

# **NUMERICAL SIMULATIONS OF OCEAN WAVES**

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# MOTIVATIONS

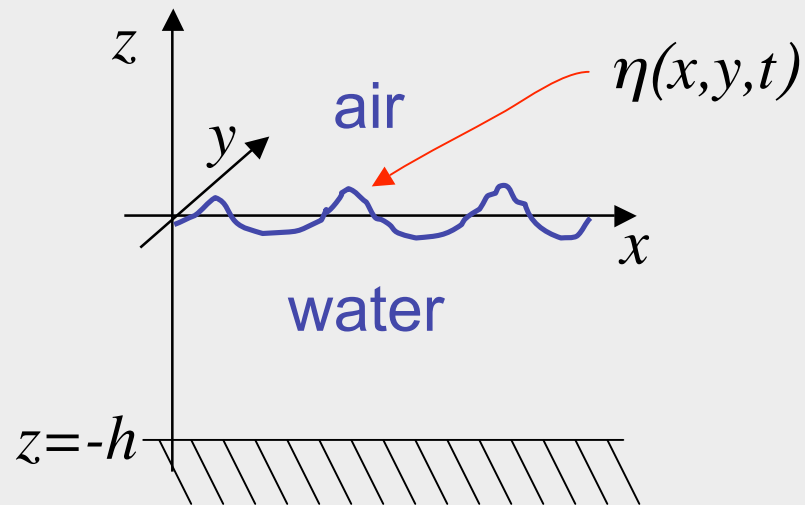
- **Verify some issues related to some theoretical prediction for wave spectra**
- **Understand the statistical properties of the surface elevations and the formation of extreme waves**

# OUTLINE OF THE PRESENTATION

- **Introduce the equations for water waves and the Hamiltonian formulation**
- **Discuss the model used for numerical simulations**
- **Applications**
- **Conclusions**

# WATER WAVES PROBLEM:

two fluids (air and water) separated by an interface



## Hypothesis:

- incompressible
- inviscid
- irrotational
- flat bottom

$$\begin{cases} \nabla^2 \phi = 0 & -h < z < \eta(x, y, t) \\ \phi_t + \frac{1}{2} |\nabla \phi|^2 + g\eta = 0 & z = \eta(x, y, t) \\ \eta_t + \nabla \phi \cdot \nabla \eta - \phi_z = 0 & z = \eta(x, y, t) \\ \phi_z = 0 & z = -h \end{cases}$$

# HAMILTONIAN FORMULATION

(Zakharov 1968)

Introduce the potential at the surface

$$\psi(\mathbf{x}, t) = \phi(\mathbf{x}, z = \eta, t)$$

Then  $\eta$  and  $\psi$  are canonically conjugated variables

$$\frac{\partial \eta(\mathbf{x}, t)}{\partial t} = \frac{\delta H}{\delta \psi(\mathbf{x}, t)}$$

$$\frac{\partial \psi(\mathbf{x}, t)}{\partial t} = - \frac{\delta H}{\delta \eta(\mathbf{x}, t)}$$

With

$$H = \frac{1}{2} \int \int_{-h}^{\eta} \left[ (\nabla \phi)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] d\mathbf{x} dz + \frac{1}{2} g \int \eta^2 d\mathbf{x}$$

## INTRODUCTION OF THE COMPLEX VARIABLE

$$a(\mathbf{k}, t) = \sqrt{\frac{g}{2\omega_k}} \eta(\mathbf{k}, t) + i \sqrt{\frac{\omega_k}{2g}} \psi(\mathbf{k}, t)$$

New pair of canonical variables  $a(\mathbf{k}, t)$ ,  $ia^*(\mathbf{k}, t)$

$$i \frac{\partial a(\mathbf{k}, t)}{\partial t} = \frac{\delta H}{\delta a^*(\mathbf{k}, t)}$$

## WEAK NONLINEARITY!!

Expansion of the Hamiltonian in powers of  $a$  and  $a^*$

Truncation at third order in nonlinearity  
(four wave interactions)

$$H = H_o + H_1 + H_2 + \dots$$

$$H_o = \int |a_k|^2 dk$$

$$H_1 \sim \int U_{0,1,2} (a_0^* a_1 a_2 + a_0 a_1^* a_2^*) \delta(k_0 - k_1 - k_2) dk_{0,1,2}$$

$$H_2 \sim \int W_{0,1,2,3} (a_0^* a_1^* a_2 a_3 + a_0 a_1 a_2^* a_3^*) \delta(k_0 + k_1 - k_2 - k_3) dk_{0,1,2,3}$$

# EVOLUTION EQUATIONS FOR THE ORIGINAL VARIABLES

$$\frac{\partial \eta_0}{\partial t} - |k_0| \psi_0 = \int U_{0,1,2} \psi_1 \eta_2 \delta(k_0 - k_1 - k_2) dk_{1,2} + \int W_{0,1,2,3} \psi_1 \eta_2 \eta_3 \delta(k_0 + k_1 - k_2 - k_3) dk_{1,2,3}$$

$$\frac{\partial \psi_0}{\partial t} + g \eta_0 = \int V_{0,1,2} \psi_1 \psi_2 \delta(k_0 - k_1 - k_2) dk_{1,2} + \int Z_{0,1,2,3} \psi_1 \psi_2 \eta_3 \delta(k_0 + k_1 - k_2 - k_3) dk_{1,2,3}$$

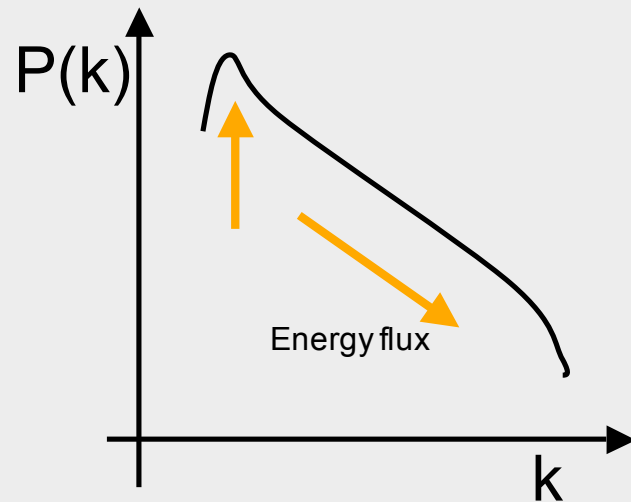
$$U_{0,1,2} = \mathbf{k}_0 \cdot \mathbf{k}_1 - |\mathbf{k}_0| |\mathbf{k}_1|$$

$$W_{0,1,2,3} = -\frac{1}{2} |\mathbf{k}_0| |\mathbf{k}_1| (|\mathbf{k}_0| + |\mathbf{k}_1| - |\mathbf{k}_1 + \mathbf{k}_3| - |\mathbf{k}_1 + \mathbf{k}_2|)$$

