

# Asymptotics for a special solution to the second member of the Painlevé I hierarchy

T. Claeys

April 22, 2010

## Abstract

We study the asymptotic behavior of a special smooth solution  $y(x, t)$  to the second member of the Painlevé I hierarchy. This solution arises in random matrix theory and in the study of Hamiltonian perturbations of hyperbolic equations. The asymptotic behavior of  $y(x, t)$  if  $x \rightarrow \pm\infty$  (for fixed  $t$ ) is known and relatively simple, but it turns out to be more subtle when  $x$  and  $t$  tend to infinity simultaneously. We distinguish a region of algebraic asymptotic behavior and a region of elliptic asymptotic behavior, and we obtain rigorous asymptotics in both regions. We also discuss two critical transitional asymptotic regimes.

## 1 Introduction

We study asymptotics for a solution to the second member in the Painlevé I hierarchy, also called the  $P_I^2$  equation,

$$x = ty - \left( \frac{1}{6}y^3 + \frac{1}{24}(y_x^2 + 2yy_{xx}) + \frac{1}{240}y_{xxxx} \right). \quad (1.1)$$

This is an ODE in the  $x$ -variable, but the equation depends also on the parameter  $t$ . Given  $t$ , general solutions to (1.1) have an infinite number of poles in the complex  $x$ -plane, but in [7], a special real solution to (1.1) was constructed which has no poles for real values of  $x$  and  $t$ , and which has the asymptotics

$$y(x, t) = \frac{1}{2}z_0|x|^{1/3} + \mathcal{O}(|x|^{-2}), \quad \text{as } x \rightarrow \pm\infty, \quad (1.2)$$

for fixed  $t \in \mathbb{R}$ , where  $z_0 = z_0(x, t)$  is the real solution of

$$z_0^3 = -48\text{sgn}(x) + 24z_0|x|^{-2/3}t. \quad (1.3)$$

It is remarkable but well-known that  $y(x, t)$  is also a solution to the KdV equation

$$y_t + yy_x + \frac{1}{12}y_{xxx} = 0. \quad (1.4)$$

In the case  $t = 0$ , the existence and uniqueness of a real solution  $y_0(x)$  with asymptotics (1.2) were proved by Moore in [25]. Thus any real pole-free solution  $y(x, t)$  to (1.1) with asymptotics given by (1.2) solves the Cauchy problem for the KdV equation with initial data  $y(x, 0) = y_0(x)$ . Since the Cauchy problem for KdV is uniquely solvable with initial data  $y_0$  (see [24]),  $y(x, t)$  can be characterized as the unique solution to the Cauchy problem for KdV with initial data  $y_0$ . The Whitham equations corresponding to this KdV solution were already studied by Gurevich and Pitaevskii in [18] and by Potemin in [26].

The equation (1.1) appeared for  $t = 0$  in a work by Brézin, Marinari, and Parisi [3], where the authors gave physical arguments supporting the existence of a real pole-free solution to (1.1) (for  $t = 0$ ) with asymptotics given by

$$y(x) \sim \mp |6x|^{1/3}, \quad \text{as } x \rightarrow \pm\infty, \quad (1.5)$$

see also [2]. The interest in this solution was renewed after Dubrovin [12] conjectured the existence and uniqueness of a real solution to (1.1) which is pole-free for real  $x, t \in \mathbb{R}$ . The uniqueness part of this conjecture remains open until now. Another part of Dubrovin's conjecture suggests that  $y(x, t)$  describes the universal asymptotics for Hamiltonian perturbations of hyperbolic equations near the point of gradient catastrophe for the unperturbed equation. This family of equations includes among others the KdV equation and KdV hierarchy, the de-focusing nonlinear Schrödinger equation, and the Camassa-Holm equation. In the particular case of the KdV equation, the conjecture was proved in [4]. Recurrence coefficients for certain critical orthogonal polynomials (related to unitary random matrix ensembles) have a similar type of asymptotics involving  $y(x, t)$  [2, 8].

In the two above-mentioned applications the  $P_I^2$  solution  $y(x, t)$  describes a singular transition (when letting  $x$  and  $t$  vary) between a region of simple algebraic asymptotics and a region of more complicated oscillatory asymptotics involving the Jacobi elliptic  $\theta$ -function. One may thus expect that  $y(x, t)$  itself also exhibits two different types of asymptotics. However in the asymptotics (1.2), there is no trace of elliptic or oscillatory asymptotic behavior. The question arises whether elliptic asymptotics can be observed when letting  $x$  and  $t$  tend to infinity simultaneously. In this paper we will prove that indeed, depending on the precise scaling of  $x$  and  $t$ ,  $y(x, t)$  admits either an algebraic or an elliptic asymptotic expansion.

The type of asymptotics we obtain depends on the value of  $s = x|t|^{-3/2}$  as indicated in Figure 1. If we let  $x \rightarrow \pm\infty$ ,  $t \rightarrow -\infty$  or if we let  $x \rightarrow \pm\infty$ ,  $t \rightarrow +\infty$  in such a way that  $s = xt^{-3/2}$  remains bounded away from the interval  $[-2\sqrt{3}, \frac{2\sqrt{15}}{27}]$ , an algebraic asymptotic expansion similar to (1.2) holds. The leading term of this expansion will be determined by the equation  $x = ty - \frac{1}{6}y^3$ , which is equation (1.1) if we ignore  $x$ -derivatives. If  $x \rightarrow \pm\infty$ ,  $t \rightarrow +\infty$  in such a way that  $s \in (-2\sqrt{3}, \frac{2\sqrt{15}}{27})$  (and remaining bounded away from the endpoints), the asymptotic expansion for  $y$  involves a system of modulation equations and elliptic  $\theta$ -functions. This type of behavior was already suggested in [18, 26], see also the recent paper [16]. For  $s$  near  $-2\sqrt{3}$  and near  $\frac{2\sqrt{15}}{27}$ , two different transitions in the asymptotics for  $y$  take place.

**Remark 1.1** Asymptotics for  $y(x, t)$  where both  $x$  and  $t$  tend to infinity in such a way that  $s$  is bounded can be seen as small dispersion asymptotics for  $y$  as a solution to the KdV equation. Indeed for  $\epsilon > 0$ , by (1.4),  $u(x, t) := \epsilon^{2/7}y(\epsilon^{-6/7}x, \epsilon^{-4/7}t)$  solves the KdV equation normalized as follows,

$$u_t + uu_x + \frac{\epsilon^2}{12}u_{xxx} = 0. \quad (1.6)$$

**Theorem 1.2 (Algebraic region)** *Let  $y$  be the real pole-free solution to the  $P_I^2$  equation defined before. Suppose that either  $t < 0$ ,  $s \in \mathbb{R}$  or  $t > 0$ ,  $s \in \mathbb{R} \setminus [-2\sqrt{3}, \frac{2\sqrt{15}}{27}]$ . Let  $z_0(s)$  be the real zero (which is unique under the above conditions on  $s$  and  $t$ ) of the equation*

$$z_0^3 = 24\text{sgn}(t)z_0 - 48s. \quad (1.7)$$

*In the limit where  $t \rightarrow -\infty$ , we have*

$$y(s|t|^{3/2}, t) = \frac{z_0(s)}{2}|t|^{1/2} + \mathcal{O}(|t|^{-1}), \quad (1.8)$$

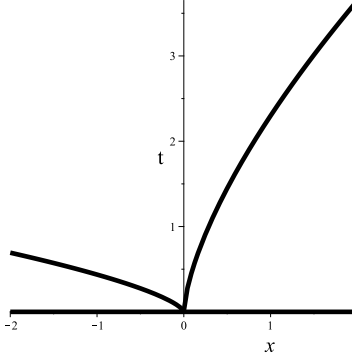


Figure 1:  $y(x, t)$  has elliptic asymptotics if  $(x, t)$  goes to infinity between the two curves and algebraic asymptotics if  $(x, t)$  tends to infinity outside this region.

uniformly for  $s \in \mathbb{R}$ . If  $t \rightarrow +\infty$  and  $s \in \mathbb{R} \setminus [-2\sqrt{3}, \frac{2\sqrt{15}}{27}]$ , the expansion (1.8) holds as well, uniformly for  $s$  bounded away from the interval  $[-2\sqrt{3}, \frac{2\sqrt{15}}{27}]$ .

**Remark 1.3** If we only take into account the terms without  $x$ -derivatives in (1.1), we obtain  $x = ty - \frac{1}{6}y^3$ . Changing variables  $y \mapsto \frac{z_0}{2}|t|^{1/2}$ , this leads to (1.7). This change of variables will be convenient later on.

