Control of nonlinear periodic waves: numerical aspects

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Sine-Gordon (SG) equation:

$$u_{tt} - u_{xx} + sin\, u = \epsilon f(x,t), \qquad \epsilon \ll 1$$

$$x \in [0, L], \qquad u(x+L) = u(x).$$

$\epsilon = 0, \quad u(x,t) = u(\Theta_1, ..., \Theta_N),\ \Theta_i = \kappa_i x - \omega_i t + \Theta_{0i},$

κ_i are multiples of $k_0 = 2\pi/L$.

Problem: How to excite high amplitude ($u \sim$ O(1)) multiphase waves with a priory given parameters by a small perturbation and which types of perturbations should be appropriate?

Parameters of control: N, κ_i, amplitudes

$u = 0 \qquad \rightarrow \qquad u = u(\Theta_1, ..., \Theta_N) + O(\epsilon)$

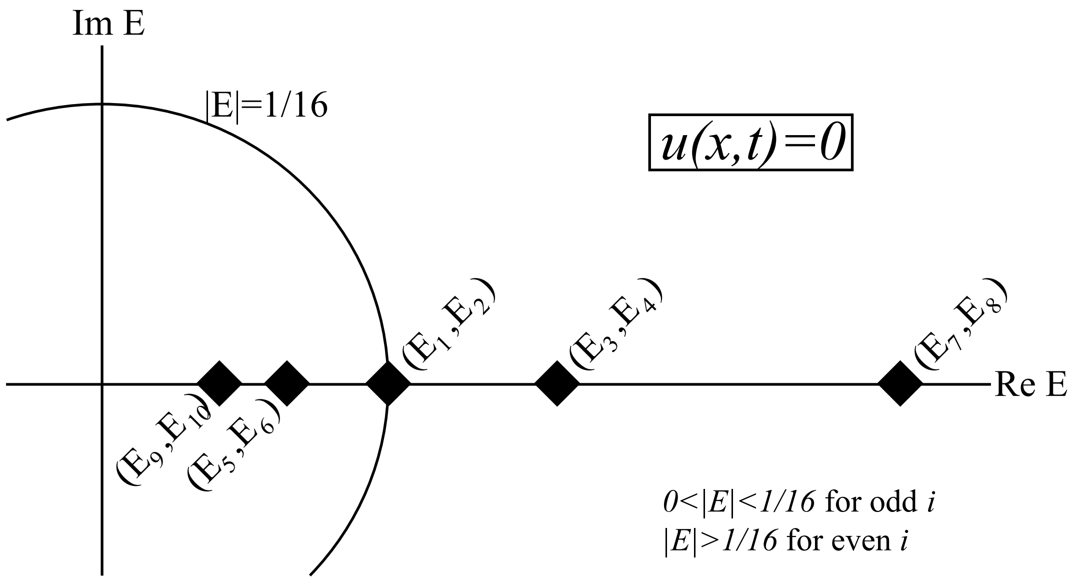
$\boxed{\epsilon = 0}$

N-phase wave associates with the discrete spectrum of $2N$ complex egenvalues $\{E_{2i-1}, E_{2i}\}_{i=1}^{i=N}$ in the scattering problem

$$\frac{dF}{dx} = UF \; ,$$

$$U(E,x) = \frac{i}{4}(u_t + u_x)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{16\sqrt{E}}\begin{pmatrix} 0 & -e^{-iu} \\ e^{iu} & 0 \end{pmatrix} - \sqrt{E}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

"nonsoliton" part of the spectrum: $E_{2i-1} = E_{2i}^*$



Each spectral point ♦ is doubly degenerated and represents *closed gaps* ($E_{2i-1} = E_{2i}$) indicating inactive degrees of freedom associated with phases Θ_i.

<u>Our goal</u> is to open the initially closed gaps by the perturbations

In our studying we adopt the usual assumption of the perturbation theory: $E \to E(t)$ (e.g. I.M. Krichever, 1983)

Autoresonant approach:

$$f(x,t) = cos(\kappa_m x - \int \Omega_m(t)dt), \qquad \kappa_m = \frac{2\pi}{L}m$$

$\Omega_m(t)$ is a slow function of time:

$$\Omega_m(t) = \Omega_{0m} - \alpha(t - t_0)$$

Ω_{0m} - some resonant frequency for the system.

For small amplitude waves: $\Omega_{0m} = \sqrt{1 + \kappa_m^2} > 0$
The number of affected gap is

$$i = \begin{cases} 2m & , m > 0 \\ -2m + 1 & , m \leq 0 \end{cases}$$

i.e. the perturbation will cross the resonance of the phase with $\kappa_i = \kappa_m$, $\omega_i = \Omega_{0m}$.