**CONVOLUTION INTEGRALS CAN BE COMPUTED USING THE FFT ALGORITHM!!!**

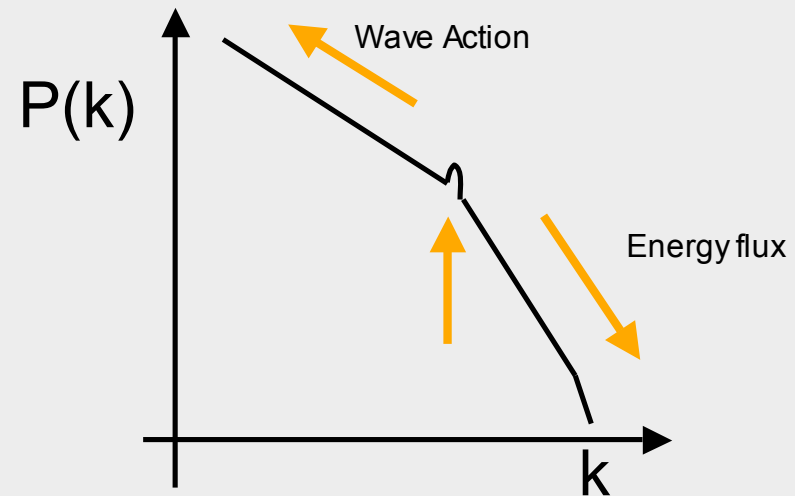
# CONSTANT FLUXES IN WAVE SPECTRUM

3D turbulence:  
direct cascade (Kolmogorov 41)



constant flux of energy

Surface gravity waves:  
double cascade (Zakharov 1966)



constant flux of energy  
and wave action

# STATISTICAL DESCRIPTION OF THE SURFACE ELEVATION

The goal is to write an evolution equation for the Wave Action Spectrum  $P(\mathbf{k}, t)$ :

$$\langle a(\mathbf{k}_1, t) a^*(\mathbf{k}_2, t) \rangle = N(\mathbf{k}_1, t) \delta(\mathbf{k}_1 - \mathbf{k}_2)$$

Related to the Wave Spectrum:

$$P(\mathbf{k}, t) = N(\mathbf{k}, t) \omega(\mathbf{k})$$

Quasi-Gaussian approximation

$$\langle a_1^* a_2^* a_3 a_4 \rangle = \langle a_1^* a_2^* \rangle \langle a_3 a_4 \rangle + \langle a_1^* a_3 \rangle \langle a_2^* a_4 \rangle + \langle a_1^* a_4 \rangle \langle a_2^* a_3 \rangle + D_{1,2,3,4}$$

## RESULT: the wave kinetic equation

$$\frac{\partial N_1}{\partial t} = J(\mathbf{k}_1)$$

$J(\mathbf{k})$  is the collision integral

$$J(\mathbf{k}_1) = \int |T_{1,2,3,4}|^2 N_1 N_2 N_3 N_4 \left( \frac{1}{N_1} + \frac{1}{N_2} - \frac{1}{N_3} - \frac{1}{N_4} \right) \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) d\mathbf{k}_{2,3,4}$$

$T_{1,2,3,4}$  is the scattering matrix

Resonant condition

$$\begin{cases} \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 \\ \omega_1 + \omega_2 = \omega_3 + \omega_4 \end{cases}$$

Collision invariants:

$$E = \int \omega_{\mathbf{k}} N_{\mathbf{k}} d\mathbf{k}$$

Total energy

$$\mathbf{M} = \int \mathbf{k} N_{\mathbf{k}} d\mathbf{k}$$

Total momentum

$$N = \int N_{\mathbf{k}} d\mathbf{k}$$

Total wave action

# STATIONARY SOLUTIONS OF THE HOMOGENEOUS WAVE KINETIC EQUATIONS

Thermodynamic solutions:

$$N_{\mathbf{k}} = \frac{1}{a + b\omega_{\mathbf{k}} + \mathbf{c} \cdot \mathbf{k}}$$

$a = 0 \rightarrow$  equipartition of energy

$b = 0 \rightarrow$  equipartition of wave action

solutions are relevant for a close system  
(without forcing or dissipation)

For ocean waves such solutions are never observed!!

# STATIONARY SOLUTIONS OF THE HOMOGENEOUS WAVE KINETIC EQUATIONS

Non equilibrium solutions (Kolmogorov-Zakharov spectra):

Additional hypothesis:

$\omega(\mathbf{k}), T_{1,2,3,4}$  invariant respect rotations and homogeneous functions of their arguments, i.e.

$$\begin{cases} \omega(\mathbf{k}) = Ak^\alpha \\ T(\lambda\mathbf{k}_1, \lambda\mathbf{k}_2, \lambda\mathbf{k}_3, \lambda\mathbf{k}_4) = \lambda^\beta T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \end{cases}$$

$$N_{\mathbf{k}} \sim k^{-\nu} \quad \nu = 2 + \frac{2\beta}{3} \quad \text{direct energy cascade}$$

$$N_{\mathbf{k}} \sim k^{-\nu} \quad \nu = 2 + \frac{2\beta - \alpha}{3} \quad \text{inverse wave action cascade}$$

**DIRAC MEDAL 2003**



# EVIDENCE OF POWER LAW FOR WATER WAVES:

$$P(k) \sim k^{-2.5}$$

First experimental evidence:

Y. Toba, J. Ocean Soc., Jpn. 1973

Numerical evidence from numerical computation:

Freely decaying (direct cascade)