For  $s \in (-2\sqrt{3}, \frac{2\sqrt{15}}{27})$ , the asymptotics for  $y(x, t)$  involve the third Jacobi elliptic  $\theta$ -function, elliptic integrals, and a system of modulation equations given by

$$(\beta_3 + \beta_2 + \beta_1)^2 + 2(\beta_3^2 + \beta_2^2 + \beta_1^2) = 120, \quad (1.9)$$

$$(\beta_3 + \beta_2 + \beta_1)^3 - 4(\beta_3^3 + \beta_2^3 + \beta_1^3) = 360s, \quad (1.10)$$

$$\int_{\beta_3}^{\beta_2} \sqrt{\xi - \beta_3}(\xi - \alpha)\sqrt{\beta_2 - \xi}\sqrt{\beta_1 - \xi}d\xi = 0, \quad \alpha = -\frac{1}{2}(\beta_3 + \beta_2 + \beta_1), \quad (1.11)$$

where we are interested in solutions  $\beta_3 < \alpha < \beta_2 < \beta_1$ . For the interested reader, we mention that the variables  $\tilde{\beta}_j = t^{\frac{1}{2}}\beta_j$ ,  $\tilde{\alpha} = t^{\frac{1}{2}}\alpha$  solve the elliptic Whitham equations appearing in the asymptotic theory for the KdV equation [17, 27, 28, 29] in the particular case of initial data

$$\tilde{\beta}_1(x, 0) = \tilde{\beta}_2(x, 0) = \tilde{\beta}_3(x, 0) = -(48x)^{\frac{1}{3}},$$

see [26, 17]. Those Whitham equations are given by

$$\frac{\partial}{\partial t}\tilde{\beta}_i + v_i \frac{\partial}{\partial x}\tilde{\beta}_i = 0, \quad v_i = \frac{2 \prod_{k \neq i}(\tilde{\beta}_i - \tilde{\beta}_k)}{3 \tilde{\beta}_i + a} + \frac{1}{3}(\tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{\beta}_3), \quad i = 1, 2, 3,$$

with

$$a(\tilde{\sigma}) = -\tilde{\beta}_1 + (\tilde{\beta}_1 - \tilde{\beta}_3) \frac{E(\tilde{\sigma})}{K(\tilde{\sigma})}, \quad \tilde{\sigma}^2 = \frac{\tilde{\beta}_2 - \tilde{\beta}_3}{\tilde{\beta}_1 - \tilde{\beta}_3},$$

and  $K(\tilde{\sigma})$ ,  $E(\tilde{\sigma})$  are the complete elliptic integrals of the first and second kind. Solvability of the system (1.9)-(1.11) was obtained by Potemin [26] for  $s \in (-2\sqrt{3}, \frac{2\sqrt{15}}{27})$ , but can also be deduced from the solvability of the Whitham equations [27, 17]. The asymptotics for  $y(x, t)$  in the elliptic region are expressed in terms of the solution to this system. We obtain asymptotics of the same type as in [10], although only rapidly decaying initial data for KdV were considered there.

**Theorem 1.4 (Elliptic region)** *Let  $s \in (-2\sqrt{3}, \frac{2\sqrt{15}}{27})$  and let  $\beta_3 < \alpha < \beta_2 < \beta_1$  solve the system of modulation equations (1.9)-(1.11). Then  $y(x, t)$  has the following expansion if we let  $t \rightarrow +\infty$ ,*

$$y(st^{3/2}, t) = t^{1/2} \frac{\beta_3 + \beta_2 - \beta_1}{2} + t^{1/2} (\beta_1 - \beta_3) \frac{E(\sigma)}{K(\sigma)} + \frac{t^{1/2}}{C^2} (\log \theta)'' \left( \frac{t^{7/4}}{2\pi} \Omega \right) + \mathcal{O}(t^{-1/2}). \quad (1.12)$$

Here  $\theta$  is the third Jacobi  $\theta$ -function

$$\theta(z; \tau) = \sum_{m=-\infty}^{+\infty} e^{2\pi imz + \pi i \tau m^2}, \quad (1.13)$$

and

$$C = \frac{2K(\sigma)}{\sqrt{\beta_1 - \beta_3}}, \quad (1.14)$$

$$\Omega = \frac{1}{15} \int_{\beta_2}^{\beta_1} \sqrt{\xi - \beta_3} (\xi - \alpha) \sqrt{\xi - \beta_2} \sqrt{\beta_1 - \xi} d\xi, \quad (1.15)$$

$$\tau = \frac{iK'(\sigma)}{K(\sigma)}, \quad \sigma = \sqrt{\frac{\beta_2 - \beta_3}{\beta_1 - \beta_3}}, \quad (1.16)$$

and  $K(\sigma), E(\sigma)$  are the complete elliptic integrals of the first and second kind. The expansion (1.12) holds uniformly for  $s \in (-2\sqrt{3} + \delta, \frac{2\sqrt{15}}{27} - \delta)$  for any  $\delta > 0$ .

If we take asymptotics for  $y(x, t)$  where  $t \rightarrow +\infty$ ,  $x \rightarrow \pm\infty$  in such a way that  $s = xt^{-3/2}$  tends to  $-2\sqrt{3}$  or  $\frac{2\sqrt{15}}{27}$  sufficiently fast, neither the algebraic expansion nor the elliptic one is valid. In this case one can obtain two types of critical asymptotics. Similar transitional asymptotics have been proved for decaying negative smooth solutions to the KdV equation in [5, 6]. Because the proofs are very similar as for KdV and for conciseness of the paper, we will not prove those results here, but we will indicate below the procedure that can be followed in order to prove them.

The first critical regime appears when  $s \rightarrow -2\sqrt{3}$ , which corresponds to the left edge of the oscillatory zone in the  $(x, t)$ -plane, see Figure 1. The asymptotics involve the Hastings-McLeod solution [19] to the second Painlevé equation. This is the unique solution to the Painlevé II equation

$$q_{\xi\xi}(\xi) = \xi q(\xi) + 2q(\xi)^3, \quad (1.17)$$

with asymptotics given by

$$q(\xi) \sim \text{Ai}(\xi), \quad \text{as } \xi \rightarrow +\infty, \quad q(\xi) \sim \sqrt{\frac{-\xi}{2}}, \quad \text{as } \xi \rightarrow -\infty. \quad (1.18)$$

For  $\xi \in \mathbb{R}$ , we have

$$y(-2\sqrt{3}t^{3/2} + c_0 t^{1/3} \xi, t) = 2\sqrt{3}t^{1/2} - \frac{1}{c_1 t^{1/12}} q(\xi) \cos(t^{7/4} \omega) + \mathcal{O}(t^{-2/3}), \quad \text{as } t \rightarrow +\infty, \quad (1.19)$$

where  $c_0 = 5^{1/3}$ ,  $c_1 = \frac{5^{-1/6} \cdot 3^{-1/4}}{2}$ , and

$$\omega = \frac{80}{21} \sqrt{5} \cdot 3^{3/4} - 2 \cdot 3^{1/4} \cdot 5^{5/6} \xi t^{-1}.$$

In other words the leftmost oscillations can be modeled by a rapidly oscillating cosine and the amplitude develops proportional to the Hastings-McLeod solution.

The second critical regime is the one where  $s \rightarrow \frac{2\sqrt{15}}{27}$ , i.e. the right edge of the oscillatory zone in the  $(x, t)$ -plane, see Figure 1. Here the asymptotic expansion for  $y$  involves a sum of  $\text{sech}^2$ -terms: we have

$$\begin{aligned} y \left( \frac{2\sqrt{15}}{27} t^{3/2} - c_2 t^{-1/4} \ln t \cdot \xi, t \right) \\ = -\frac{2}{3} \sqrt{15} t^{1/2} + \frac{7}{3} \sqrt{15} t^{1/2} \sum_{k=0}^{\infty} \text{sech}^2(X_k) + \mathcal{O}(t^{-5/4} \ln^2 t), \end{aligned} \quad (1.20)$$

where  $c_2 = \frac{\sqrt{7} \cdot 3^{3/4}}{8 \cdot 5^{1/4}}$ , and

$$X_k = -\frac{7}{8} \left( \frac{1}{2} + k - \xi \right) \ln t - \ln(\sqrt{2\pi} h_k) - \left( k + \frac{1}{2} \right) \ln \gamma, \quad (1.21)$$

$$h_k = \frac{2^{k/2}}{\pi^{1/4} \sqrt{k!}}, \quad \gamma = 4\sqrt{2} \cdot 5^{3/8} \cdot 7^{5/4} \cdot 3^{-11/8}. \quad (1.22)$$

Recall that the KdV equation admits soliton solutions of the form  $a \text{sech}^2(bx - ct)$ . The expansion (1.20) can thus be seen as a superposition of solitons. The amplitude of the rightmost oscillations is of the same order as the leading order term of  $y$  outside the oscillatory region, and of the same order as the amplitude of the oscillations in the elliptic region. Because of the term with  $\ln t$  in (1.21), every soliton is sharply localized near a positive half integer value of  $\xi$ .