One-phase solutions: N=1

Physical meaning of the autoresonance for weakly nonlinear SG problem:

$$L = \frac{1}{2}(u_t^2 - \Omega_{0m}^2 u^2) + \frac{1}{24}u^4 + \epsilon u \cos\theta_d, \qquad \theta_d = \kappa_m x - \int \Omega(t)dt.$$

Using the ansatz $u = a\cos\theta$, where the amplitude $a(t)$ and the frequency $\omega(t) = -\theta_t$ are slow function of time, one obtains the averaged over θ Lagrangian

$$\Lambda(a,\theta,\theta_t) = \frac{1}{4}(\omega^2 - \Omega_{0m}^2)a^2 + \frac{1}{64}a^4 - \frac{1}{2}\epsilon a\cos\Phi, \qquad \Phi = \theta - \theta_d.$$

The Lagrangian equations are

$$\boxed{A_\tau = \mu\sin\Phi, \qquad \Phi_\tau = \tau - A^2 + \frac{\mu}{A}\cos\Phi,}$$

where it is used $|\omega^2 - \Omega_{0m}^2| \ll \Omega_{0m}^2$ and rescaling

$$\tau = \alpha^{1/2}t, \qquad A = \frac{1}{4}\alpha^{-1/4}\Omega_{0m}^{-1/2}a, \qquad \mu = \frac{\epsilon}{8}(\Omega_{0m}^2\alpha)^{-3/4}$$

$$\Phi_{\tau\tau} = -\frac{\partial U_{eff}(\Phi, A)}{\partial \Phi} - \Gamma(A)\Phi_{\tau} \; ,$$

$$U_{eff}(\Phi, A) = -\Phi - 2\mu A \cos\Phi - \frac{1}{4}\frac{\mu^2}{A^2}\cos 2\Phi \; .$$

Threshold condition for phase-locking:

$$\mu > \mu_{cr} = 0.41 \qquad \longrightarrow \qquad \boxed{\epsilon > \epsilon_{cr} = 3.28\Omega_{0m}^{3/2}\alpha^{3/4}}$$

(the detailes of the studying can be found also in A.G. Shagalov, J.J. Rasmussen, V. Naulin, J.Phys.A: Math.Gen. 42 (2009) 045502)

Numerical aspects:

1. Scattering problem.

Introduce the grid in the interval $[0, l]$: $x_n = h\,n$, $h = L/M$, $n = 0, ..., M$.

Calculation of the transfer matrix

$$F^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad F^{n+1} = F^n + \frac{h}{2}[U(x_n)F^n + U(x_{n+1})(1 + hU(x_n))F^n],$$

$$\Delta(E) = tr(F^M).$$

The discrete spectrum $\{E_j\}$ corresponds to the solutions of the implicit equation $\Delta(E) = \pm 2$.

2. Calculation of frequences ω_i for the multiphase waves.
(R. Flesch, M.G. Forest, A.Sinha, Physica D 48(1991) 169)

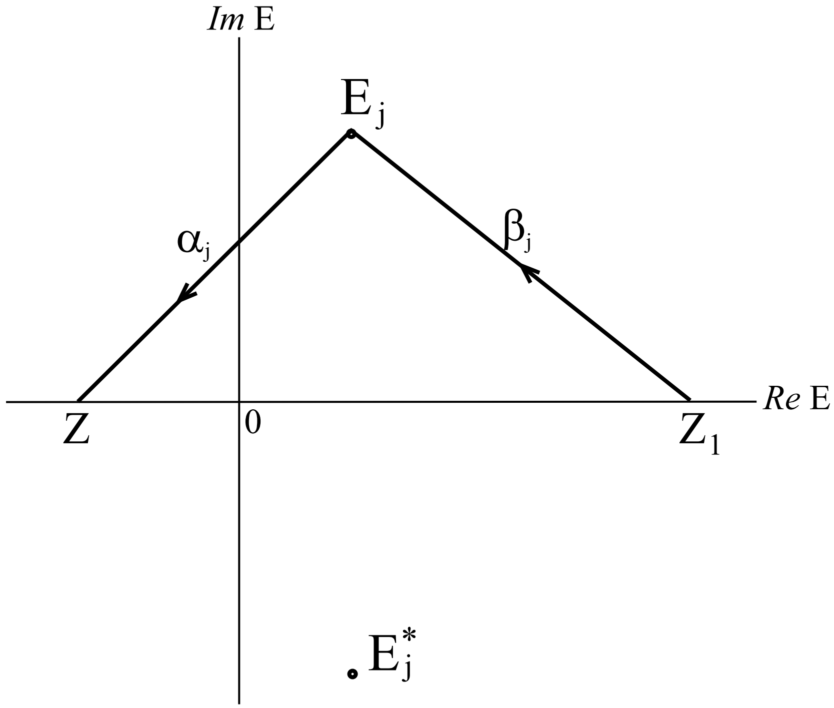
$$A_{kj} = 4\ Re\left(\int_{\alpha_j} \frac{E^{N-k}dE}{R(E)}\right), \qquad k,j = 1,...,N,$$