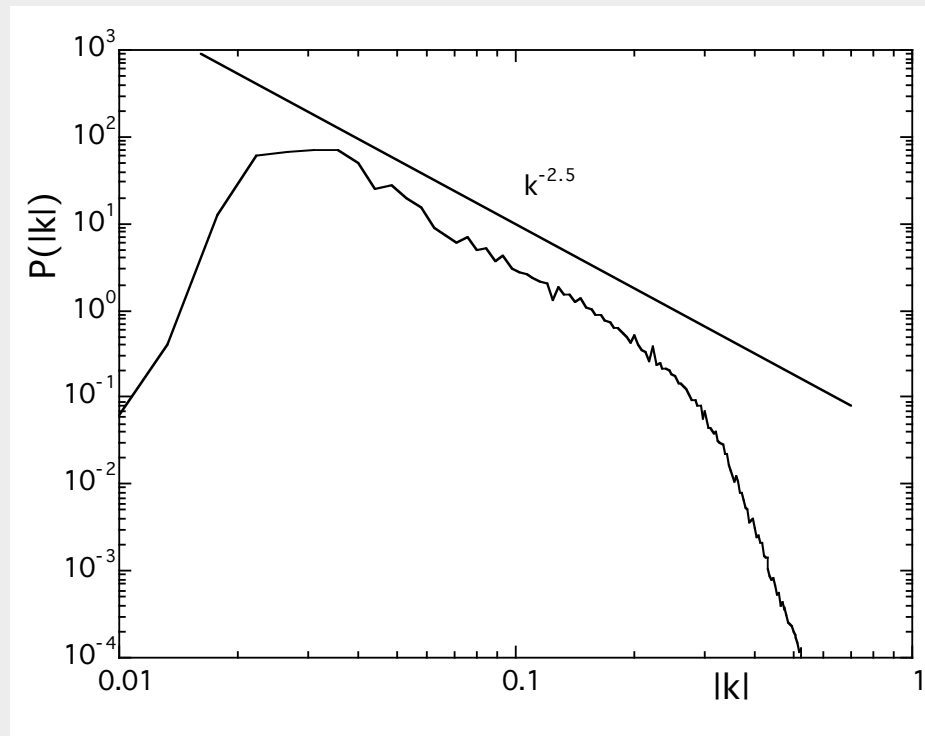
M.O. et al. Phys Rev. Lett. 2001

Forced (direct and inverse cascade)

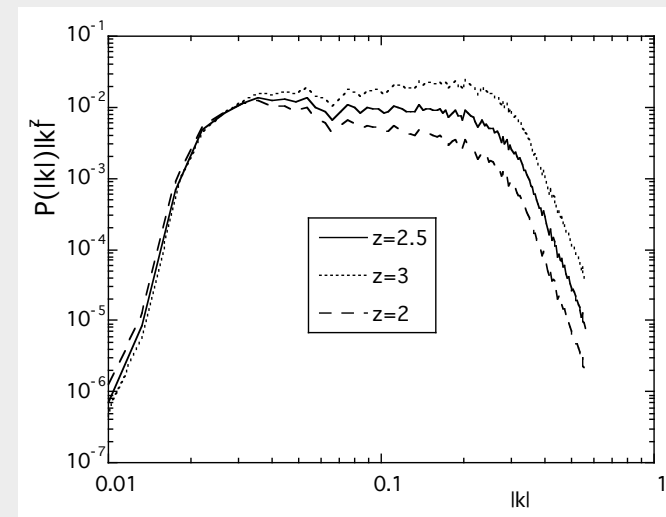
A. Korotkevich Phys Rev. Lett. 2008

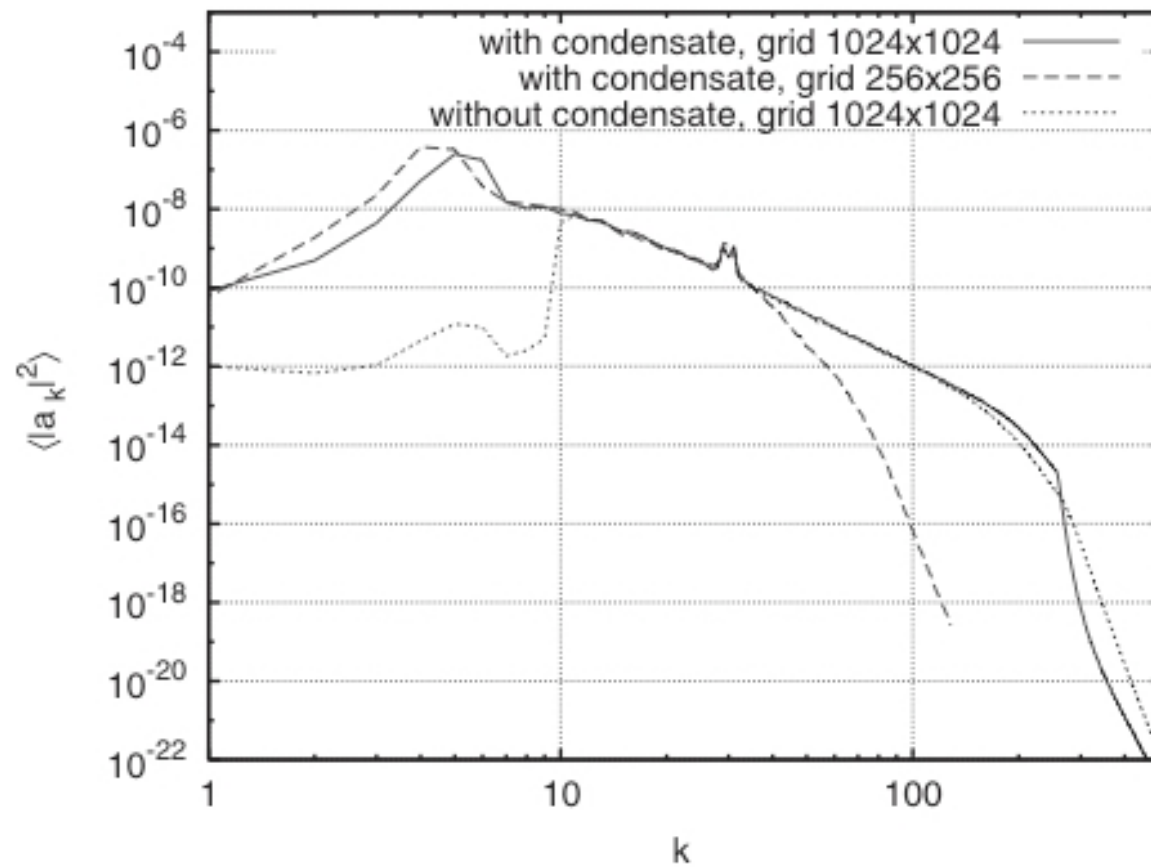
# NUMERICAL SIMULATION OF TRUNCATED EULER EQUATIONS

## WAVE SPECTRUM



## COMPENSATED WAVE SPECTRUM





From A. Korotkevich PRL 2008

# WAVE FORECASTING

**What we can predict today:**

- i) Significant wave height ( $H_s = 4 \sigma$ )**
- ii) Dominant period**
- iii) Dominant direction of propagation of the waves**