### 1.1 Riemann-Hilbert problem for $P_I^2$

In order to prove our results, we will perform an asymptotic analysis of the Riemann-Hilbert (RH) problem associated to the real pole-free solution to the  $P_I^2$  equation. For a more general description of Riemann-Hilbert problems for Painlevé equations we refer to [15].

Consider the following RH problem for given complex parameters  $x$  and  $t$ , on a contour  $\Gamma = \mathbb{R}^- \cup \bigcup_{j=0}^6 e^{\frac{j2\pi i}{7}} \mathbb{R}^+$  consisting of eight straight rays orientated from left to right.

#### RH problem for $Y$ :

- (a)  $Y$  is analytic in  $\mathbb{C} \setminus \Gamma$ .
- (b)  $Y$  satisfies the following jump relations on  $\Gamma$ ,

$$Y_+(\zeta) = Y_-(\zeta) \begin{pmatrix} 1 & s_j \\ 0 & 1 \end{pmatrix}, \quad \text{for } \arg \zeta = \frac{j2\pi}{7} \text{ with } j \text{ even}, \quad (1.23)$$

$$Y_+(\zeta) = Y_-(\zeta) \begin{pmatrix} 1 & 0 \\ s_j & 1 \end{pmatrix}, \quad \text{for } \arg \zeta = \frac{j2\pi}{7} \text{ with } j \text{ odd}, \quad (1.24)$$

$$Y_+(\zeta) = Y_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } \zeta \in \mathbb{R}^-. \quad (1.25)$$

- (c)  $Y$  has an asymptotic expansion of the form

$$Y(\zeta) = \left( I + \sum_{k=1}^{\infty} A_k(x, t) \zeta^{-k} \right) \zeta^{-\frac{1}{4}\sigma_3} N e^{-\theta(\zeta; x, t)\sigma_3}, \quad \text{as } \zeta \rightarrow \infty, \quad (1.26)$$

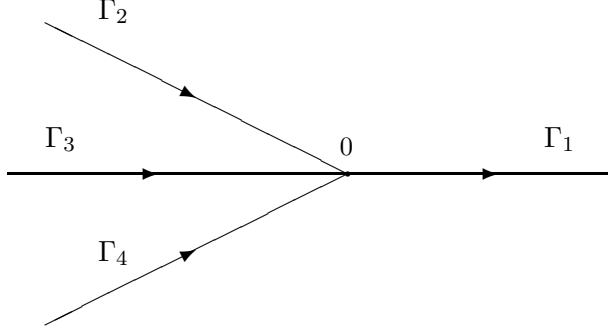


Figure 2: The oriented contour  $\Gamma$  consisting of the four straight rays  $\Gamma_1$ ,  $\Gamma_2$ ,  $\Gamma_3$ , and  $\Gamma_4$ .

where

$$N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{1}{4}\pi i \sigma_3}, \quad \theta(\zeta; x, t) = \frac{1}{105} \zeta^{7/2} - \frac{1}{3} t \zeta^{3/2} + x \zeta^{1/2}, \quad (1.27)$$

$$\text{and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is the RH problem for a general solution to the  $P_I^2$  equation (1.1). The RH problem can only be solvable if the Stokes multipliers  $s_j$  satisfy the relation

$$\begin{pmatrix} 1 & 0 \\ -s_4 & 1 \end{pmatrix} \begin{pmatrix} 1 & -s_5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_6 & 1 \end{pmatrix} \begin{pmatrix} 1 & s_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -s_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -s_3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (1.28)$$

For any set of Stokes multipliers  $s_0, \dots, s_6$  (independent of  $\zeta, x, t$ ) satisfying this condition, it was proved in [7] that the function

$$y(x, t) = 2A_{1,11}(x, t) - A_{1,12}^2(x, t), \quad (1.29)$$

with  $A_1$  given as in (1.26), is a solution to (1.1). However,  $y$  can have poles at certain isolated values of  $(x, t)$ . Those points corresponds to the (isolated) values of  $x$  and  $t$  at which the RH problem for  $Y$  is not solvable. Since  $y$  solves (1.1), it also solves the KdV equation (1.4). Even in the more general situation where  $s_0, \dots, s_6$  depend on  $\zeta$  (but not on  $x$  and  $t$ ),  $y(x, t)$  still solves the KdV equation (locally near  $(x, t)$ ) if the RH problem is solvable at  $(x, t)$ .

We are interested in one particular solution to the  $P_I^2$  equation. This solution corresponds to the case where

$$s_1 = s_2 = s_5 = s_6 = 0, \quad s_3 = s_4 = s_0 = 1, \quad (1.30)$$

which means that there are only jumps for  $Y$  on  $\Gamma = \cup_{j=1}^4 \Gamma_j$ , with

$$\Gamma_1 = \mathbb{R}^+, \quad \Gamma_2 = e^{\frac{6\pi i}{7}} \mathbb{R}^+, \quad \Gamma_3 = \mathbb{R}^-, \quad \Gamma_4 = e^{-\frac{6\pi i}{7}} \mathbb{R}^+, \quad (1.31)$$

see Figure 2. For this choice of Stokes multipliers, it was proved in [7] that the RH problem for  $Y$  is uniquely solvable for all real values of  $x$  and  $t$ , and that  $y$  defined by (1.29) is the unique real pole-free solution to equation (1.1) with asymptotics given by (1.2). From now on, we refer to  $Y$  as the solution to the RH problem with Stokes multipliers given by (1.30). One can verify from the RH conditions that  $Y(\zeta)$  has analytic continuations from each of the four sectors determined by  $\Gamma_1, \dots, \Gamma_4$  to the entire complex plane. We denote  $Y_j$ ,  $j = 1, \dots, 4$  for the analytic continuation from the sector in between  $\Gamma_j$  and  $\Gamma_{j \bmod 4+1}$ . The jump conditions tell us that

$$Y_1 = Y_2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad Y_2 = Y_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y_3 = Y_4 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad Y_1 = Y_4 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (1.32)$$

## Outline

In the next sections, we will obtain asymptotics for  $Y = Y(\zeta; x, t)$  when  $x, t \rightarrow \infty$  by applying the Deift/Zhou steepest descent method [11] on the RH problem. Using (1.29), this will also lead to asymptotics for  $y(x, t)$ . In Section 2, we will prove the algebraic expansion for  $y$  given in Theorem 1.2. In Section 3, we will prove the elliptic asymptotics stated in Theorem 1.4. In Section 4, we will give a brief overview of the procedure that can be followed to prove the critical asymptotics (1.19) and (1.20). In addition we will make some comments about special solutions to higher members of the Painlevé I hierarchy and about unbounded solutions of the KdV equation.

## 2 Algebraic region

Here we generalize the asymptotic analysis done in [7] for  $t$  fixed, which was inspired by [21, 22]. We assume  $t \neq 0$  in what follows.

### 2.1 Construction of the $g$ -function

In order to transform the RH problem for  $Y$  into a RH problem normalized at infinity and with 'sufficiently simple' jump conditions, we need to construct a so-called  $g$ -function. This function will be of the following form,

$$g(\zeta) = c_1(\zeta - z_0)^{7/2} + c_2(\zeta - z_0)^{5/2} + c_3(\zeta - z_0)^{3/2}, \quad (2.1)$$

where we take the branch of the roots which is analytic in  $\mathbb{C} \setminus (-\infty, z_0]$  and positive for  $\zeta > z_0$ . Let us write

$$s = x|t|^{-3/2}. \quad (2.2)$$

If  $z_0 = z_0(s)$  is the real root (which is unique for  $t < 0$  and for  $t > 0$ ,  $s \in \mathbb{R} \setminus [-2\sqrt{3}, \frac{2\sqrt{15}}{27}]$ ) of the third degree equation

$$z_0^3 = 24\text{sgn}(t)z_0 - 48s, \quad (2.3)$$

and if we let

$$c_1 = \frac{1}{105}, \quad c_2 = \frac{1}{30}z_0, \quad c_3 = \frac{1}{24}z_0^2 - \frac{\text{sgn}(t)}{3}, \quad (2.4)$$

it is straightforward to check that, with  $\theta$  given by (1.27),

$$|t|^{7/4}g(\zeta; s) = \theta(|t|^{1/2}\zeta; x, t) + d_1\zeta^{-1/2} + \mathcal{O}(\zeta^{-3/2}), \quad \text{as } \zeta \rightarrow \infty, \quad (2.5)$$

where  $d_1$  does not depend on  $\zeta$ ; its explicit value can be calculated but is unimportant.