$$B_{kj} = 2\ Re\left(\int_{\alpha_j} \frac{E^{N-k}dE}{R(E)}\right) + 2i\ Im\left(\int_{\beta_j} \frac{E^{N-k}dE}{R(E)}\right),$$

where $R^2(E) = E\ \Pi_{l=1}^{2N}(E - E_l)$

$$A^T C = I\ ,$$

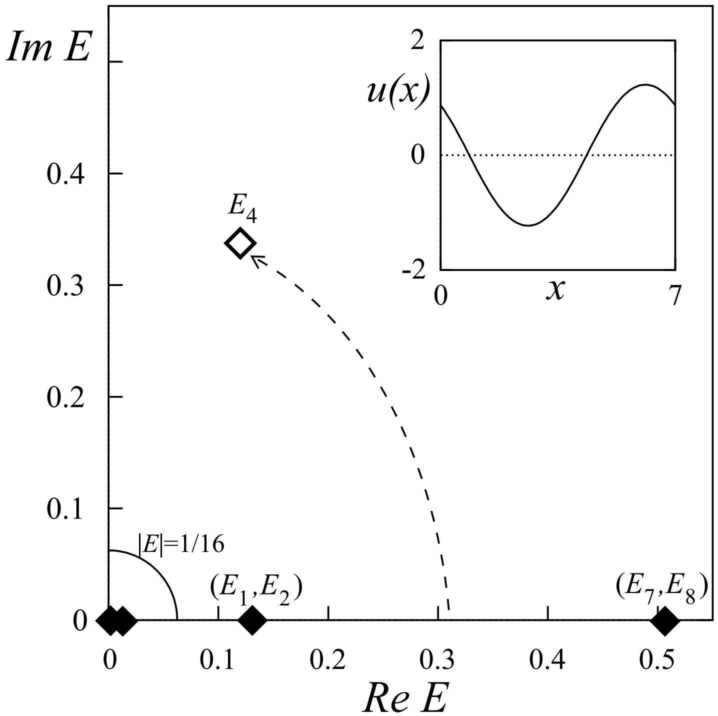
$$\omega_j = \sum_{k=1}^{N} 4\pi i (F^{-1})_{kj} \left(C_{k1} + \frac{(-1)^{N-1} C_{kN}}{16\sqrt{\Pi_{l=1}^{2N} E_l}} \right)\ , \qquad F = C(A-2B)\ .$$



$Z < 0$ and $Z_1 > Re(E_j)$ are arbitrary.

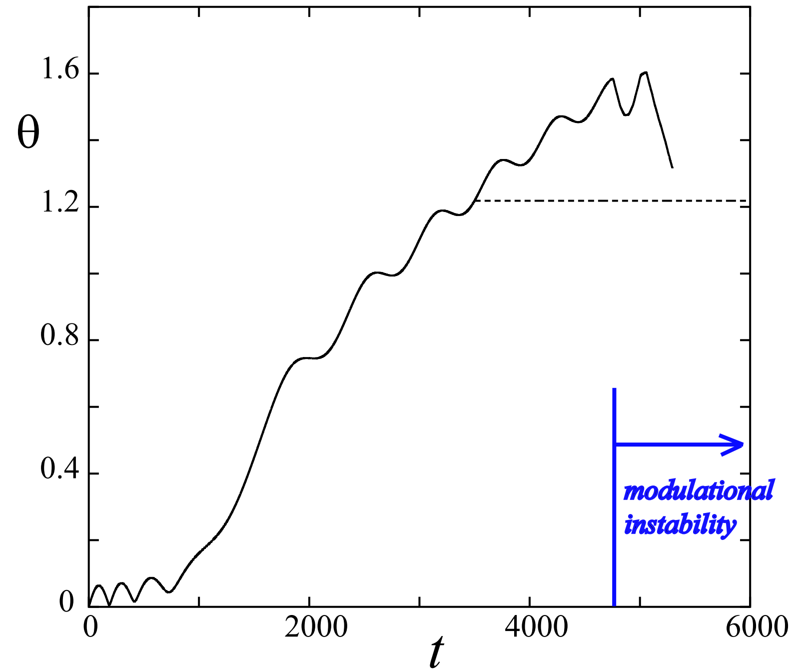
$$A_{kj} = 4Re\left(\int_{E_j}^{Z} \frac{\Phi(E,k)dE}{\sqrt{E-E_j}}\right) \quad \xrightarrow{E \to \theta} \quad A_{kj} = 8\ Re(\sqrt{Z-E_j}\int_0^1 \Phi(\theta,k)d\theta)$$