**These information are included in the wave spectrum**

## THE WAM (WAVE Model)

**Model developed in Reading, U.K. (ECMWF), operational from 1992**

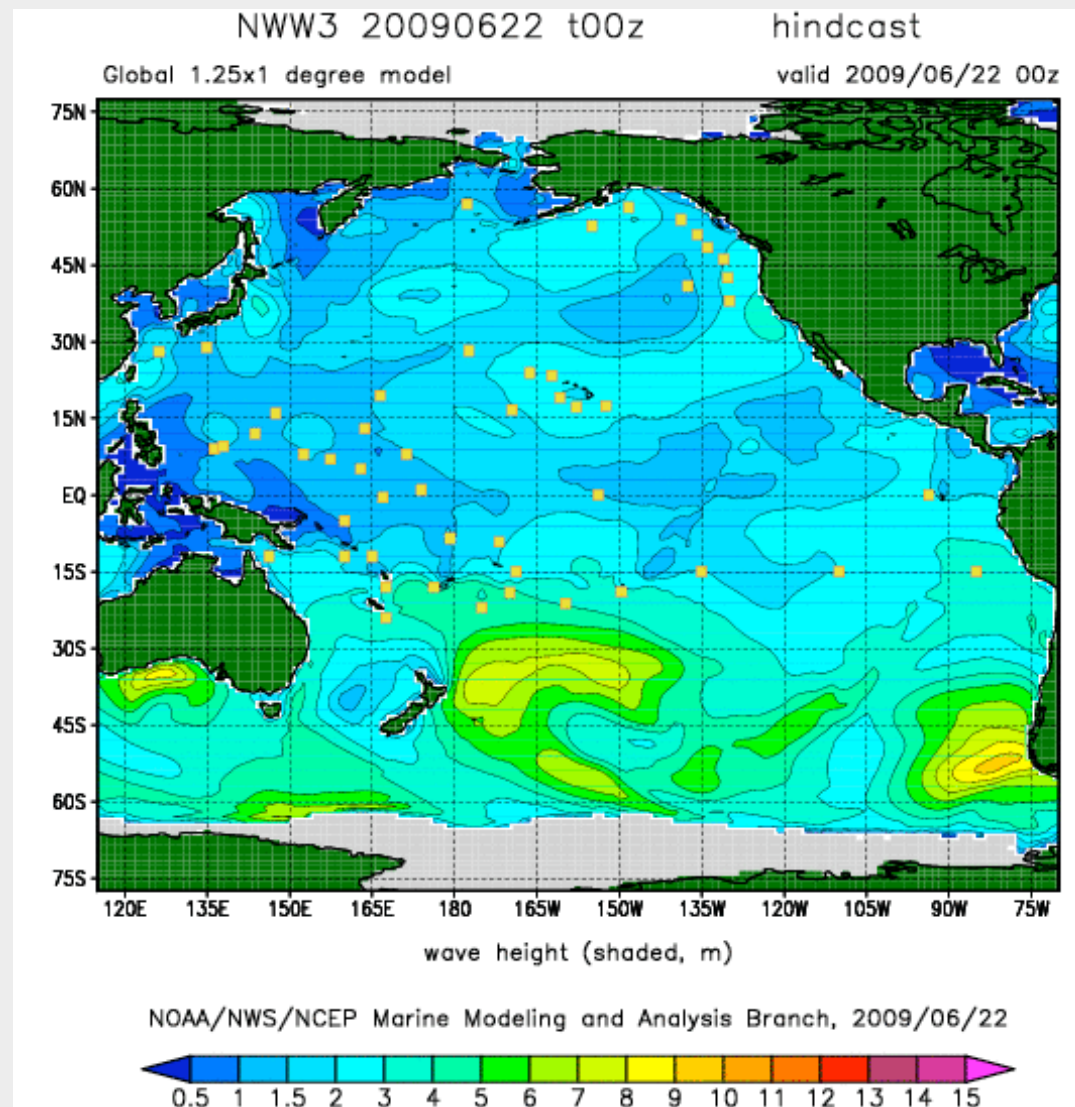
**It is a global model coupled with the atmospheric model.**

**24 directions and 30 frequencies,  
resolution: 40 km**

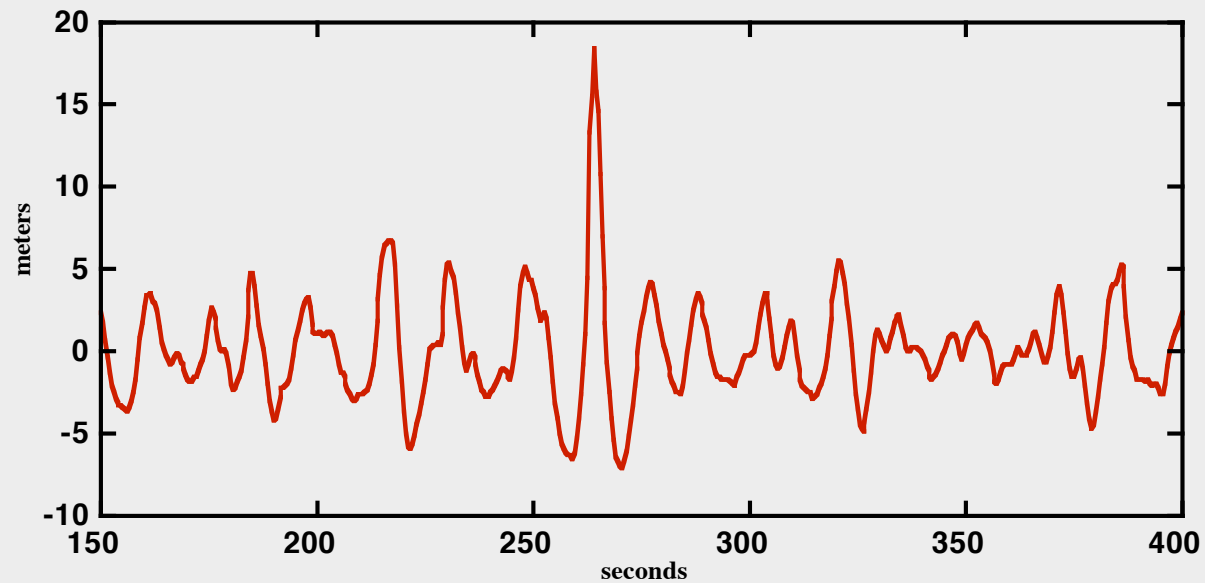
**Similar model was developed by NOAA (USA) some years later**

$$\frac{\partial N_1}{\partial t} + \mathbf{C}_1 \cdot \nabla N_1 = J(\mathbf{k}_1) + S_{wind} + S_{diss}$$

