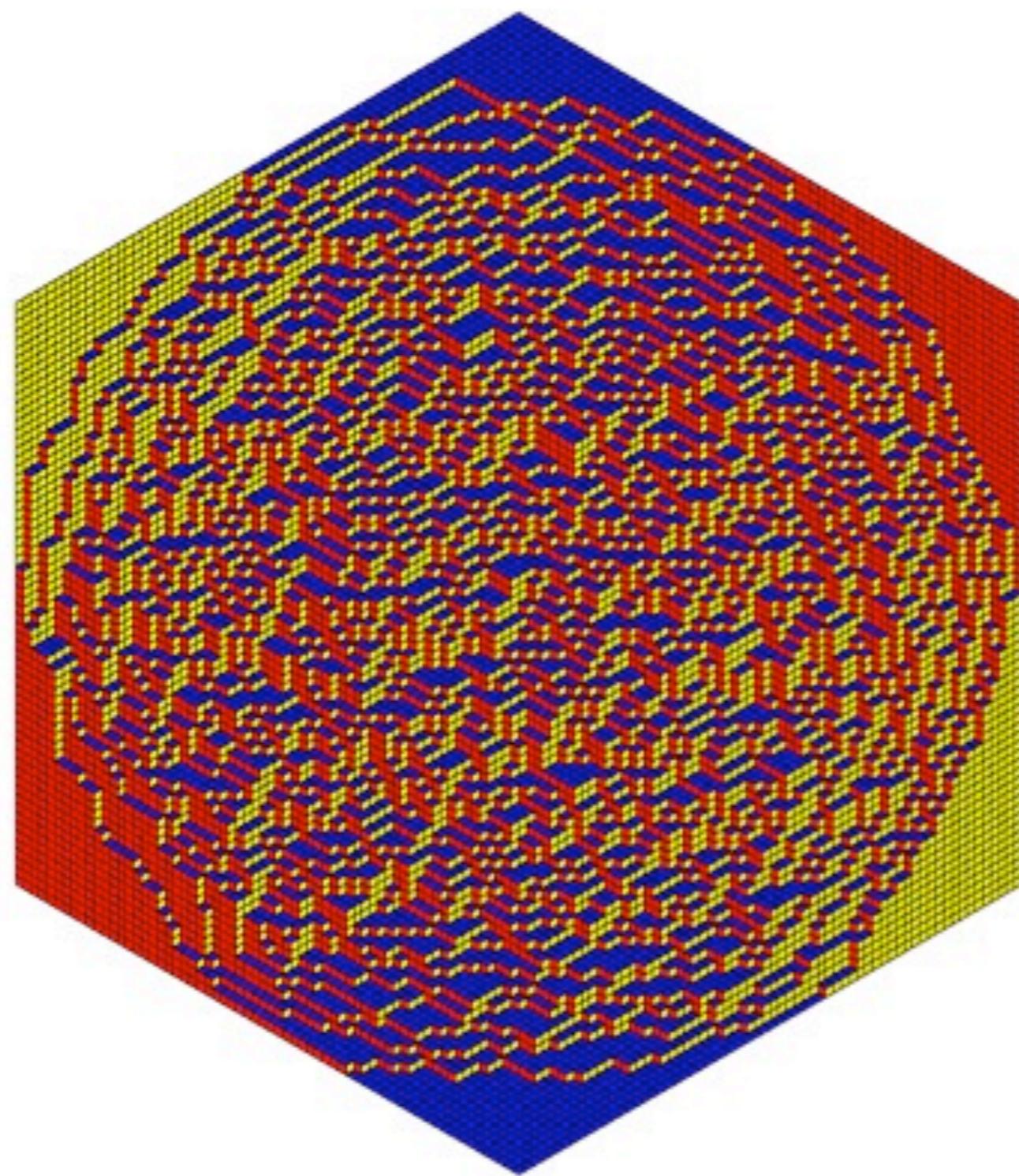
Vertical rhombi



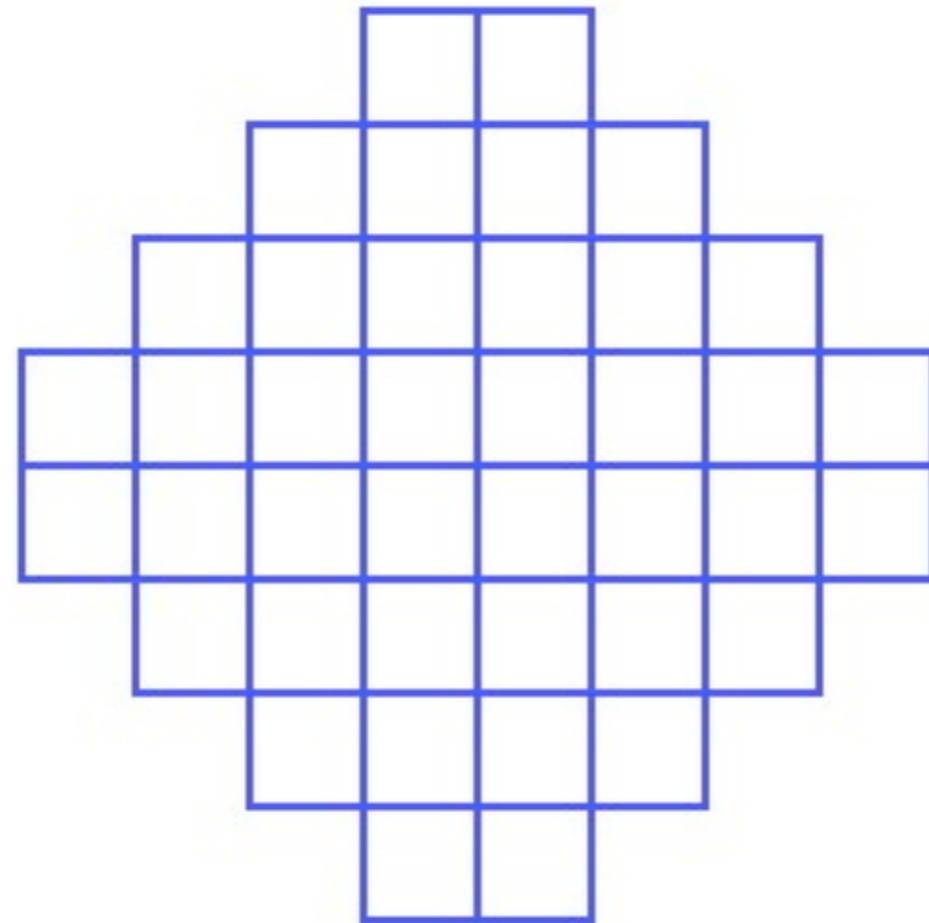