where $E = E_j + (Z - E_j)\theta^2,\ \theta \in [0,1]$.

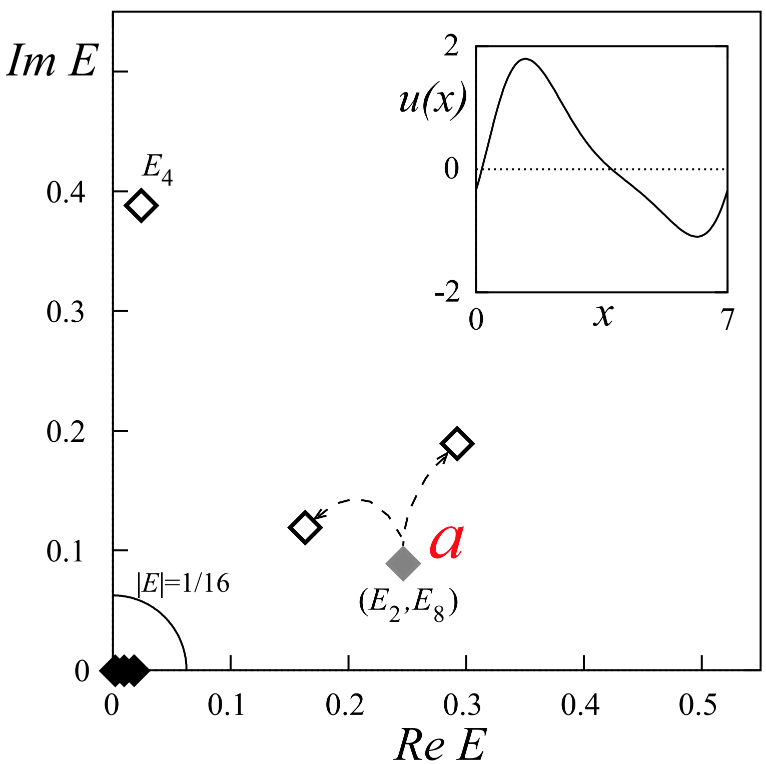


The spectrum of the excited one-phase traveling wave at $t_* = 3500$; $\Diamond$ − non-degenerate spectral points, ♦ − double degenerate spectral points; $L = 7$, $m = 1$, $\Omega_{01} = 1.344$, $t_0 = 1000$, $\epsilon = 0.003$, $\alpha = 0.00003$.

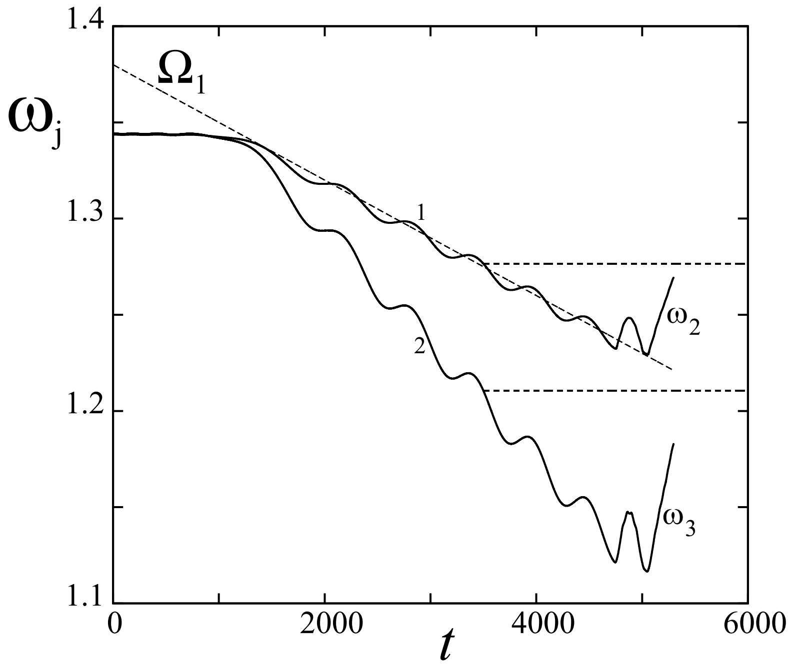
The insert shows the shape of the wave at $t = t_*$.