**Proposition 2.1** *Suppose that either  $t < 0$  or  $t > 0$  and  $s \in \mathbb{R} \setminus [-2\sqrt{3}, \frac{2\sqrt{15}}{27}]$ . Then*

$$g(\zeta; s) > 0, \quad \text{for } \zeta > z_0, \quad (2.6)$$

$$\text{Im } g'_+(\zeta; s) > 0, \quad \text{for } \zeta < z_0. \quad (2.7)$$

**Proof.** The function  $(\zeta - z_0)^{-3/2}g(\zeta; s)$  is quadratic in  $\zeta$  and has no real zeros if  $c_2^2 - 4c_1c_3 < 0$ . Since  $c_1 > 0$ , it is positive for large  $\zeta$  and thus for all  $\zeta$  if  $c_2^2 - 4c_1c_3 < 0$ . Now by (2.4),

$$c_2^2 - 4c_1c_3 = -\frac{1}{315} \left( \frac{3z_0^2}{20} - 4\text{sgn}(t) \right), \quad (2.8)$$

which is negative if either  $t$  is negative or  $t > 0$  and  $|z_0| > 4\sqrt{\frac{5}{3}}$ . If  $z_0 \in (2\sqrt{2}, 4\sqrt{\frac{5}{3}})$ , one verifies that both zeros are smaller than  $z_0$  so that (2.6) still holds. In a similar way one shows that  $\text{Im} g'_+(\zeta)$  has no real zeros apart from  $z_0$  if  $\frac{25}{4}c_2^2 - 21c_1c_3 < 0$ , which is true if  $t < 0$  and if  $t > 0$ ,  $|z_0| > 4\sqrt{3}$ . Moreover in  $(-4\sqrt{3}, -2\sqrt{2})$  both zeros lie to the right of  $z_0$  and (2.7) still holds.

We can conclude that (2.6) and (2.7) hold for  $t < 0$  and for  $t > 0$  if  $z_0 \in \mathbb{R} \setminus [-4\sqrt{\frac{5}{3}}, 4\sqrt{3}]$ . By (2.3) this is equivalent to  $s \in \mathbb{R} \setminus [-2\sqrt{3}, \frac{2\sqrt{15}}{27}]$ .  $\square$

By (2.1) and the above proposition, a straightforward complex analysis argument using the Cauchy-Riemann conditions leads to the following corollary.

**Corollary 2.2** *Suppose that either  $t < 0$  or  $t > 0$  and  $s \in \mathbb{R} \setminus [-2\sqrt{3}, \frac{2\sqrt{15}}{27}]$ . For  $\theta_0 > 0$  sufficiently small, we have*

$$\text{Re } g(\zeta) < 0, \quad \text{for } \arg(\zeta - z_0) = \pi \pm \theta_0, \quad (2.9)$$

$$\text{Re } g(\zeta) > 0, \quad \text{for } \arg(\zeta - z_0) = 0. \quad (2.10)$$

These inequalities enable us to transform the RH problem for  $Y$  to a RH problem for which the jumps decay exponentially fast to  $I$  as  $t \rightarrow \infty$ , except on  $(-\infty, z_0)$  and in a small neighborhood of  $z_0$ .

## 2.2 Normalization of the RH problem

Fix  $0 < \theta_0 < \frac{2\pi}{7}$  such that Corollary 2.2 holds. Let us define  $T$  as follows,

$$T(\zeta; x, t) = \begin{pmatrix} 1 & 0 \\ d_1 |t|^{1/4} & 1 \end{pmatrix} \times \begin{cases} Y_1(|t|^{1/2}\zeta; x, t)e^{|t|^{7/4}g(\zeta; x, t)\sigma_3}, & \text{if } 0 < \arg(\zeta - z_0) < \pi - \theta_0, \\ Y_2(|t|^{1/2}\zeta; x, t)e^{|t|^{7/4}g(\zeta; x, t)\sigma_3}, & \text{if } \pi - \theta_0 < \arg(\zeta - z_0) < \pi, \\ Y_3(|t|^{1/2}\zeta; x, t)e^{|t|^{7/4}g(\zeta; x, t)\sigma_3}, & \text{if } \pi < \arg(\zeta - z_0) < \pi + \theta_0, \\ Y_4(|t|^{1/2}\zeta; x, t)e^{|t|^{7/4}g(\zeta; x, t)\sigma_3}, & \text{if } \pi + \theta_0 < \arg(\zeta - z_0) < 2\pi, \end{cases} \quad (2.11)$$

with  $d_1$  given by (2.5), and with the  $Y_j$ 's the analytic extensions of  $Y$  as explained in Section 1.1. Then  $T$  satisfies the following RH problem.

### 2.2.1 RH problem for $T$

(a)  $T$  is analytic in  $\mathbb{C} \setminus \Sigma$ , where  $\Sigma = z_0 + (\mathbb{R} \cup e^{\pi \pm \theta_0} \mathbb{R}^+)$ .

(b)  $T$  satisfies the following jump relations on  $\Sigma$ ,

$$T_+(\zeta) = T_-(\zeta) \begin{pmatrix} 1 & e^{-2|t|^{7/4}g(\zeta; x, t)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } \arg(\zeta - z_0) = 0, \quad (2.12)$$

$$T_+(\zeta) = T_-(\zeta) \begin{pmatrix} 1 & 0 \\ e^{2|t|^{7/4}g(\zeta; x, t)} & 1 \end{pmatrix}, \quad \text{for } \arg(\zeta - z_0) = \pi \pm \theta_0, \quad (2.13)$$

$$T_+(\zeta) = T_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } \arg(\zeta - z_0) = \pi. \quad (2.14)$$



(c)  $T$  has the following asymptotic behavior as  $\zeta \rightarrow \infty$ ,

$$T(\zeta) = (I + B_1(x, t)\zeta^{-1} + \mathcal{O}(\zeta^{-2})) |t|^{-\frac{1}{8}\sigma_3} \zeta^{-\frac{1}{4}\sigma_3} N, \quad (2.15)$$

with

$$B_1 = \begin{pmatrix} \frac{d_1^2}{2} + |t|^{-1/2} A_{1,11} - d_1 |t|^{-1/4} A_{1,12} & |t|^{-1/2} A_{1,12} - d_1 |t|^{-1/4} \\ * & * \end{pmatrix}, \quad (2.16)$$

where the values of the \*-entries can be calculated but are unimportant for us, and with  $A_1$  as in (1.26).

Using (1.29), we can recover  $y(x, t)$  from the identity

$$y(x, t) = 2t^{1/2} B_{1,11} - t B_{1,12}^2. \quad (2.17)$$

By Corollary 2.2, it follows that the jump matrices for  $T$  decay to the identity matrix when  $|t| \rightarrow \infty$  except on  $(-\infty, z_0)$  and in a small fixed neighborhood of  $z_0$ . Ignoring a neighborhood of  $z_0$  and ignoring exponential decay, we have jump conditions which can be solved explicitly.

### 2.3 Outside parametrix

We define the outside parametrix by

$$P^{(\infty)}(\zeta) = |t|^{-\frac{\sigma_3}{8}} (\zeta - z_0)^{-\frac{\sigma_3}{4}} N, \quad (2.18)$$

which is analytic in  $\mathbb{C} \setminus (-\infty, z_0]$ , and satisfies the jump condition

$$P_+^{(\infty)}(\zeta) = P_-^{(\infty)}(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for  $\zeta \in (-\infty, z_0)$  if  $N$  is given by (1.27). Note also that

$$T(\zeta) P^{(\infty)}(\zeta)^{-1} = I + \frac{C_1}{\zeta} + \mathcal{O}(\zeta^{-2}), \quad \text{as } \zeta \rightarrow \infty, \quad (2.19)$$

with

$$C_1 = B_1 - \frac{z_0}{4} \sigma_3. \quad (2.20)$$

The leading order asymptotics for  $T$  and  $y(x, t)$  will be determined by the outside parametrix.

### 2.4 Local parametrix near $z_0$

In order to obtain asymptotics for  $T$  uniformly for  $\zeta \in \mathbb{C}$ , we need to construct a local parametrix in a small fixed neighborhood  $U$  of  $z_0$ . This local parametrix has been constructed in [7, Section 3.4] using the Airy function, and it solves the following RH problem.