Type III

$a = \alpha n, b = \beta n, c = \gamma n$ , with fixed  $\alpha, \beta, \gamma$ , and  $n \rightarrow \infty$

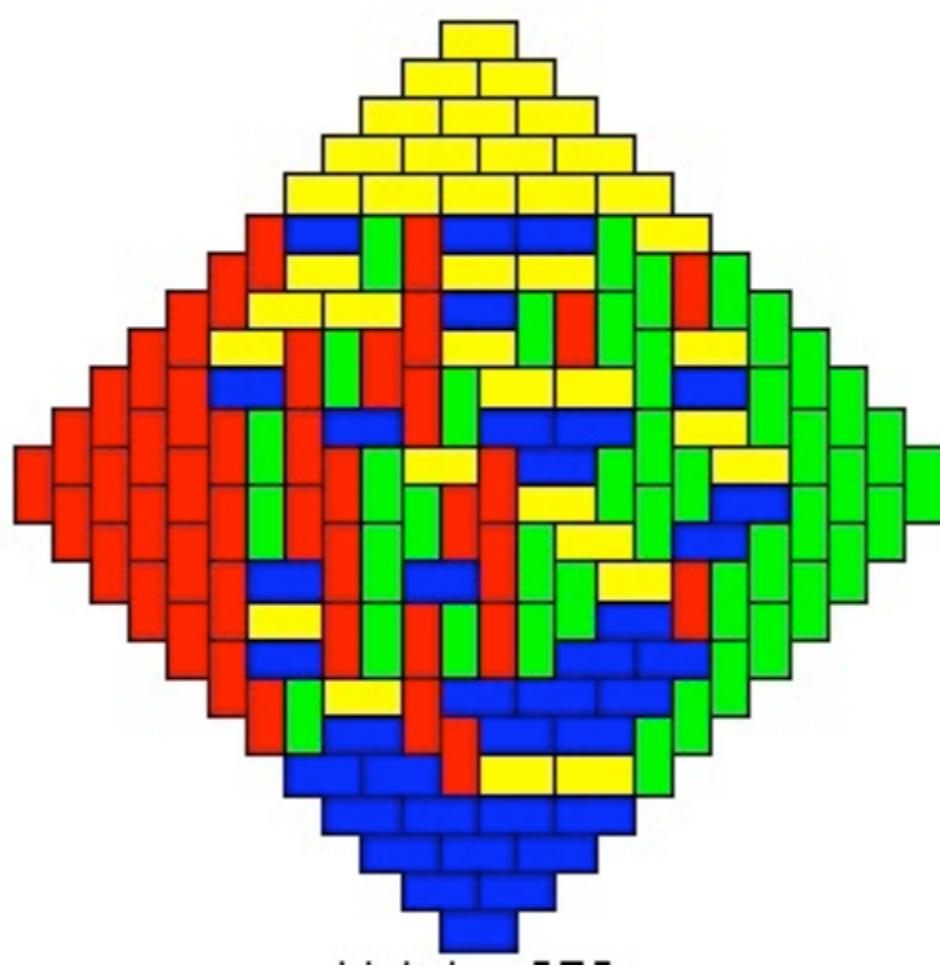
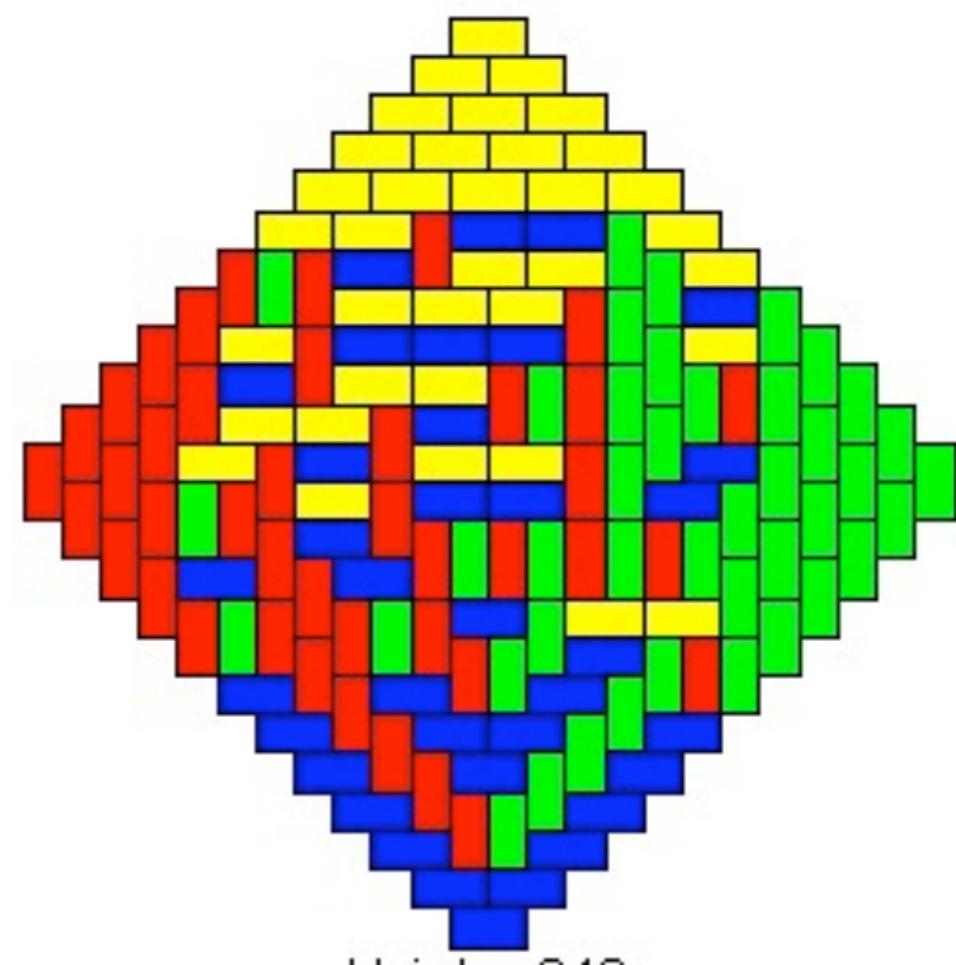