# WAVE HEIGHT



# Time series from Draupner Stat-Oil Platform (North Sea) January, 1 - 1995



# **SOME MECHANISMS OF FORMATION OF EXTREME WAVES**

- **Linear superposition**
- **Nonlinear Focusing (modulational instability)**
- **Crossing Seas in deep water**
- **Crossing seas in shallow water**

# NONLINEAR FOCUSING (MODULATIONAL INSTABILITY)

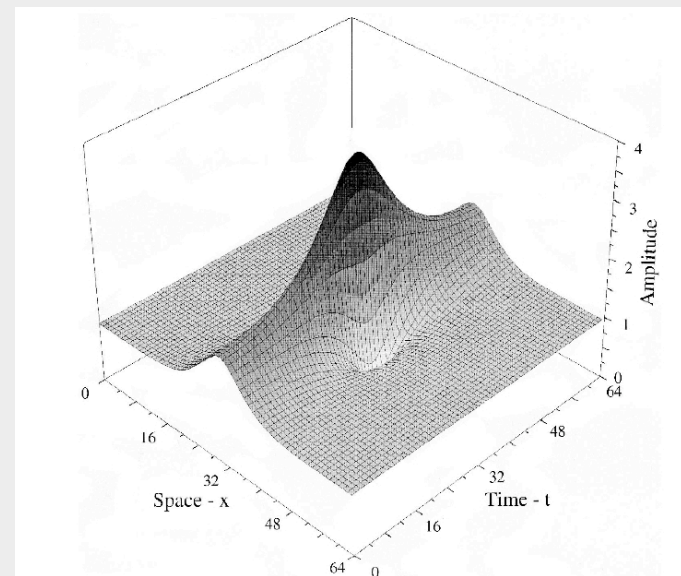
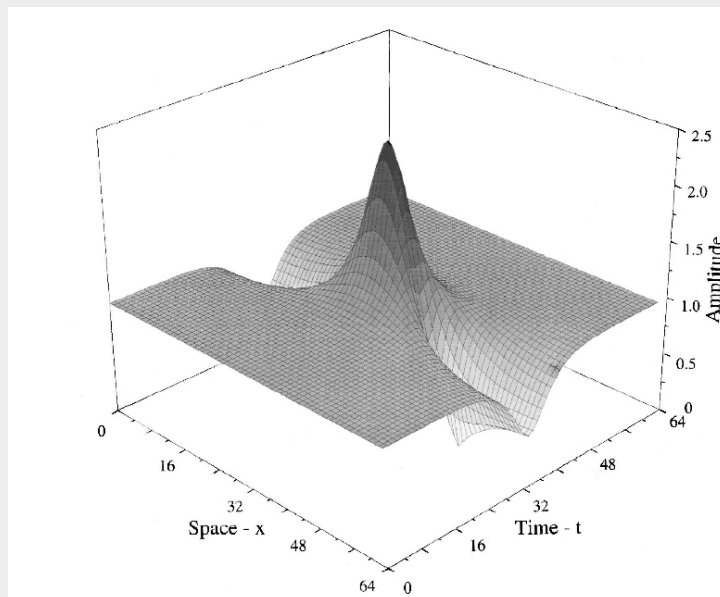
## Hypothesis:

- Waves propagate in one direction
- Narrow band approximation
- Weakly nonlinearity