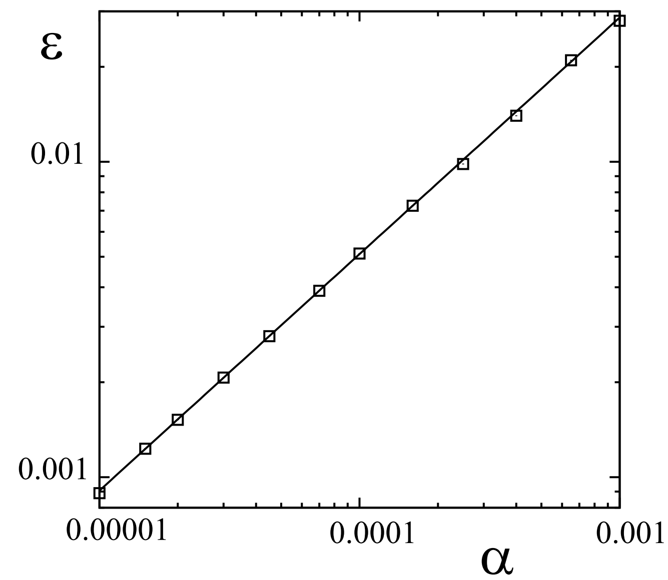


The opening of the driven gap. The solid line represents the angular width of the opening gap $\theta = \arg(E_4)$, the dotted line corresponds to the stationary wave solution established after the driving perturbation was switched off at $t = 3500$.



The spectrum of the three-phase wave at $t = 4800$ after the development of the modulational instability. The point a is the doubly degenerate eigenvalue in the initial stage of the modulational instability at $t = 4200$; $\Diamond$ describes nondegenerate spectral points and ♦ represents doubly degenerate spectral points. The parameters are $L = 7$, $m = 1$, $\Omega_{01} = 1.344$, $t_0 = 1000$, $\epsilon = 0.003$, and $\alpha = 0.00003$. The insert shows the waveform at time $t = 4800$.





The evolution of the frequencies ω_j of two adjoint waves. Line 1 is $\omega_2(t)$ of the driven phase-locked wave, line 2 is $\omega_3(t)$, and the dotted lines correspond to the stationary wave established after the driving perturbation is switch off at $t = 3500$.

The threshold for autoresonant phase-locking found numerically (boxes) and the theoretical prediction $\epsilon = 5.1\alpha^{3/4}$ (solid line).

Excitation of multiphase wave.

"Step-by-step" scenario: to drive the system *successively* by the small amplitude, chirped frequency wave

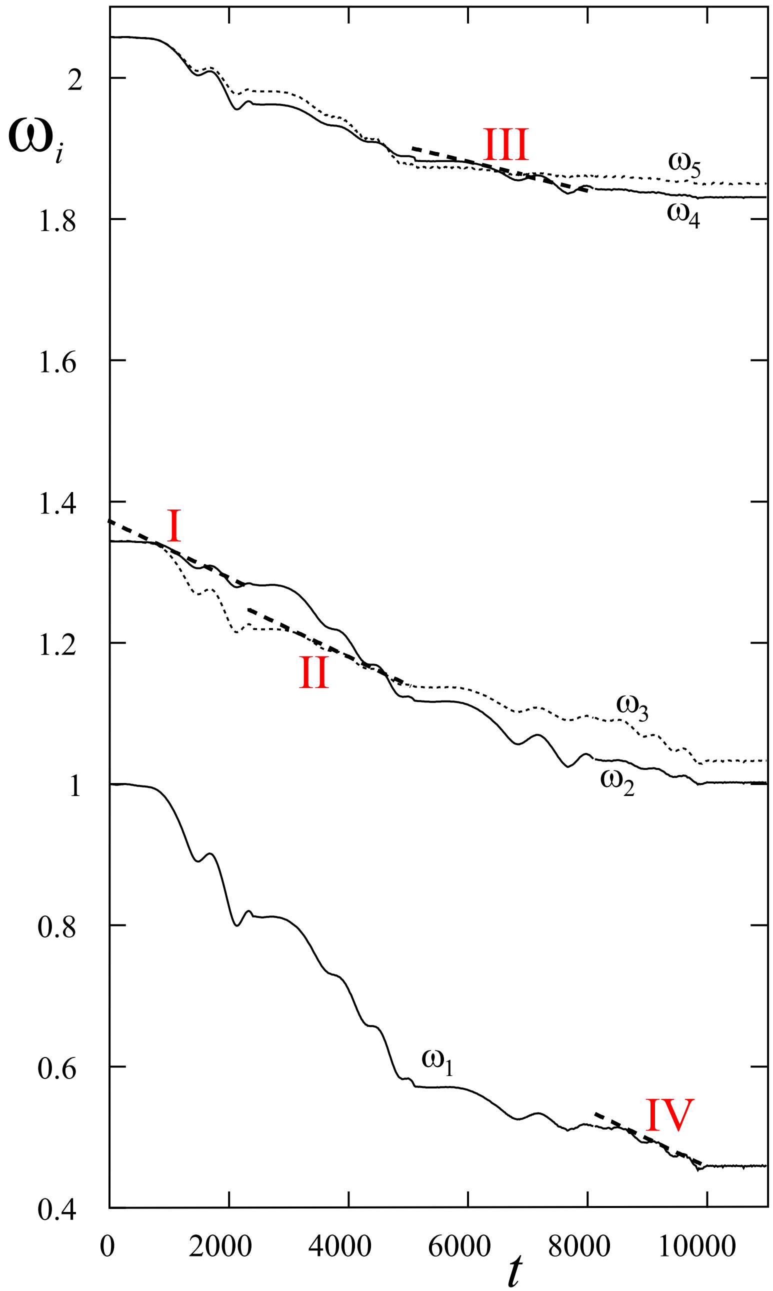
$$\epsilon f(x,t) = \epsilon\, cos(\kappa_m x - \int \Omega_m(t)dt)$$