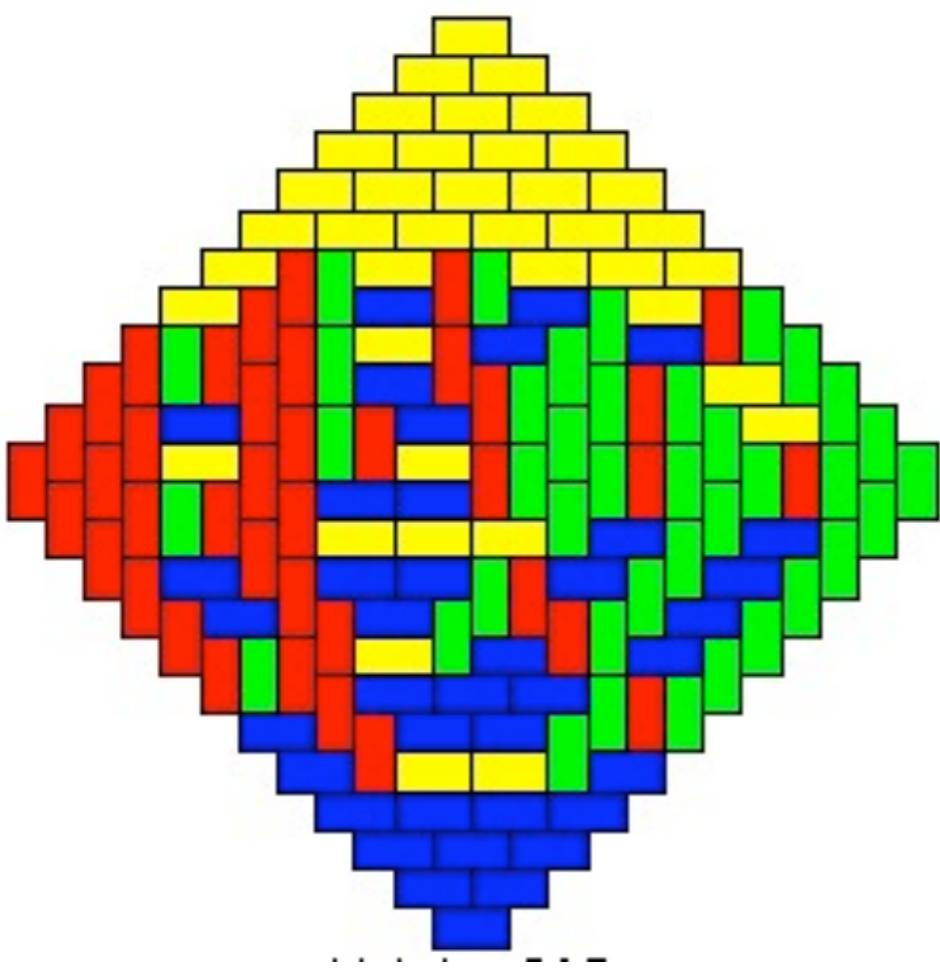


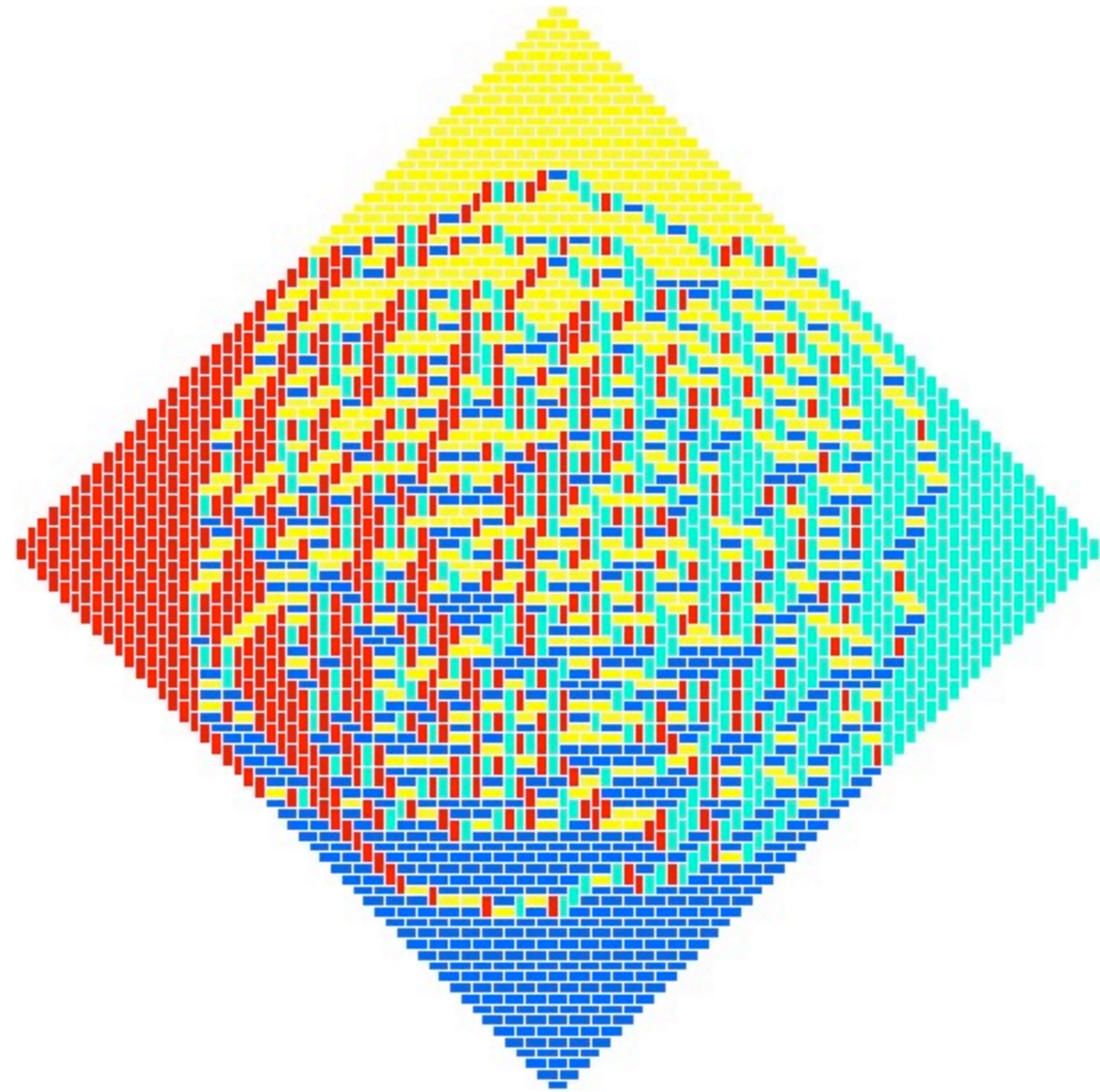


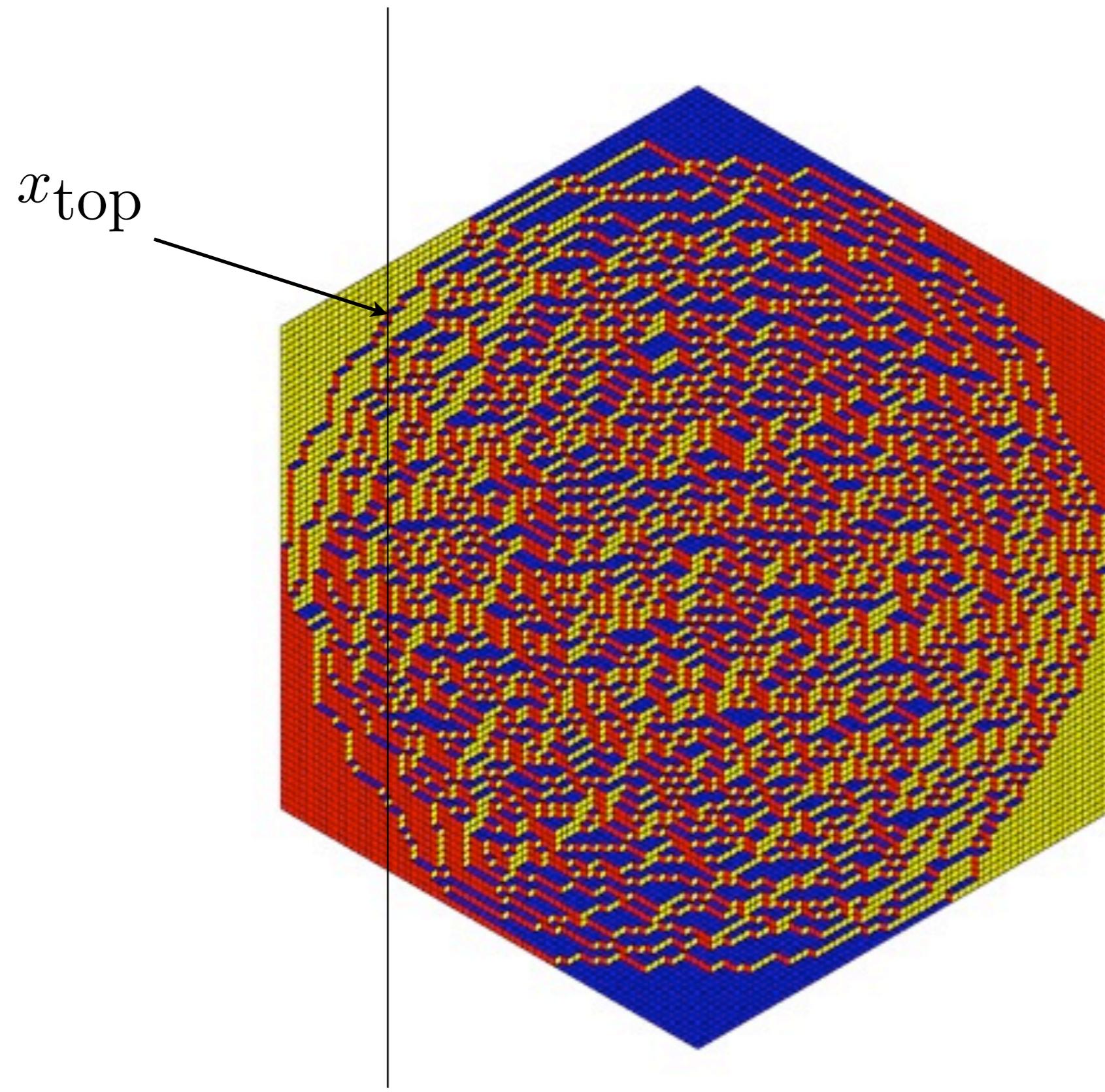