**RH problem for  $P$ :**

- (a)  $P$  is analytic in  $U \setminus \Sigma$ .
- (b)  $P_+(\zeta) = P_-(\zeta) v_T(\zeta)$  for  $\zeta \in \Sigma \cap U$ , where  $v_T$  is the jump matrix for  $T$  given by (2.12)-(2.14).
- (c)  $P(\zeta) P^{(\infty)}(\zeta)^{-1} = I + \mathcal{O}(|t|^{-3/2})$ , as  $t \rightarrow -\infty$  and if  $t \rightarrow +\infty$  in such a way that  $s$  is bounded away from  $[-2\sqrt{3}, \frac{2\sqrt{15}}{27}]$ , uniformly for  $\zeta \in \partial U$ .

The local parametrix is needed for the rigor of the RH analysis, but its explicit expression in terms of the Airy function will not be needed; it does not contribute to the leading order asymptotics for  $y(x, t)$ .

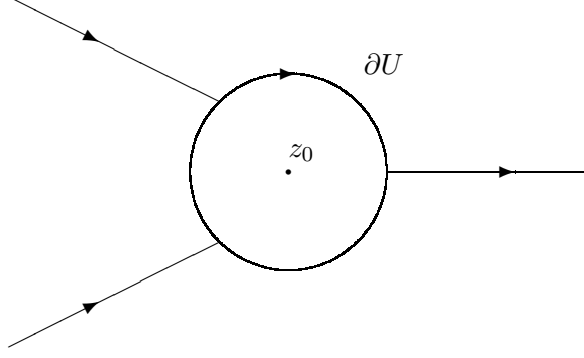


Figure 3: The jump contour  $\Sigma_R$ .

## 2.5 Final transformation

Define

$$R(\zeta; x, t) = \begin{cases} T(\zeta; x, t)P^{(\infty)}(\zeta)^{-1}, & \text{for } \zeta \in \mathbb{C} \setminus U, \\ T(\zeta; x, t)P(\zeta)^{-1}, & \text{for } \zeta \in U. \end{cases} \quad (2.21)$$

Then one verifies that

- (a)  $R$  is analytic in  $\mathbb{C} \setminus \Sigma_R$ , with  $\Sigma_R$  as given in Figure 3,
- (b) for  $\zeta \in \Sigma_R$ , we have  $R_+(\zeta) = R_-(\zeta)v_R(\zeta)$ , where  $v_R(\zeta) = I + \mathcal{O}(|t|^{-3/2})$  in the limit where  $t \rightarrow -\infty$  or where  $t \rightarrow +\infty$  and  $s$  remains bounded away from  $[-2\sqrt{3}, \frac{2\sqrt{15}}{27}]$ ,
- (c) as  $\zeta \rightarrow \infty$ , we have  $R(\zeta) = I + \mathcal{O}(\zeta^{-1})$ .

This is a small-norm RH problem which can be solved by a series expansion [9], and it follows from this standard procedure that

$$R(\zeta; x, t) = I + \mathcal{O}(|t|^{-3/2}), \quad (2.22)$$

uniformly in  $\zeta$  as  $t \rightarrow -\infty$  and also as  $t \rightarrow +\infty$  with  $s$  bounded away from the interval  $[-2\sqrt{3}, \frac{2\sqrt{15}}{27}]$ . Furthermore as  $\zeta \rightarrow \infty$  we can expand  $R$ :

$$R(\zeta) = I + \frac{R_1}{\zeta} + \mathcal{O}(\zeta^{-2}), \quad \text{as } \zeta \rightarrow \infty, \quad (2.23)$$

which implies that

$$R_1(x, t) = \mathcal{O}(|t|^{-3/2}). \quad (2.24)$$

Thus by (2.20) we have

$$B_1 = \frac{z_0}{4}\sigma_3 + \mathcal{O}(|t|^{-3/2}), \quad (2.25)$$

which results in the asymptotics (1.8) for  $y$  by (2.17).

### 3 Elliptic region

In this section we consider the case where  $t > 0$  and  $s \in (-2\sqrt{3}, \frac{2\sqrt{15}}{27})$ . The construction done in the previous section fails in this case because Proposition 2.1 does not hold, which would result in exponentially growing jump matrices for  $T$  at certain parts of the contour. In order to prevent this, we are forced to modify the  $g$ -function and afterwards to open lenses not only along  $(-\infty, z_0)$  but also on an additional interval.

#### 3.1 Construction of the $g$ -function and modulation equations

We search for a  $g$ -function in the form

$$g(\zeta) = \frac{1}{30} \int_{\beta_1}^{\zeta} (z - \beta_3)^{1/2} (z - \alpha) (z - \beta_2)^{1/2} (z - \beta_1)^{1/2} dz, \quad (3.1)$$

analytic in  $\mathbb{C} \setminus (-\infty, \beta_1]$  and positive for  $\zeta > \beta_1$ , where

$$\alpha = -\frac{1}{2}(\beta_3 + \beta_2 + \beta_1), \quad (3.2)$$

with  $\beta_3 \leq \alpha \leq \beta_2 \leq \beta_1$  real and depending on  $s$  but not on  $\zeta$ . As  $\zeta \rightarrow \infty$ , the  $g$ -function can be expanded as follows

$$g(\zeta; x, t) = \frac{1}{105} \zeta^{7/2} + c_1 \zeta^{3/2} + c_2 \zeta^{1/2} + d_1 t^{-7/4} \zeta^{-1/2} + \mathcal{O}(\zeta^{-3/2}), \quad (3.3)$$

with

$$c_1 = -\frac{1}{360}((\beta_3 + \beta_2 + \beta_1)^2 + 2(\beta_3^2 + \beta_2^2 + \beta_1^2)), \quad (3.4)$$

$$c_2 = \frac{1}{360}((\beta_3 + \beta_2 + \beta_1)^3 - 4(\beta_3^3 + \beta_2^3 + \beta_1^3)). \quad (3.5)$$

Now we want to choose  $\beta_3$ ,  $\beta_2$ , and  $\beta_1$  in such a way that

$$t^{7/4} g(\zeta; s) = \theta(t^{1/2} \zeta; x, t) + d_1 \zeta^{-1/2} + \mathcal{O}(\zeta^{-3/2}), \quad \text{as } \zeta \rightarrow \infty. \quad (3.6)$$

This is true if

$$(\beta_3 + \beta_2 + \beta_1)^2 + 2(\beta_3^2 + \beta_2^2 + \beta_1^2) = 120, \quad (3.7)$$

$$(\beta_3 + \beta_2 + \beta_1)^3 - 4(\beta_3^3 + \beta_2^3 + \beta_1^3) = 360s. \quad (3.8)$$

Furthermore, in order to be able to do a steepest descent analysis, we also require

$$\int_{\beta_3}^{\beta_2} \sqrt{\xi - \beta_3} (\xi - \alpha) \sqrt{\beta_2 - \xi} \sqrt{\beta_1 - \xi} d\xi = 0, \quad (3.9)$$

or in other words  $g_{\pm}(\beta_2) = g_{\pm}(\beta_3)$ . It will become clear later on why we need to impose the latter condition. We look for solutions to the modulation equations (3.7)-(3.9) for which  $\beta_3 \leq \alpha \leq \beta_2 \leq \beta_1$ . For two special values of  $s$ , the system of equations can be solved easily. The first one corresponds to the confluent case where  $\beta_3 = \beta_2 = \alpha$ . Then (3.9) is automatically satisfied, and (3.7)-(3.8) are solved uniquely by

$$\beta_3 = \beta_2 = \alpha = -\sqrt{3}, \quad \beta_1 = 4\sqrt{3}, \quad s = -2\sqrt{3}. \quad (3.10)$$

In the case where  $\beta_2 = \beta_1$ , we have the unique solution

$$\beta_3 = -\frac{4}{3}\sqrt{15}, \quad \alpha = -\frac{1}{3}\sqrt{15}, \quad \beta_2 = \beta_1 = \sqrt{15}, \quad s = \frac{2\sqrt{15}}{27}. \quad (3.11)$$

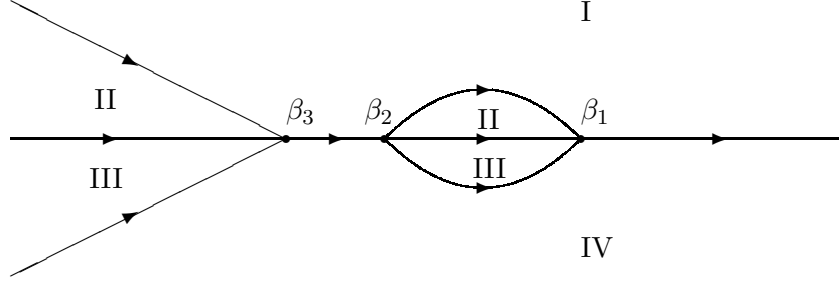


Figure 4: The jump contour  $\Sigma$ .