## NLS

Focusing for  $kh > 1.36$

Defocusing for  $kh \leq 1.36$

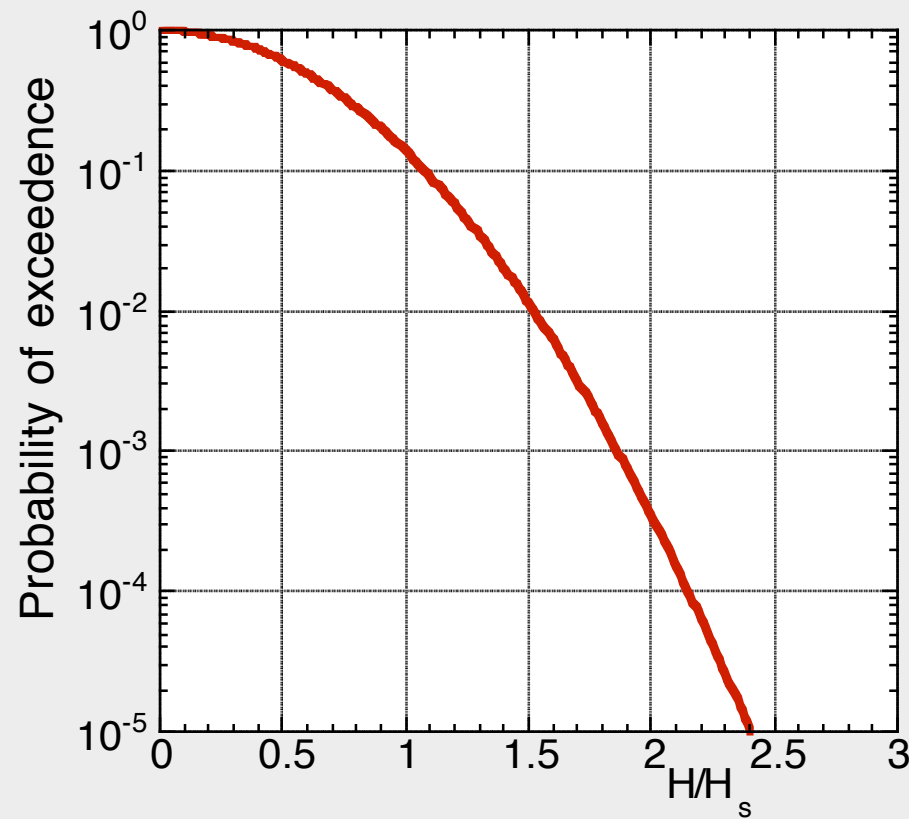


**Are these solutions of any relevance for real ocean waves?**

**Do these solutions change the statistical properties of the surface elevation?**

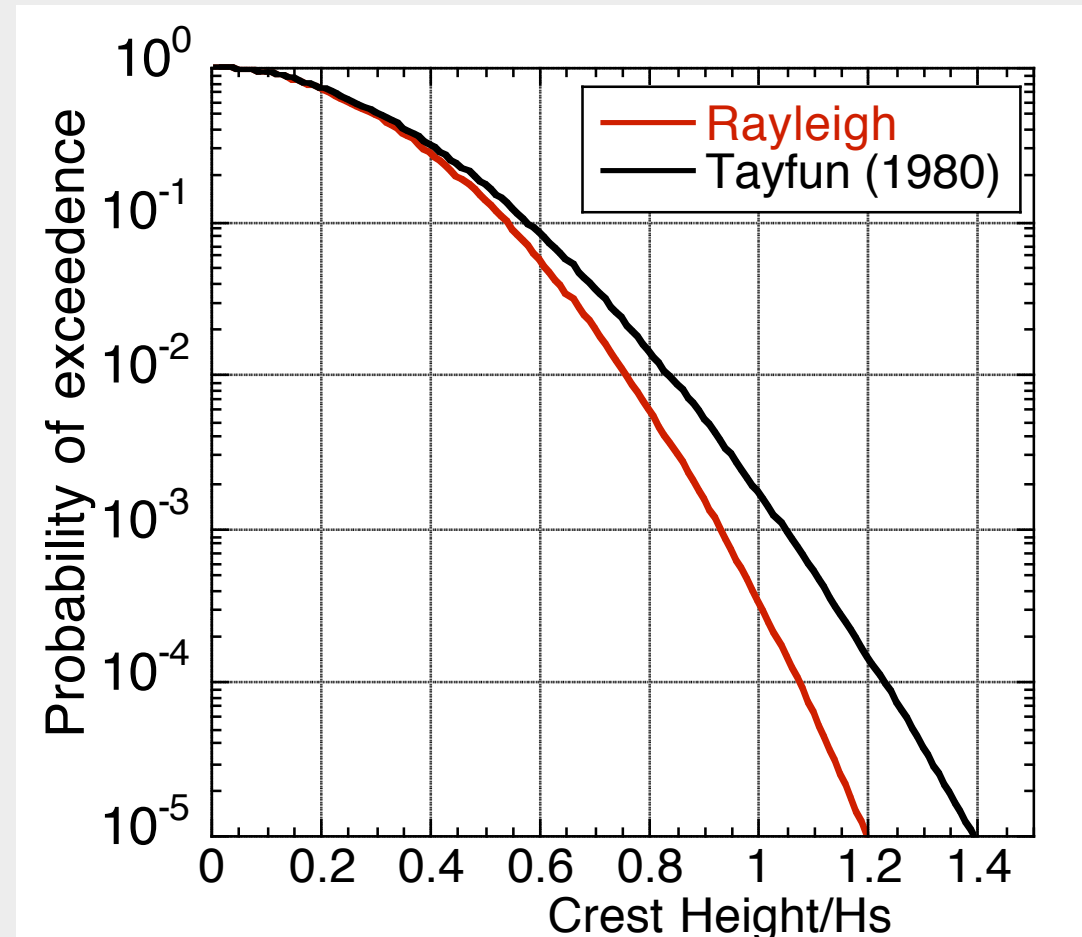
# THE STATISTICAL PROPERTIES OF SURFACE GRAVITY WAVES: LINEAR THEORY

**Longuet-Higgins (1952):** the surface elevation is described by a Gaussian process; in the narrow band approximation wave heights and wave crests are described by a Rayleigh distribution