with the different vectors

$$\kappa_m = \frac{2\pi}{L} m\ .$$

In the following example of four steps process we will use successively $m = 1, -1, 2, 0$

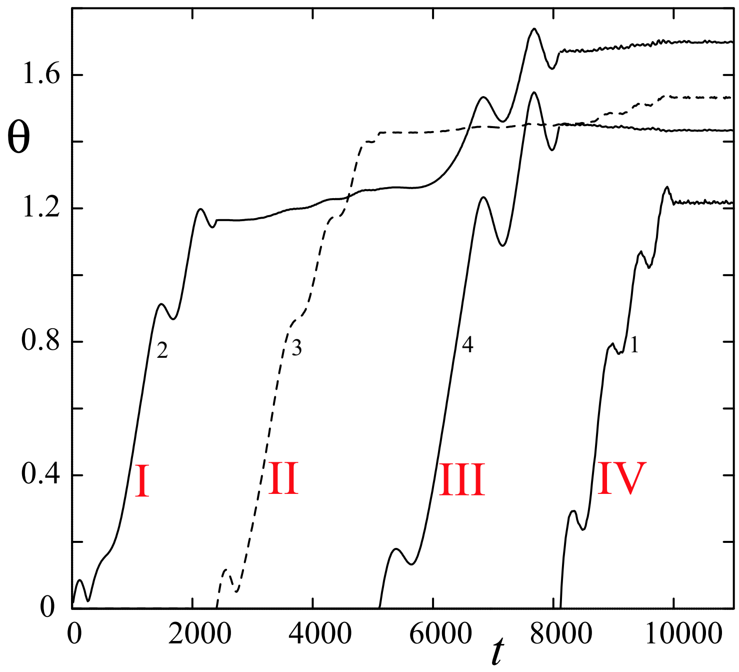
(i.e. we will successively open the gaps of numbers $i = 2, 3, 4, 1$)