Those confluent cases correspond exactly to the values of  $s$  which are at the border between the algebraic and the elliptic region. For  $s$  in between the two critical values  $-2\sqrt{3}$  and  $\frac{2\sqrt{15}}{27}$ , the equations (3.7)-(3.9) are solvable [26]. Note that the different variable  $z = \frac{s}{\sqrt{6}}$  was used in [26].

Before proceeding with the RH analysis, let us write down some useful properties of the  $g$ -function, relying on (3.1) and (3.9):

$$g_+(\zeta) + g_-(\zeta) = 0, \quad \text{for } \zeta \in (-\infty, \beta_3) \cup (\beta_2, \beta_1), \quad (3.12)$$

$$g_+(\zeta) - g_-(\zeta) = 2g_+(\beta_2) = 2g_+(\beta_3) = -i\Omega, \quad \text{for } \zeta \in (\beta_3, \beta_2), \quad (3.13)$$

with

$$\Omega = \frac{1}{15} \int_{\beta_2}^{\beta_1} \sqrt{\xi - \beta_3}(\xi - \alpha) \sqrt{\xi - \beta_2} \sqrt{\beta_1 - \xi} d\xi. \quad (3.14)$$

### 3.2 Normalization of the RH problem and contour deformation

We will now use the  $g$ -function to normalize the RH problem at infinity in a suitable way. We define  $T$  in such a way that it has jumps on a lens-shaped contour as shown in Figure 4:

$$T(\zeta; x, t) := \begin{pmatrix} 1 & 0 \\ d_1 t^{1/4} & 1 \end{pmatrix} \times \begin{cases} Y_1(t^{1/2}\zeta; x, t) e^{t^{7/4}g(\zeta; x, t)\sigma_3}, & \text{in region I,} \\ Y_2(t^{1/2}\zeta; x, t) e^{t^{7/4}g(\zeta; x, t)\sigma_3}, & \text{in region II,} \\ Y_3(t^{1/2}\zeta; x, t) e^{t^{7/4}g(\zeta; x, t)\sigma_3}, & \text{in region III,} \\ Y_4(t^{1/2}\zeta; x, t) e^{t^{7/4}g(\zeta; x, t)\sigma_3}, & \text{in region IV,} \end{cases} \quad (3.15)$$

with  $d_1$  given by (3.3). Then  $T$  satisfies the following RH problem.

#### 3.2.1 RH problem for $T$

- (a)  $T$  is analytic in  $\mathbb{C} \setminus \Sigma$ , where  $\Sigma$  is as shown in Figure 4.

(b)  $T$  satisfies the following jump relations on  $\Sigma$ ,

$$T_+(\zeta) = T_-(\zeta) \begin{pmatrix} 1 & e^{-2t^{7/4}g(\zeta;x,t)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } \zeta > \beta_1,$$

$$T_+(\zeta) = T_-(\zeta) \begin{pmatrix} 1 & 0 \\ e^{2t^{7/4}g(\zeta;x,t)} & 1 \end{pmatrix}, \quad \text{for } \zeta \text{ off the real axis,}$$

$$T_+(\zeta) = T_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } \zeta \in (-\infty, \beta_3) \cup (\beta_2, \beta_1),$$

$$T_+(\zeta) = T_-(\zeta) \begin{pmatrix} e^{-it^{7/4}\Omega} & e^{-t^{7/4}(g_+(\zeta)+g_-(\zeta))} \\ 0 & e^{it^{7/4}\Omega} \end{pmatrix}, \quad \text{for } \zeta \in (\beta_3, \beta_2).$$

(c)  $T$  has asymptotics of the following form as  $\zeta \rightarrow \infty$ ,

$$T(\zeta) = (I + B_1(x, t)\zeta^{-1} + \mathcal{O}(\zeta^{-2})) t^{-\frac{1}{8}\sigma_3} \zeta^{-\frac{1}{4}\sigma_3} N. \quad (3.16)$$

Away from the branch points  $\beta_3, \beta_2, \beta_1$ , it is straightforward to verify that the off-diagonal entries in the jump matrices for  $T$  decay on  $\Sigma \setminus ((-\infty, \beta_3) \cup (\beta_2, \beta_1))$  as  $t \rightarrow \infty$ , if the lenses are chosen sufficiently close to the real axis.

### 3.3 Outside parametrix

Ignoring the small jumps and the branch points, we obtain the following RH problem.

**RH problem for  $P^{(\infty)}$**

(a)  $P^{(\infty)}$  is analytic in  $\mathbb{C} \setminus (-\infty, \beta_1]$ .

(b) On  $(-\infty, \beta_1)$ , we have

$$P_+^{(\infty)}(\zeta) = P_-^{(\infty)}(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } \zeta \in (-\infty, \beta_3) \cup (\beta_2, \beta_1), \quad (3.17)$$

$$P_+^{(\infty)}(\zeta) = P_-^{(\infty)}(\zeta) e^{-it^{7/4}\Omega\sigma_3}, \quad \text{for } \zeta \in (\beta_3, \beta_2). \quad (3.18)$$

(c)  $P^{(\infty)}$  behaves at  $\infty$  as

$$P^{(\infty)}(\zeta) = (I + D_1\zeta^{-1} + \mathcal{O}(\zeta^{-2})) t^{-\frac{1}{8}\sigma_3} \zeta^{-\frac{1}{4}\sigma_3} N. \quad (3.19)$$

Similar RH problems have been solved many times in the literature in terms of  $\theta$ -functions and meromorphic differentials, see e.g. [10, 9, 1, 23]. The only minor difference in our case is that  $\infty$  is a branch point. This RH problem can be solved explicitly using the Jacobi  $\theta$ -function and elliptic integrals. We will construct the parametrix explicitly and refer to [1] for the construction in more general settings.

In addition to the RH conditions stated above, we need to construct the outside parametrix in such a way that

$$P^{(\infty)}(\zeta) = \mathcal{O}(|\zeta - \beta_j|^{1/4}), \quad \text{as } \zeta \rightarrow \beta_j, \quad j = 1, 2, 3. \quad (3.20)$$

If this is not the case, the construction of local parametrices later on would fail.  $P^{(\infty)}$  has the form

$$P^{(\infty)}(\zeta) = \begin{pmatrix} 1 & 0 \\ dt^{1/4} & 1 + \frac{c_1}{\zeta - \beta_2} \end{pmatrix} \frac{1}{\sqrt{2}} t^{-\frac{\sigma_3}{8}} \gamma(\zeta)^{-\sigma_3} N \begin{pmatrix} h(\zeta) & 0 \\ 0 & \hat{h}(\zeta) \end{pmatrix}. \quad (3.21)$$

Here  $h, \hat{h}$  are scalar functions we will determine below, and

$$\gamma(\zeta) = \left( \frac{(\zeta - \beta_1)(\zeta - \beta_3)}{\zeta - \beta_2} \right)^{1/4}. \quad (3.22)$$

If we construct  $h, \hat{h}$  in such a way that they are analytic in  $\mathbb{C} \setminus (-\infty, \beta_1]$  and that they satisfy the jump conditions

$$h_+(\zeta) = h_-(\zeta)e^{-it^{7/4}\Omega}, \quad \text{for } \zeta \in (\beta_3, \beta_2), \quad (3.23)$$

$$\hat{h}_+(\zeta) = \hat{h}_-(\zeta)e^{it^{7/4}\Omega}, \quad \text{for } \zeta \in (\beta_3, \beta_2), \quad (3.24)$$

$$h_{\pm}(\zeta) = \hat{h}_{\mp}(\zeta), \quad \text{for } \zeta \in (-\infty, \beta_3) \cup (\beta_2, \beta_1), \quad (3.25)$$

it follows from (3.21) and (3.22) that the jump conditions (3.17)-(3.18) are satisfied. Moreover if  $h, \hat{h}$  admit expansions of the form

$$h(\zeta) = 1 + h_1\zeta^{-1/2} + h_2\zeta^{-1} + \mathcal{O}(\zeta^{-3/2}), \quad \text{as } \zeta \rightarrow \infty, \quad (3.26)$$

$$\hat{h}(\zeta) = 1 - h_1\zeta^{-1/2} + h_2\zeta^{-1} + \mathcal{O}(\zeta^{-3/2}), \quad \text{as } \zeta \rightarrow \infty, \quad (3.27)$$

and if we take  $d = h_1$ ,  $P^{(\infty)}$  has the expansion (3.19) with

$$D_{1,11} = \frac{\beta_3 - \beta_2 + \beta_1}{4} + h_2, \quad D_{1,12} = -h_1t^{-1/4}. \quad (3.28)$$

