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Pole-free solutions of the first Painlevé hierarchy and non-generic critical behavior for the KdV equation

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Dedicated to Boris Dubrovin on the occasion of his sixtieth birthday

Abstract

We establish the existence of real pole-free solutions to all even members of the Painlevé I hierarchy. We also obtain asymptotics for those solutions and describe their relevance in the description of critical asymptotic behavior of solutions to the KdV equation in the small dispersion limit. This was understood in the case of a generic critical point, and we generalize it here to the case of non-generic critical points.

1 Introduction and statement of results

We first introduce the Painlevé I hierarchy, which has the Painlevé I equation

$$q_{ss} = s + 6q^2 \tag{1.1}$$

as its first member. The m-th equation of the hierarchy is of order $2m$ and is defined recursively. It has the form

$$s + \mathcal{L}_m(q) + \sum_{j=1}^{m-1} t_j \mathcal{L}_{j-1}(q) = 0, \qquad t_1, \ldots, t_{m-1} \in \mathbb{R}, \tag{1.2}$$

where $\mathcal{L}_m$ is the Lenard-Magri recursion operator defined by

$$\mathcal{L}_0(q) = -4q, \tag{1.3}$$

$$\frac{d}{ds}\mathcal{L}_{k+1}(q) = \left(\frac{1}{4}\frac{d^3}{ds^3} - 2q\frac{d}{ds} - q_s\right)\mathcal{L}_k(q), \qquad \text{for } k = 0, \ldots, m-1. \tag{1.4}$$

The constants of integration in (1.4) are fixed by the requirements $\mathcal{L}_1(0) = \cdots = \mathcal{L}_m(0) = 0$. One could also add a term $t_m\mathcal{L}_{m-1}$ in (1.2), but this term cancels after a simple

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transformation of the other variables $q, s, t_1, \dots, t_{m-1}$. The first equations in the hierarchy are given by (up to multiplication by a non-zero constant)

$$
\begin{aligned}
m = 0: &\qquad s - 4q = 0, \\
m = 1: &\qquad q_{ss} = s + 6q^2, \\
m = 2: &\qquad q_{ssss} = 4s - 40q^3 + 10q_s^2 + 20qq_{ss} - 16t_1 q, \\
m = 3: &\qquad q^{(6)} = 16s + 28qq_{ssss} + 56q_s q_{sss} + 42q_{ss}^2 - 280(q^2 q_{ss} + qq_s^2 - q^4) \\
&\qquad\qquad + 16t_2(6q^2 - q_{ss}) - 64t_1 q, \\
m = 4: &\qquad q^{(8)} = 64s + 36qq^{(6)} + 108q_s q^{(5)} + 228q_{ss}q_{ssss} - 504q^2 q_{ssss} \\
&\qquad\qquad + 138q_{sss}^2 - 1512qq_{ss}^2 - 1848q_s^2 q_{ss} - 2016qq_s q_{sss} \\
&\qquad\qquad - 2016q^5 + 3360q^3 q_{ss} + 5040q^2 q_s^2 \\
&\qquad + 16t_3(-40q^3 + 10q_s^2 + 20qq_{ss} - q^{(4)}) \\
&\qquad\qquad + 64t_2(6q^2 - q_{ss}) - 256t_1 q.
\end{aligned}
$$

In the above equations we have written $q^{(j)}$ for the j-th derivative of q with respect to s. We will call equation (1.2) the P_{I}^m equation, and we refer to [34, 38, 40, 43, 26] for more information about the first Painlevé hierarchy. Given $t_1, \dots, t_{m-1}$, solutions to these equations are meromorphic functions in the complex s-plane, with in general an infinite number of poles [40]. Heuristic arguments supporting the existence of real pole-free solutions to the even members of the hierarchy were already given in [3] in the case where $t_1 = \dots = t_{m-1} = 0$. Their asymptotic behavior was discussed in [37]. A particular solution to the second member of the hierarchy appeared in [41, 42] in relation with the Gurevich-Pitaevskii special solutions to the KdV equation [27, 39]. This solution corresponds to the solution studied in [3] for $t_1 = 0$, and has applications in the study of ideal incompressible liquids [32, 33] and quantum gravity [15], see also [25].

After the rescalings

$$U = -60^{2/7} \cdot q, \qquad X = 60^{-1/7} \cdot s, \qquad T = -4 \cdot 60^{-3/7} \cdot t_1,$$

the second member of the hierarchy P_{I}^2 becomes

$$X = TU - \left(\frac{1}{6}U^3 + \frac{1}{24}(U_X^2 + 2UU_{XX}) + \frac{1}{240}U_{XXXX} \right). \tag{1.5}$$

This equation was studied by Dubrovin in [16], where he conjectured the existence and uniqueness of a real solution without poles for real values of X and T. The existence of such a solution was proved in [6] together with the asymptotic behavior $U(X,T) \sim \mp(6|X|)^{1/3}$ as $X \to \pm\infty$. In addition it is known that $U(X,T)$ is also a solution to the KdV equation $U_T + UU_X + \frac{1}{12}U_{XXX} = 0$ [27, 39, 41, 42].