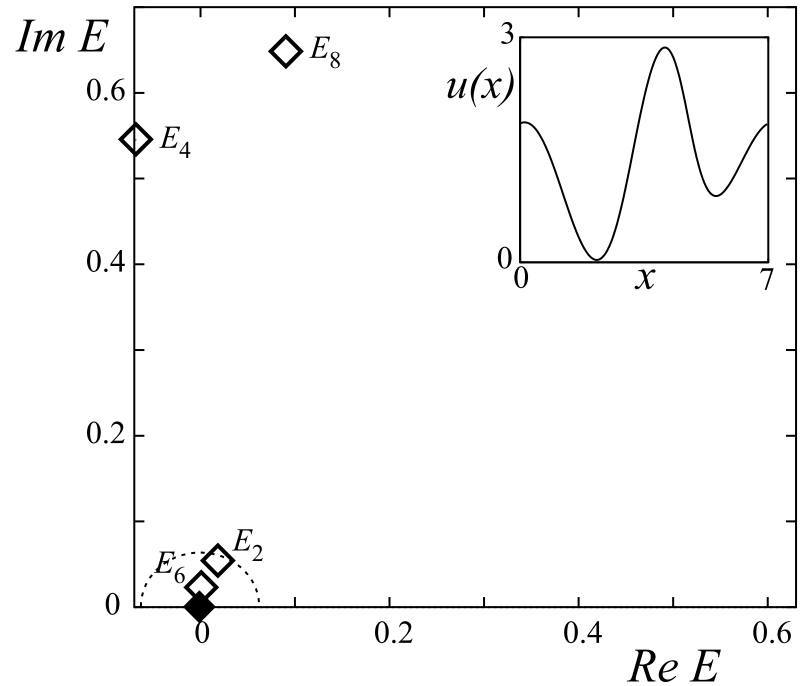


The evolution of the frequencies ω_i in four successive stages of excitation of a four-phase wave.

In stage I, $m = 1$, $\epsilon = 0.003$, $\alpha = 0.00004$, $\Omega_{0m} = 1.34$, and $t_0 = 800$.
In stage II, $m = -1$, $\epsilon = 0.003$, $\alpha = 0.00004$, $\Omega_{0m} = 1.22$, and $t_0 = 3000$.
In stage III, $m = 2$, $\epsilon = 0.003$, $\alpha = 0.00002$, $\Omega_{0m} = 1.88$, and $t_0 = 6000$.
In stage IV, $m = 0$, $\epsilon = 0.002$, $\alpha = 0.00004$, $\Omega_{0m} = 0.51$, and $t_0 = 8700$.

The thick dashed lines show the evolution of the driving frequencies $\Omega_m(t)$.





The opening of the driven gaps in the process of successive excitation of four phases. The numbers of the curves correspond to the numbers of the frequencies in previous figure.

The spectrum of the four-phase wave at $t = 10000$. $\Diamond$ represents nondegenerate spectral points and $\blacklozenge$ are doubly degenerate spectral points. The insert shows the actual waveform at the same time.

Multiphase driving and control of the modulational instability.

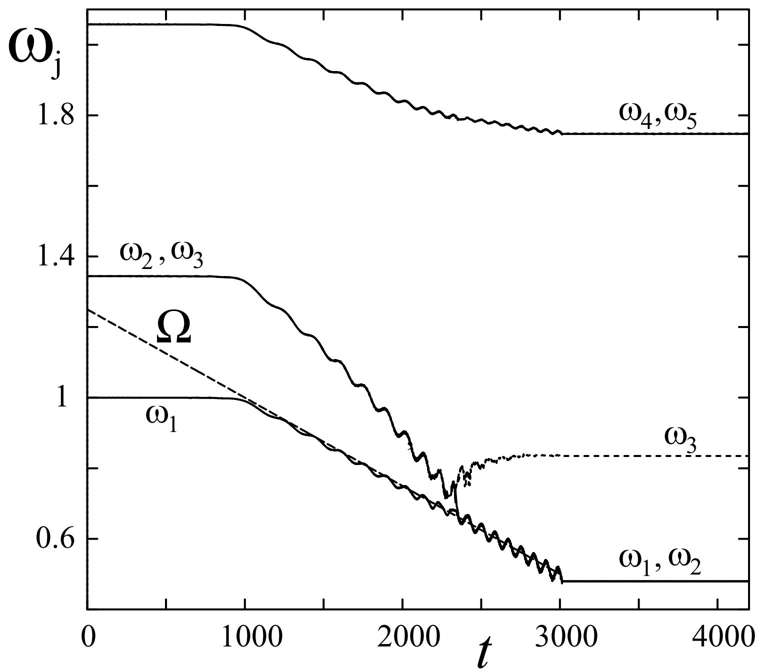
Three-phase drive (m=0,1,-1):

$$\epsilon f(x,t) = \epsilon[1 + r\cos(\kappa x)]\cos\left(\int \Omega(t)dt\right) \ ,$$