We will now construct  $h, \hat{h}$  explicitly. Let us consider the third Jacobi  $\theta$ -function

$$\theta(z; \tau) = \sum_{m=-\infty}^{+\infty} e^{2\pi imz + \pi i\tau m^2}, \quad (3.29)$$

which is symmetric in  $z$  and has the periodicity properties

$$\theta(z + 1; \tau) = \theta(z; \tau), \quad \theta(z + \tau; \tau) = e^{-\pi i\tau - 2\pi iz}\theta(z; \tau). \quad (3.30)$$

Now we let  $h, \hat{h}$  be of the form

$$h(\zeta) = \frac{\theta(0)}{\theta(c)} \frac{\theta(u(\zeta) + c)}{\theta(u(\zeta))}, \quad \hat{h}(\zeta) = \frac{\theta(0)}{\theta(c)} \frac{\theta(c - u(\zeta))}{\theta(u(\zeta))}, \quad (3.31)$$

where

$$u(\zeta) = \frac{1}{2C} \int_{\infty}^{\zeta} \frac{dz}{\sqrt{(z - \beta_3)(z - \beta_2)(z - \beta_1)}}, \quad (3.32)$$

$$C = \int_{\beta_3}^{\beta_2} \frac{dz}{\sqrt{(z - \beta_3)(z - \beta_2)(z - \beta_1)}} = \frac{2K(\sigma)}{\sqrt{\beta_1 - \beta_3}}, \quad (3.33)$$

$$\tau = i \int_{-\infty}^{\beta_3} \frac{dz}{C\sqrt{(\beta_3 - \zeta)(\beta_2 - \zeta)(\beta_1 - \zeta)}} = \frac{iK'(\sigma)}{K(\sigma)}, \quad (3.34)$$

$$\sigma = \sqrt{\frac{\beta_2 - \beta_3}{\beta_1 - \beta_3}}, \quad (3.35)$$

and  $K$  is the complete elliptic integral of the second kind. We then have  $u_+(\zeta) \equiv -u_-(\zeta) \pmod{\mathbb{Z}}$  for  $\zeta \in (-\infty, \beta_3) \cup (\beta_2, \beta_1)$ , and using (3.30) we obtain (3.25). For  $\zeta \in (\beta_3, \beta_2)$  we have

$u_+(\zeta) - u_-(\zeta) = \tau$ . Using the second property in (3.30), we obtain (3.23)-(3.24) if  $c = \frac{t^{7/4}\Omega}{2\pi}$ , with  $\Omega$  as in (3.14). Now we can calculate  $h_1, h_2$  in (3.26)-(3.27), which leads to

$$h_1 = \frac{1}{C} \left( \frac{\theta'}{\theta}(0) - \frac{\theta'}{\theta}(c) \right), \quad (3.36)$$

$$h_2 = \frac{1}{C^2} \left( \frac{\theta''}{2\theta}(c) - \frac{\theta''}{2\theta}(0) + \frac{\theta'^2}{\theta^2}(0) - \frac{\theta'}{\theta}(0) \frac{\theta'}{\theta}(c) \right). \quad (3.37)$$

From the standard theory of  $\theta$ -functions [13] it follows that  $\theta(u(\zeta))$  has its only zero at  $\zeta = \beta_2$ , so that  $P^{(\infty)}$  has no singularities other than the branch points. As  $\zeta \rightarrow \beta_3, \beta_1$ ,  $h$  and  $\hat{h}$  is bounded and this leads to (3.20). As  $\zeta \rightarrow \beta_2$ ,  $h(\zeta)$  and  $\hat{h}(\zeta)$  are of order  $\mathcal{O}(\zeta - \beta_2)^{-1/2}$ . Here we need to exploit the freedom to choose  $c_1$  in (3.21) in such a way that (3.20) holds. The value of  $c_1$  can be computed easily but is unimportant for us. This completes the construction of the parametrix.

### 3.4 Local parametrices near $\beta_3, \beta_2$ , and $\beta_1$

The local parametrices near the branch points can be constructed in the same way as in the algebraic case using the Airy function, we again refer to [9, 15, 7] for details. We do not need the precise form of the parametrices here, it is sufficient to have the existence of parametrices satisfying the conditions

- (a)  $P$  is analytic in a fixed neighborhood  $U_j$  of  $\beta_j$ ,
- (b)  $P$  satisfies exactly the same jump conditions than  $T$  inside  $U_j$ ,
- (c) For  $\zeta \in \partial U_j$ , we have  $P(\zeta) = P^{(\infty)}(\zeta)(I + \mathcal{O}(t^{-3/2}))$  as  $t \rightarrow \infty$ .
- (d)  $T(\zeta)P^{-1}(\zeta)$  is analytic at  $\beta_j$ .

It should be noted that the construction of the local parametrix relies on (3.13) and thus indirectly on (3.9). Condition (3.20) is also crucial.

### 3.5 Final transformation

Define

$$R(\zeta; x, t) = \begin{cases} T(\zeta; x, t)P^{(\infty)}(\zeta)^{-1}, & \text{for } \zeta \in \mathbb{C} \setminus U, \\ T(\zeta; x, t)P(\zeta)^{-1}, & \text{for } \zeta \in U_1 \cup U_2 \cup U_3. \end{cases} \quad (3.38)$$

Then after a similar argument as in the algebraic case, we have

$$R(\zeta; x, t) = I + \mathcal{O}(t^{-3/2}), \quad (3.39)$$

uniformly in  $\zeta$  as  $t \rightarrow \infty$  with  $s \in (-2\sqrt{3} + \delta, \frac{2\sqrt{15}}{27} - \delta)$ ,  $\delta > 0$ . As  $\zeta \rightarrow \infty$  we have

$$R(\zeta) = I + \frac{B_1(x, t) - D_1(x, t)}{\zeta} + \mathcal{O}(\zeta^{-2}), \quad \text{as } \zeta \rightarrow \infty, \quad (3.40)$$

which implies that

$$B_1(x, t) = D_1(x, t) + \mathcal{O}(t^{-3/2}), \quad (3.41)$$

and by (2.17) and (3.28),

$$y(x, t) = 2D_{1,11}t^{1/2} - D_{1,12}^2t + \mathcal{O}(t^{-1/2}) = \frac{\beta_3 - \beta_2 + \beta_1}{2}t^{1/2} + 2h_2t^{1/2} - h_1^2t^{1/2} + \mathcal{O}(t^{-1/2}). \quad (3.42)$$

Using (3.36)-(3.37), we obtain

$$y(x, t) = \frac{\beta_3 - \beta_2 + \beta_1}{2} t^{1/2} - t^{1/2} C^{-2}(\log \theta)''(0) + t^{1/2} C^{-2}(\log \theta)''(c) + \mathcal{O}(t^{-1/2}). \quad (3.43)$$

By standard manipulations for  $\theta$ -functions and elliptic integrals, this leads to (1.12).

## 4 Critical asymptotics for $y$ and possible generalizations

In this section, we indicate how the critical expansions (1.19) and (1.20) can be obtained. Afterwards we will make some remarks about asymptotics for the Brézin-Marinari-Parisi solutions to higher members of the Painlevé I hierarchy and about asymptotics for certain unbounded KdV solutions.

### Painlevé II asymptotics

For  $s$  near  $-2\sqrt{3}$ , we can proceed as in the algebraic case, see Section 2, with some modifications. The first problem is that Proposition 2.1 does not hold for  $\zeta$  near  $-\sqrt{3}$ . This leads to jumps on the lines  $z_0 + e^{\pi \pm \theta_0}$  which are not uniformly close to the identity matrix. In order to overcome this, we need to close lenses again at the point  $-\sqrt{3}$ , so that we have a contour as in Figure 4 but with  $\beta_3 = \beta_2$ . Then the jumps will converge to the identity matrix as  $t \rightarrow \infty$  except in a small neighborhood of  $-\sqrt{3} < z_0$ . Near this point, we need to construct a local parametrix built out of  $\Psi$ -functions associated with the Hastings-McLeod solution to Painlevé II [14, 20]. This construction is essentially the same as the one in [5]. The calculations that finally lead to (1.19) are rather tedious, and as they are similar as in [5], we do not think it is appropriate to include the details in this paper.

### Solitonic asymptotics

For  $s$  near  $\frac{2\sqrt{15}}{27}$ , we again proceed as in Section 2, but now Proposition 2.1 breaks down near  $\sqrt{15} > z_0$ . There is no need to modify the jump contour for  $T$  here. It is however necessary to build a local parametrix near  $\sqrt{15}$ . This time the parametrix has to be constructed using Hermite polynomials, similarly as in [6]. The degree of the Hermite polynomials will depend on the value of  $\xi$  in (1.20). If  $\xi$  is close to a half positive integer, a transition to Hermite polynomials of higher degree takes place, and this requires a modified local parametrix. A long calculation for which we refer to [6] leads to (1.20).