The Aztec Diamond of order  $n$  is the region given by  $|x| + |y| \leq n + 1$

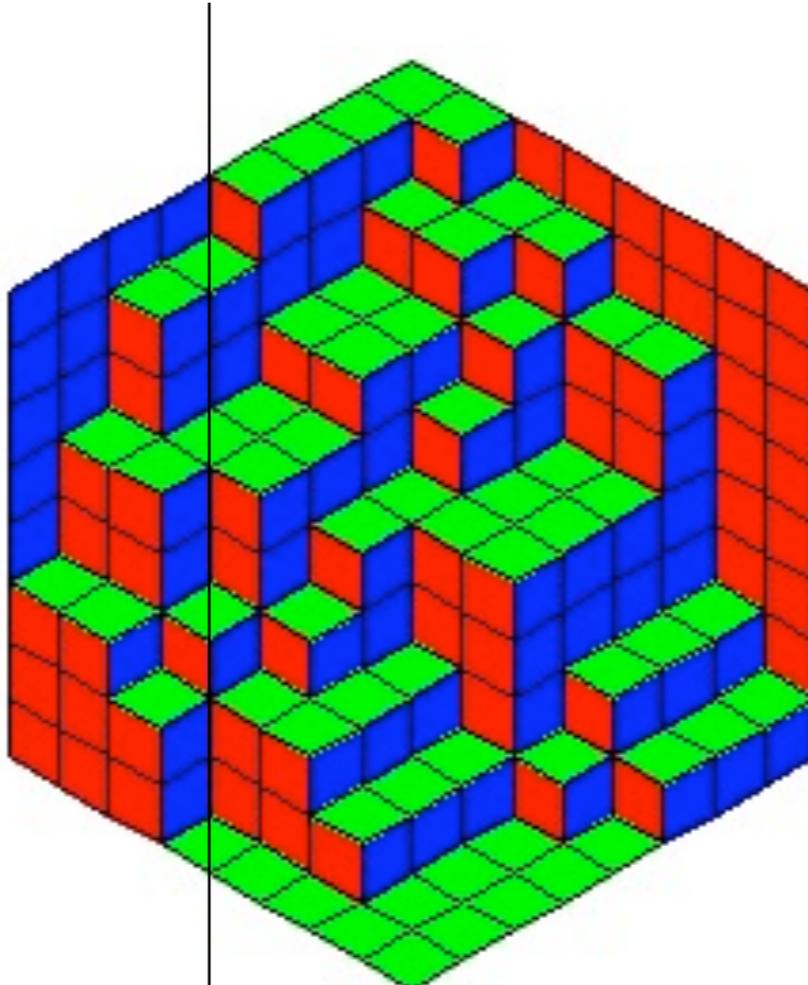
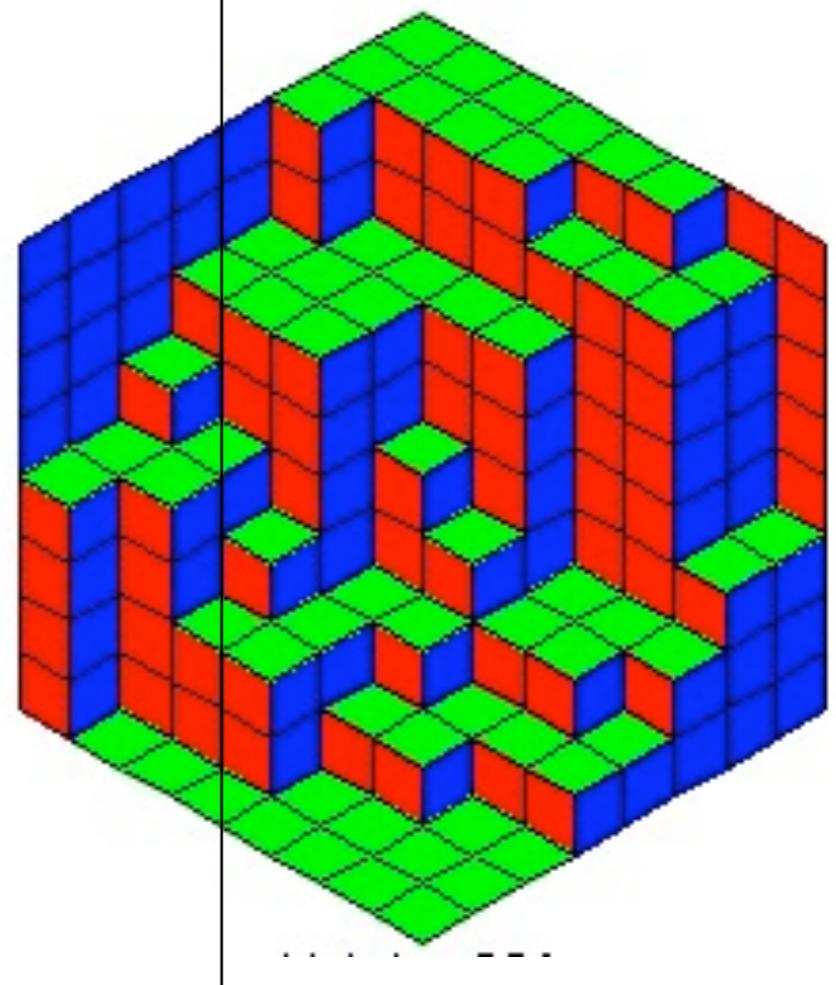
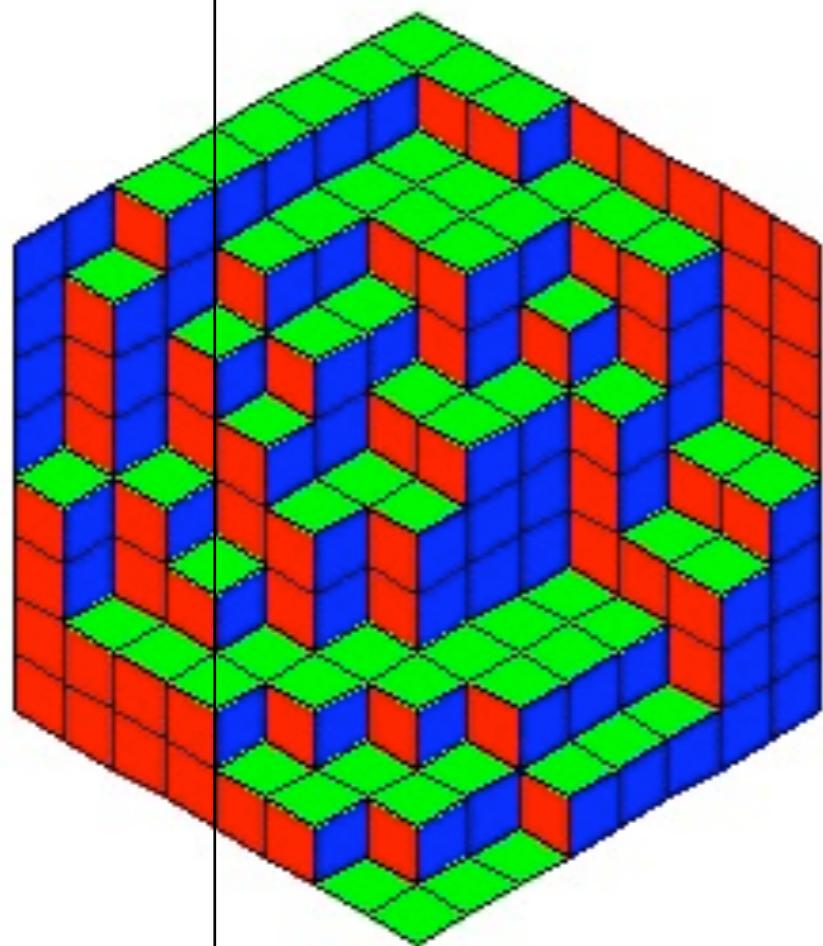