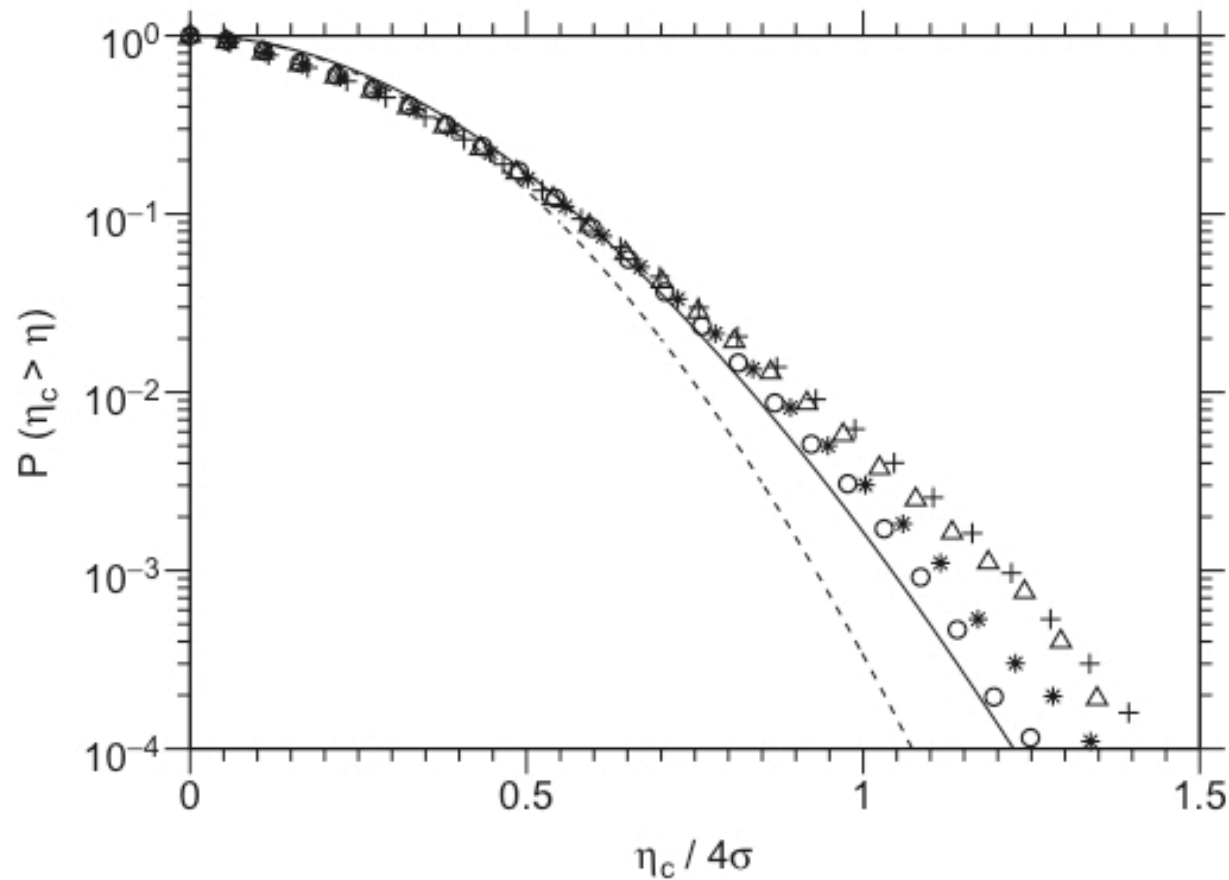


**Longuet-Higgins (1963), Tayfun (1980):**

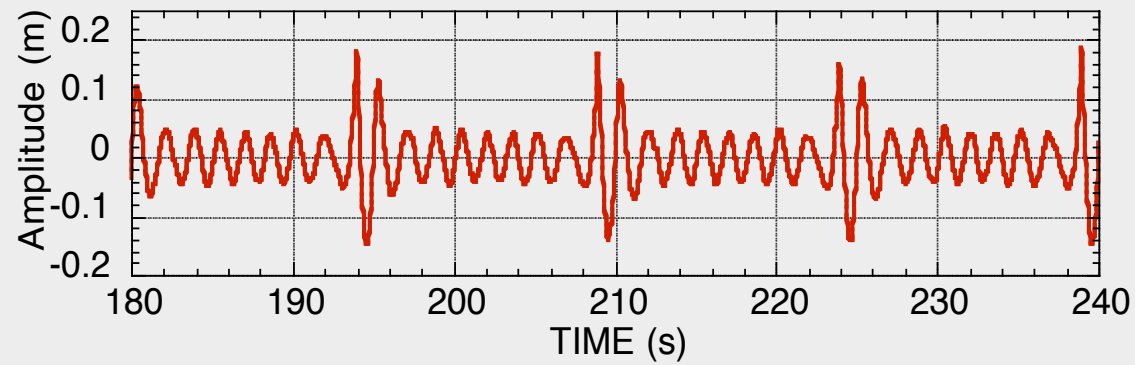
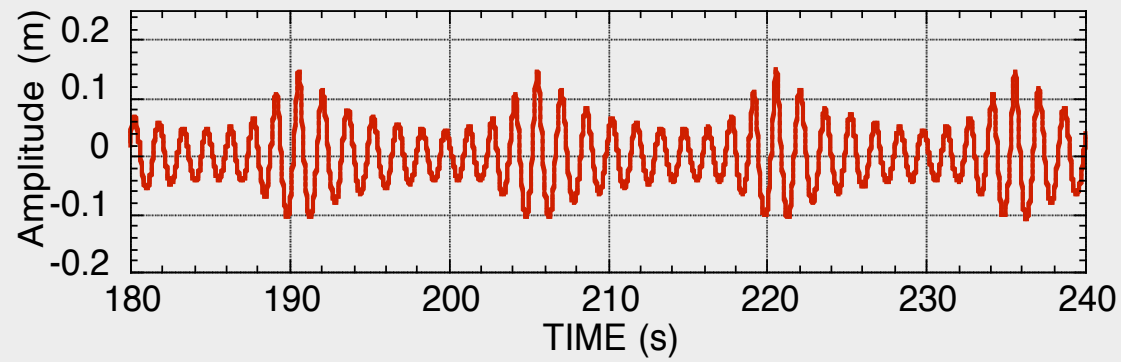
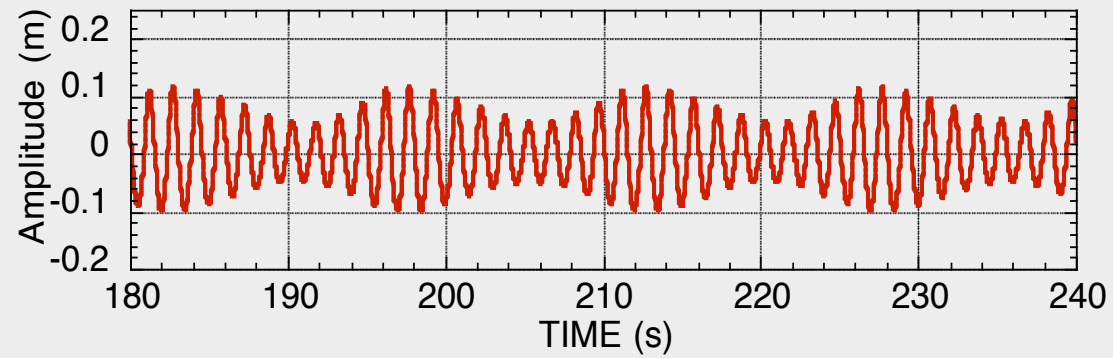
**Corrections to the Rayleigh distribution for wave crests due to the influence of Stokes-like (phase-locked) components**