### Higher members of the Painlevé I hierarchy

The Painlevé I hierarchy contains an infinite number of equations  $P_I^m$  of order  $2m$  with  $m = 1, 2, \dots$ . The equation for  $m = 1$  is the Painlevé I equation  $y_{xx} = x + 6y^2$ , and the equation for  $m = 2$  is, up to a transformation  $x \mapsto ax, t \mapsto bt$  given by (1.1). Brézin, Marinari, and Parisi [3] considered not only the case  $m = 2$ , but the general case  $m = 2k$ . They believed that for any  $k \in \mathbb{N}$ , there is a real pole-free solution  $y_k$  to the  $2k$ -th member of the hierarchy which has asymptotics of the form

$$y_k(x) \sim \mp |c_k x|^{\frac{1}{2k+1}}, \quad \text{as } x \rightarrow \pm\infty. \quad (4.1)$$

This conjecture was supported by Moore [25] when he considered the RH problem for the  $2k$ -th member of the hierarchy. Although the general RH problem for the  $P_I^{2k}$  equation has  $4k + 2$  Stokes multipliers, only three among them are non-zero for the special solution under consideration: the one corresponding to the positive real line and the ones corresponding to the two anti-Stokes lines closest to the negative real line, all three of them being equal to



1 with the orientation as in Figure 2. Adding  $2k - 1$  monodromy preserving time parameters to the RH problem as in (1.27), where  $\theta$  now has a leading order term  $c\zeta^{\frac{4k+3}{2}}$ , one can study long time/space double scaling asymptotics as we did for  $k = 1$ . Following the general procedures of the Riemann-Hilbert analysis for Painlevé equations [15], one expects again regions of algebraic and elliptic asymptotic behavior, but in addition also regions of hyperelliptic behavior. The transitions between those regions would be interesting to study as well, and might lead to more general Painlevé II hierarchy asymptotics, a more general form of solitonic asymptotics, and possibly also to critical asymptotics in terms of the special solutions to the  $P_I^{2j}$  equation with  $j < k$ .

### Unbounded solutions to the KdV equation

We already mentioned that the RH problem for  $P_I^2$  generates solutions to the KdV equation by (1.29), also in the case where the Stokes multipliers  $s_0, \dots, s_6$  depend on  $\zeta$ . For generic choices of  $s_j(\zeta)$ ,  $y$  will have poles at certain values of  $x$  and  $t$ . However, for initial data satisfying  $y(x, 0) = \mathcal{O}(|x|^{1/3})$  as  $x \rightarrow \pm\infty$ , the Cauchy problem for KdV is well-posed [24]. It would be interesting to see if such solutions can be generated by choosing appropriate Stokes multipliers  $s_j(\zeta)$ , and if those solutions have asymptotic expansions similar to the ones for  $y(x, t)$ .

### Acknowledgements

The author is grateful to T. Grava and B. Dubrovin for useful discussions and comments. He is a Postdoctoral Fellow of the Fund for Scientific Research - Flanders (Belgium), and was also supported by Belgian Interuniversity Attraction Pole P06/02, by the ESF program MISGAM, and by ERC Advanced Grant FroMPDEs.

### References

- [1] P. Bleher, Lectures on random matrix models: the Riemann-Hilbert approach, arXiv:0801.1858v2.
- [2] M.J. Bowick and E. Brézin, Universal scaling of the tail of the density of eigenvalues in random matrix models, *Phys. Lett. B* **268** (1991), no. 1, 21–28.
- [3] E. Brézin, E. Marinari, and G. Parisi, A non-perturbative ambiguity free solution of a string model, *Phys. Lett. B* **242**, no. 1, (1990), 35–38.
- [4] T. Claeys and T. Grava, Universality of the break-up profile for the KdV equation in the small dispersion limit using the Riemann-Hilbert approach. *Comm. Math. Phys.* **286** (2009), 979–1009.
- [5] T. Claeys and T. Grava, Painlevé II asymptotics near the leading edge of the oscillatory zone for the Korteweg-de Vries equation in the small dispersion limit, *Comm. Pure Appl. Math.* **63** (2010), 203–232.
- [6] T. Claeys and T. Grava, Solitonic asymptotics for the Korteweg-de Vries equation in the small dispersion limit, arXiv:0911.5686.
- [7] T. Claeys and M. Vanlessen, The existence of a real pole-free solution of the fourth order analogue of the Painlevé I equation, *Nonlinearity* **20** (2007), 1163–1184.

- [8] T. Claeys and M. Vanlessen, Universality of a double scaling limit near singular edge points in random matrix models, *Comm. Math. Phys.* **273** (2007), 499–532.
- [9] P. Deift, T. Kriecherbauer, K.T-R McLaughlin, S. Venakides, and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.* **52** (1999), 1491–1552.
- [10] P. Deift, S. Venakides, and X. Zhou, New result in small dispersion KdV by an extension of the steepest descent method for Riemann-Hilbert problems. *Internat. Math. Res. Notices* **6** (1997), 285–299.
- [11] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, *Ann. Math.* **137** (1993), no. 2, 295–368.
- [12] B. Dubrovin, On Hamiltonian perturbations of hyperbolic systems of conservation laws, II: *Comm. Math. Phys.* **267** (2006), no. 1, 117–139.
- [13] J. Fay, “Theta Functions on Riemann Surfaces”, Lecture Notes in Mathematics **352**, SpringerVerlag (1970).
- [14] H. Flaschka and A.C. Newell, Monodromy and spectrum-preserving deformations I, *Comm. Math. Phys.* **76** (1980), no. 1, 65–116.
- [15] A.S. Fokas, A.R. Its, A.A. Kapaev, and V.Yu. Novokshenov, “Painlevé transcendents: the Riemann-Hilbert approach”, AMS Mathematical Surveys and Monographs **128** (2006).
- [16] R. Garifullin, B. Suleimanov, and N. Tarkhanov, Phase Shift in the Whitham Zone for the Gurevich-Pitaevskii Special Solution of the Korteweg-de Vries Equation, arXiv:0912.4853.
- [17] T. Grava and F.-R. Tian, The generation, propagation, and extinction of multiphases in the KdV zero-dispersion limit, *Comm. Pure Appl. Math.* **55** (2002), no. 12, 1569–1639.
- [18] A. V. Gurevich, L.P. Pitaevskii, Nonstationary structure of a collisionless shock wave *Soviet Phys. JETP* **38** (1974), no. 2, 291–297.
- [19] S.P. Hastings and J.B. McLeod, A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation, *Arch. Rational Mech. Anal.* **73** (1980), 31–51.
- [20] A.R. Its and V.Yu. Novokshenov, “The isomonodromic deformation method in the theory of Painlevé equations”, Lecture Notes in Math. **1191**, Springer-Verlag, Berlin - New York (1986).
- [21] A.A. Kapaev, Weakly nonlinear solutions of the equation  $P_I^2$ , *Journal of Mathematical Sciences* **73**, no. 4, (1995), 468–481.
- [22] A.A. Kapaev, Quasi-linear Stokes phenomenon for the Painlevé first equation, *J. Phys. A* **37**, no. 46, (2004), 11149–11167.
- [23] A.B.J. Kuijlaars and M.Y. Mo, The global parametrix in the Riemann-Hilbert steepest descent analysis for orthogonal polynomials, preprint arXiv:0909.5626.
- [24] A. Menikoff, The existence of unbounded solutions of the Korteweg-de Vries equation, *Comm. Pure Appl. Math.* **25** (1972), 407–432.

- [25] G. Moore, Geometry of the string equations, *Comm. Math. Phys.* **133**, (1990), no. 2, 261–304.
- [26] G.V. Potemin, Algebro-geometric construction of self-similar solutions of the Whitham equations, *Russian Math. Surveys* **43** (1988), 252-253.
- [27] F.-R. Tian, Oscillations of the zero dispersion limit of the Korteweg-de Vries equation, *Comm. Pure Appl. Math.* **46** (1993), 1093–1129.
- [28] S. Venakides, The Korteweg de Vries equations with small dispersion: higher order Lax-Levermore theory, *Comm. Pure Appl. Math.* **43** (1990), 335–361.
- [29] G.B. Whitham, *Linear and nonlinear waves*, J. Wiley, New York (1974).

Tom Claeys  
Cité Scientifique - Laboratoire Painlevé M2  
F-59655 Villeneuve d'Ascq, FRANCE  
E-mail: tom.claeys@math.univ-lille1.fr