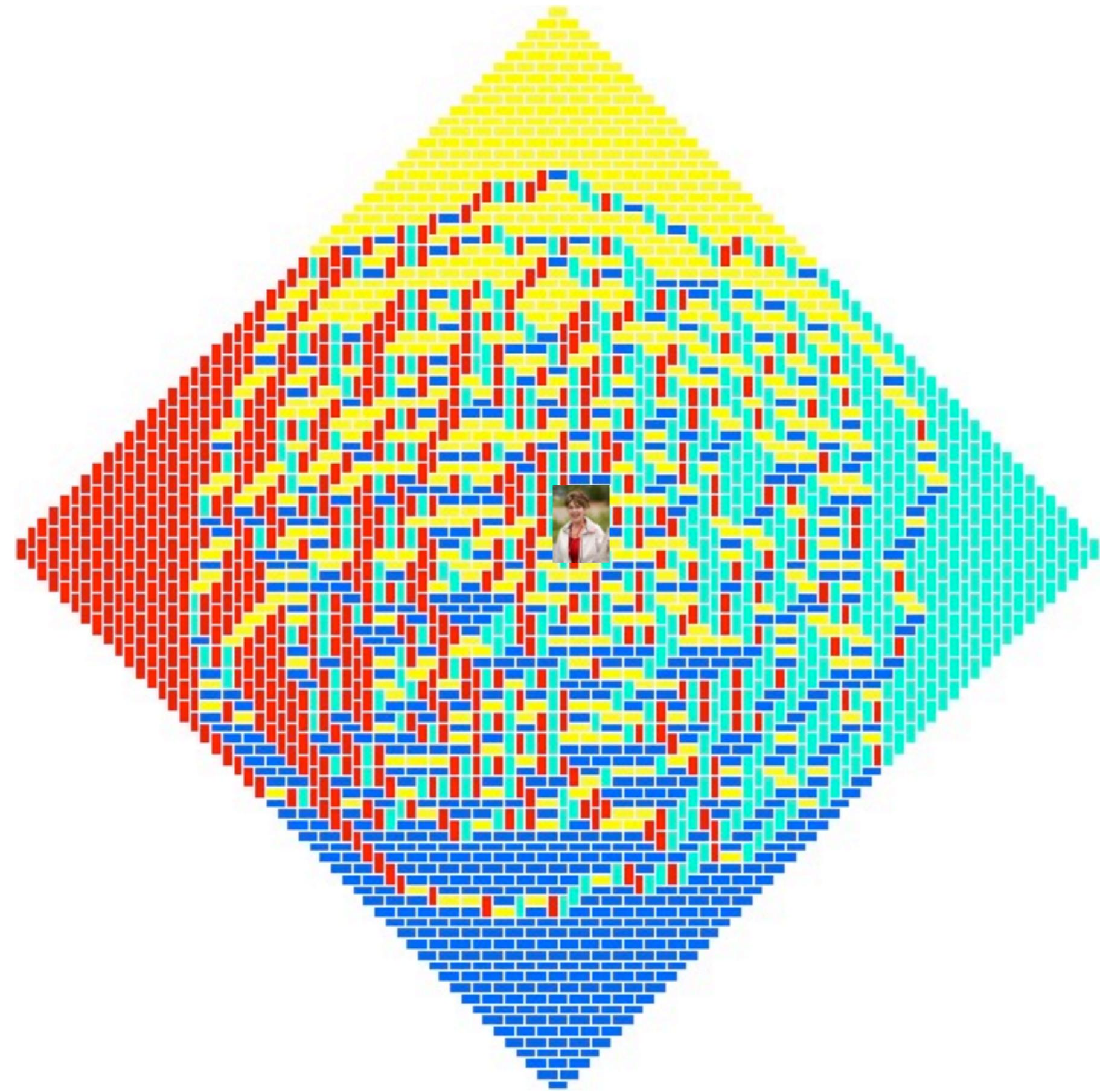
Tile this region by “dominoes” -  $2 \times 1$  rectangles.