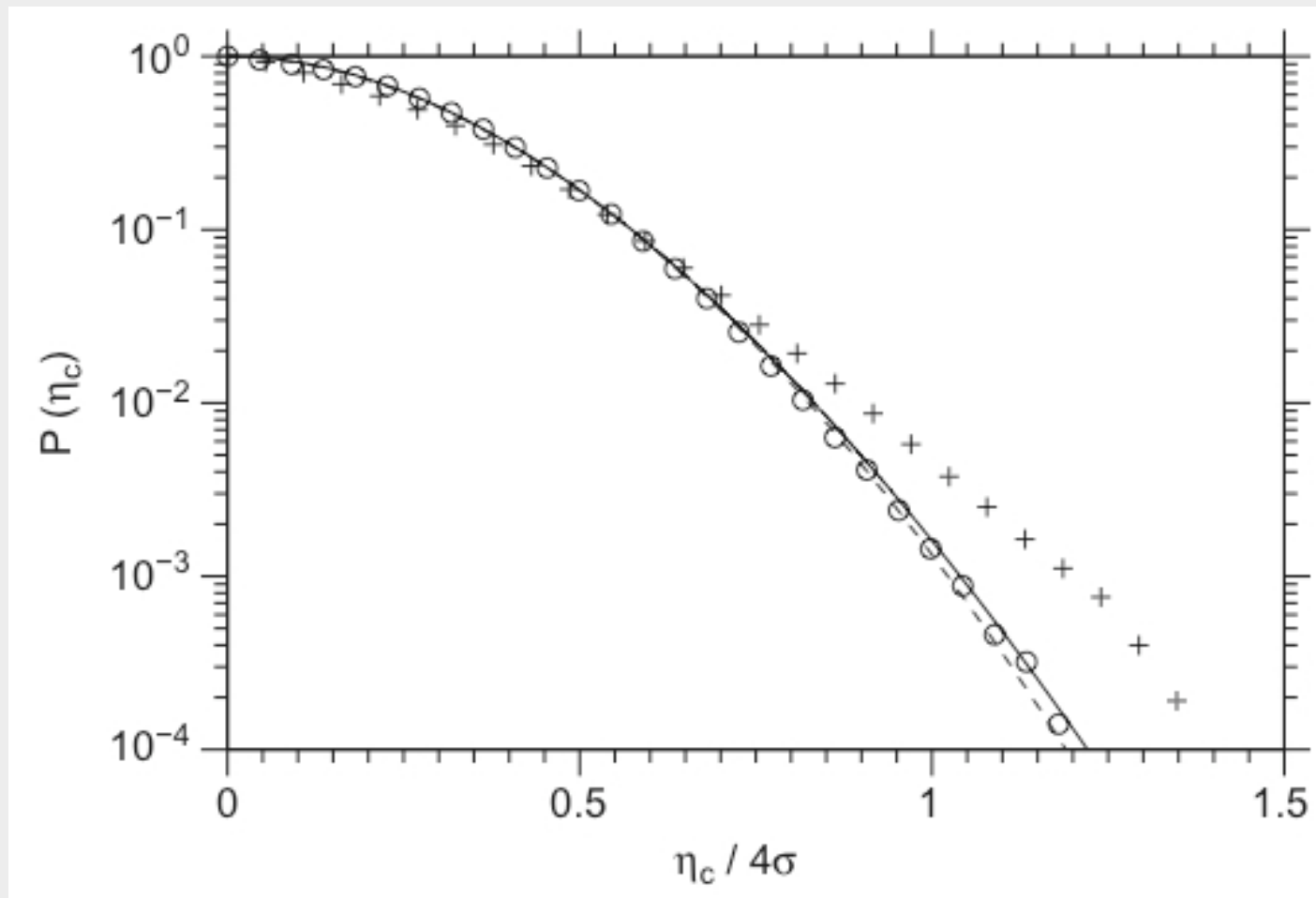
# 1D NUMERICAL SIMULATION OF TRUNCATED EULER EQUATIONS



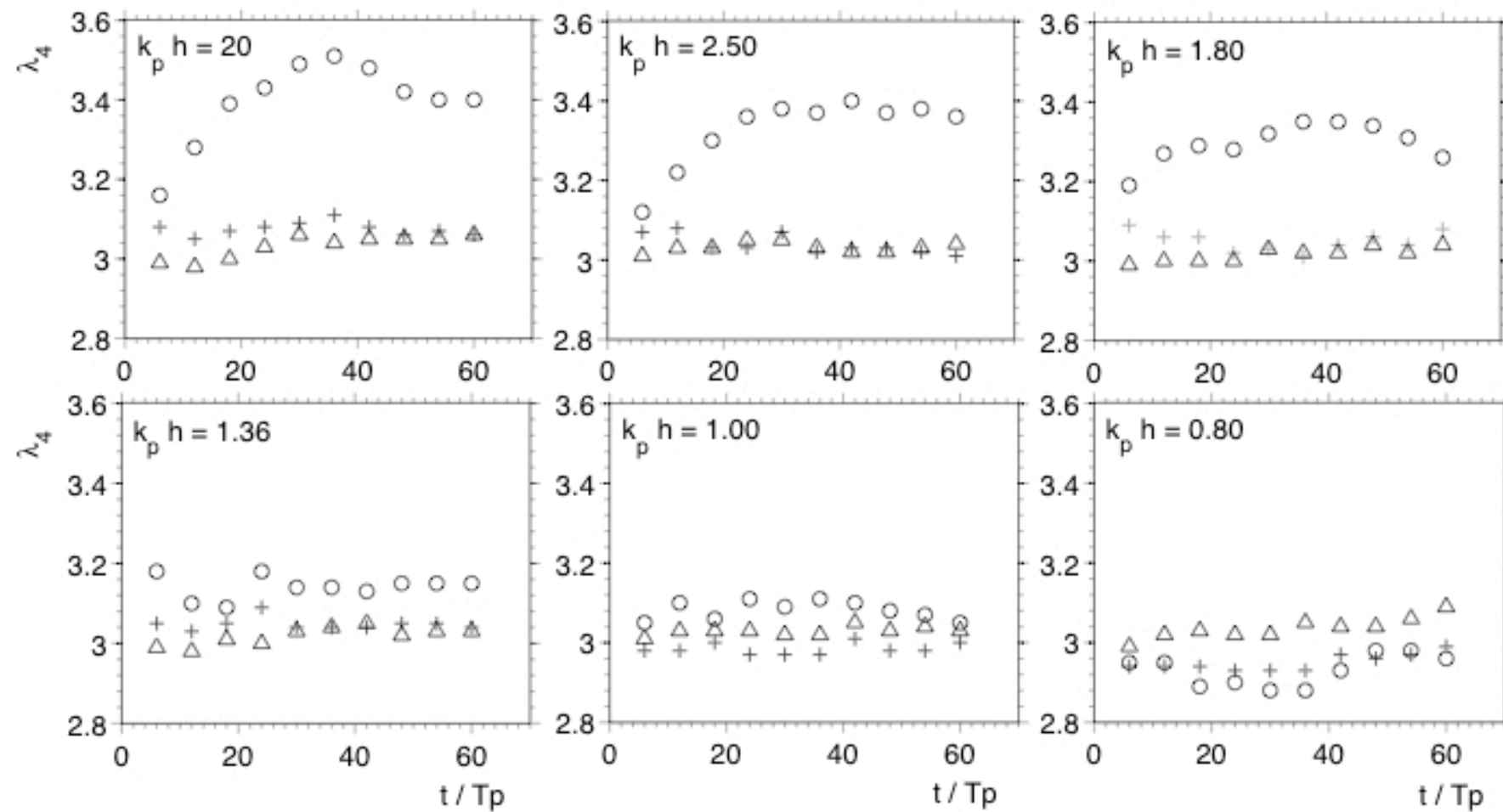
# EXPERIMENTS AT MARINTEK (NORWAY)



# OCEAN WAVES ARE NOT UNIDIMENSIONAL!!



## EVOLUTION OF KURTOSIS IN TIME



# **COLLABORATORS**

**A. Toffoli, A. R. Osborne**