As a first result in this paper, we will prove the existence of real pole-free solutions for all even members of the hierarchy, we will obtain asymptotics for them, and show that they follow, as functions of the time variables $t_1, \dots, t_{m-1}$, the time flows of the KdV hierarchy.

Theorem 1.1 *Let m be an even positive integer. There exists a solution $q = q(s, t_1, \dots, t_{m-1})$ to equation (1.2) which has the properties* *(i) q is real and has no poles for real values of $s, t_1, \ldots, t_{m-1}$; for $s, t_1, \ldots, t_{m-1}$ in a sufficiently small neighborhood of the real line, q depends analytically on each of its variables,*

(ii) q satisfies the PDE

$$q_{t_k} + \frac{1}{2k+1}\frac{d}{ds}\mathcal{L}_k = 0, \qquad \text{for } k = 1, \ldots, m-1, \tag{1.6}$$

which is (up to re-scaling) the k-th equation in the KdV hierarchy,

(iii) for $t_1, \ldots, t_{m-1} = 0$, q has the asymptotic behavior

$$q(s, 0, \ldots, 0) = c|s|^{\frac{1}{m+1}} + \mathcal{O}(|s|^{-\frac{m}{m+1}}), \qquad \text{as } s \to \pm\infty, \tag{1.7}$$

with

$$c = \frac{\operatorname{sgn}(s)}{2}\left(\frac{2^{m-1}(m+1)!}{(2m+1)!!}\right)^{\frac{1}{m+1}}. \tag{1.8}$$

Remark 1.2 The above results are not true for $m = 1$, since it is known that there do not exist real pole-free solutions to the Painlevé I equation [1]. Also for $m > 1$ odd, we do not expect that our results can be generalized. Parts (i) and (iii) are the essential parts of the theorem, part (ii) will follow from rather standard arguments that express the relation between the Painlevé I hierarchy and the KdV hierarchy.

Remark 1.3 For $t_1, \ldots, t_{m-1} \in \mathbb{R}$ fixed, our asymptotic analysis can be generalized to obtain the asymptotics

$$q(s, t_1, \ldots, t_{m-1}) = c|s|^{\frac{1}{m+1}} + \mathcal{O}(|s|^{-\frac{1}{m+1}}), \qquad \text{as } s \to \pm\infty, \tag{1.9}$$

see Remark 2.4 below. Note that the error term is weaker in the case of non-zero t_j's than in (1.7).

Remark 1.4 The leading order of the asymptotic behavior of q as $s \to \pm\infty$ for fixed $t_1, \ldots, t_{m-1}$ is relatively simple, and can be formally obtained when neglecting all derivatives of q in the equations of the hierarchy. It was proved in [8] that solutions to the Painlevé I hierarchy with this asymptotic behavior exist as $x \to \infty$ in a sector containing the positive real line. If one takes double scaling limits where $t_1, \ldots, t_{m-1}$ tend to infinity simultaneously with s, the situation becomes more complicated. Then the type of asymptotics will depend on the precise scaling of all variables. For example it can be expected that the asymptotics for q can be expressed in terms of elliptic θ-functions in some regions and, for $m > 2$, in terms of hyperelliptic θ-functions in other regions. For critical scalings of the variables, one can even expect asymptotics for $q = q_m$ in terms of the pole-free solutions $q_2, q_4, \ldots, q_{m-2}$ of the lower order equations in the Painlevé I hierarchy, and in terms of certain solutions to the Painlevé II hierarchy, see the discussion in [32] and [4].

1.1 Critical behavior for KdV solutions in the small dispersion limit

Let us consider the KdV equation

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0, \qquad \epsilon > 0. \tag{1.10}$$

For small ϵ, this is an example of a Hamiltonian perturbation of the Hopf equation $u_t + 6uu_x = 0$. Solutions to the Hopf equation exist only for small times, and develop a point of gradient catastrophe after a certain time. Indeed, consider for example initial data u_0 which are negative, smooth, tending to 0 rapidly at $\pm\infty$ and with a single local minimum. Then the method of characteristics describes the solution in terms of the initial data:

$$u(x,t) = u_0(\xi), \qquad x = 6tu_0(\xi) + \xi,$$

and the slope becomes infinite at the critical time

$$t_c = \frac{1}{\max_{\xi\in\mathbb{R}}[-6u_0'(\xi)]}.$$

The point x_c and time t_c where u_x blows up, and the value $u_c = u(x_c, t_c)$ are also determined by the equations

$$F(u;x,t) := -x + 6ut + f_L(u) = 0, \tag{1.11}$$
$$F'(u;x,t) = 6t + f_L'(u) = 0, \tag{1.12}$$
$$F''(u;x,t) = f_L''(u) = 0, \tag{1.13}$$

where f_L is the inverse of the decreasing part of the initial data u_0. For generic initial data we have $f_L'''(u_c) \neq 0$, which means that the Hopf solution behaves locally as

$$u(x,t_c) = u_c - c(x-x_c)^{1/3} + \mathcal{O}(x-x_c), \qquad \text{as } x \to x_c. \tag{1.14}$$