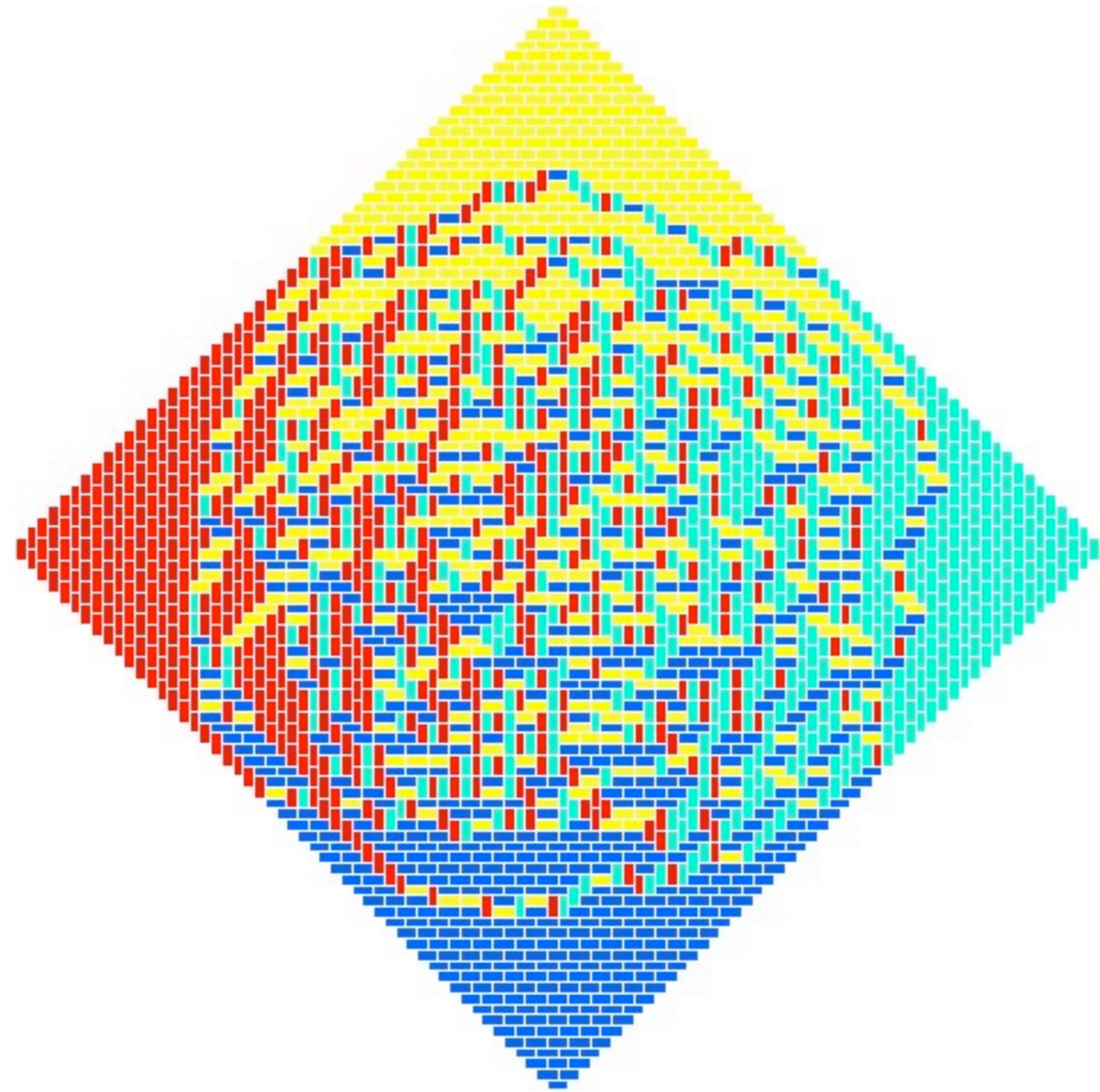












Baik, Kriecherbauer, McL, and Miller:

$$\lim_{n \rightarrow \infty} \text{Prob} \left\{ c(x_{top} - \beta)n^{2/3} \leq s \right\} = F_{TW}(s) .$$