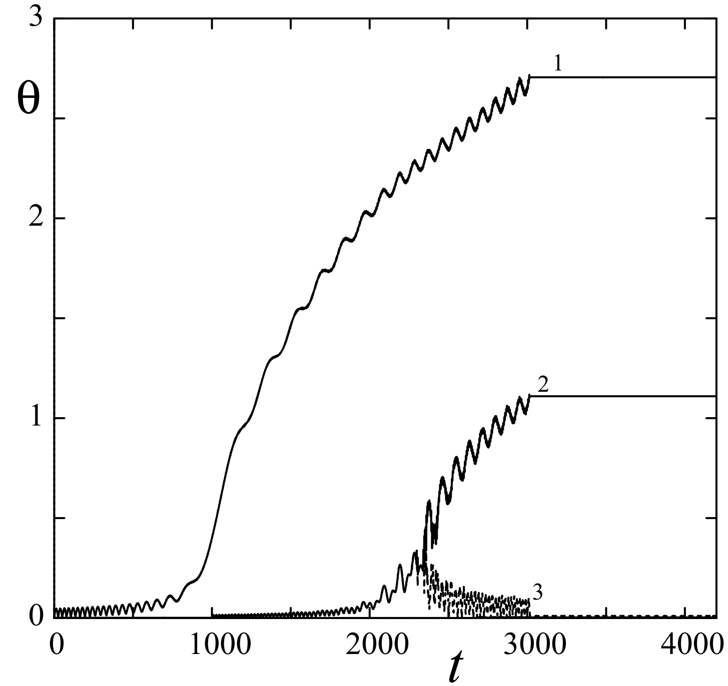
$$\Omega(t) = \Omega_0 - \alpha(t - t_0) \ ,$$

where $\Omega_0 = 1$ and

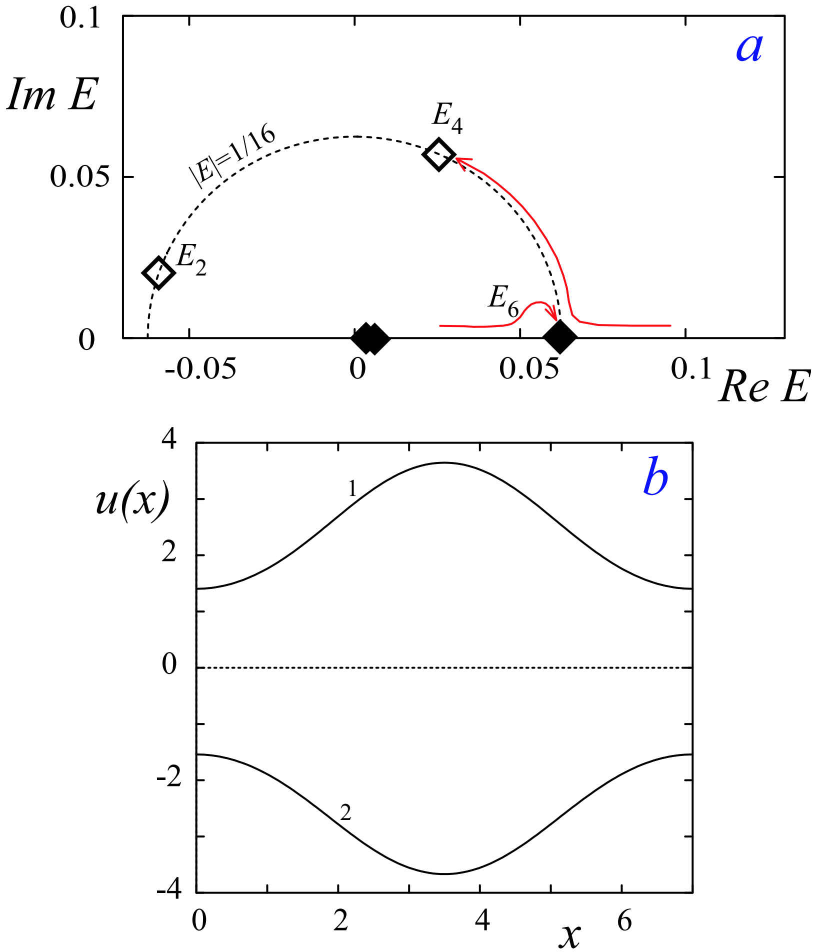
$$r = \begin{cases} 0 & , \quad t < t_0 \\ 1 & , \quad t \geq t_0 \end{cases}$$



The evolution of the frequencies ω_j in the multi-phase excitation of the periodic SG breather solution. The parameters are $\epsilon = 0.012$, $\alpha = 0.00025$, $\Omega_0 = 1$, and $t_0 = 1000$



The opening of the driven gaps in the process of multi-phase excitation of the breather solution. The numbers of the curves correspond to numbers of the frequencies in the previous figure.



The autoresonant SG breather solution.

(a) The spectrum of the breather solution at $t = 4200$, $\Diamond$ describes the nondegenerate spectral points and $\blacklozenge$ corresponds to the doubly degenerate spectral points.

(b) The actual waveform of the breather (line 1 at $t = 3973.2$ and line 2 at $t = 3978.4$.