For non-generic initial data however, it can happen that $f_L'''(u_c) = 0$ and that

$$u(x,t_c) = u_c - c(x-x_c)^{\frac{1}{m+1}} + \mathcal{O}(x-x_c), \qquad \text{as } x \to x_c, \tag{1.15}$$

for any even value of m. This is the case if the initial data are such that

$$f_L^{(3)}(u_c) = f_L^{(4)}(u_c) = \ldots = f_L^{(m)}(u_c) = 0, \qquad f_L^{(m+1)}(u_c) \neq 0. \tag{1.16}$$

It was conjectured by Dubrovin [16, 17, 18, 19, 20] that the behavior of generic solutions to any Hamiltonian perturbation of a hyperbolic equation near the critical point of the unperturbed equation is described universally in terms of the pole-free solution to the second member (of order 4) of the Painlevé I hierarchy: there should be an expansion of the form

$$u(x,t,\epsilon) = u_c + a_1\epsilon^{2/7} U\left(a_2\epsilon^{-6/7}(x - x_c - a_3(t-t_c)), a_4\epsilon^{-4/7}(t-t_c)\right) + \mathcal{O}\left(\epsilon^{4/7}\right), \tag{1.17}$$

in a double scaling limit where $\epsilon \to 0$ and simultaneously $x \to x_c$, $t \to t_c$ in such a way that $\epsilon^{-6/7}(x - x_c - a_3(t-t_c))$ and $\epsilon^{-4/7}(t-t_c)$ tend to real constants, and where U is the pole-free solution to equation (1.5). The values of $a_1, \ldots, a_4$ depend on the equation and the initial data, but not on x, t, ϵ. In the case of the KdV equation, this was proved afterwards [5] for analytic negative initial data with sufficient decay at $\pm\infty$ and with a single local minimum, under the condition that the gradient catastrophe for the Hopf equation is generic, i.e. $f_L'''(u; x_c, t_c) \neq 0$. If this condition is not satisfied, we will prove here that the KdV solution is no longer described in terms of the pole-free solution to the P_{I}^2 equation, but in terms of a solution to a higher order equation in the Painlevé I hierarchy satisfying the properties given in Theorem 1.1. The order of the equation is determined by the number of vanishing derivatives of f_L: if we have (1.16) for $m \in 2\mathbb{N}$, the pole-free solution to the P_{I}^m equation of order $2m$ will appear.

We consider initial data $u_0(x)$ in the class of negative functions with only one local minimum, and such that u_0 can be extended to an analytic function $u_0(z)$ in a region of the form

$$\mathcal{S} = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \tan\theta |\mathrm{Re}\, z|\} \cup \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \sigma\}$$

for some $0 < \theta < \pi/2$ and $\sigma > 0$. In addition we need sufficient decay at infinity,

$$u_0(x) = \mathcal{O}\left(\frac{1}{|x|^{3+s}}\right), \quad s > 0, \quad x \in \mathcal{S}, \quad x \to \infty. \tag{1.18}$$

The local minimum is localized at a point x_M and we assume that $u_0''(x_M) \neq 0$ and $u_0(x_M) = -1$.

Theorem 1.5 *Let $u_0(x)$ be initial data for the Cauchy problem of the KdV equation satisfying the conditions described above, and assume that we have (1.16) for $m \in 2\mathbb{N}$. Write $u_c = u(x_c, t_c)$ for the Hopf solution at the point x_c and time t_c of gradient catastrophe of the Hopf equation. We take a double scaling limit where we let $\epsilon \to 0$ and at the same time $x \to x_c$ and $t \to t_c$ in such a way that, for some $\tau_0, \tau_1 \in \mathbb{R}$,*

$$\lim \frac{x - x_c - 6u_c(t - t_c)}{k^{1/2}\epsilon^{\frac{2m+2}{2m+3}}} = \tau_0, \qquad -\lim \frac{3(t - t_c)}{k^{3/2}\epsilon^{\frac{2m}{2m+3}}} = \tau_1, \tag{1.19}$$

where

$$k = \left(-\frac{2^{m-1}}{(2m+1)!!} f_L^{(m+1)}(u_c)\right)^{\frac{2}{2m+3}} > 0. \tag{1.20}$$

In this double scaling limit the solution $u(x, t, \epsilon)$ of the KdV equation (1.10) admits the asymptotic expansion

$$u(x, t, \epsilon) = u_c - \frac{2}{k}\epsilon^{\frac{2}{2m+3}} q\left(\epsilon^{-\frac{2m+2}{2m+3}}\tau_0(x, t, \epsilon), \epsilon^{-\frac{2m}{2m+3}}\tau_1(t, \epsilon), 0, 0, \ldots, 0\right) + \mathcal{O}(\epsilon^{\frac{4}{2m+3}}), \tag{1.21}$$

where

$$\tau_0(x, t, \epsilon) = \frac{x - x_c - 6u_c(t - t_c)}{k^{1/2}}, \qquad \tau_1(t, \epsilon) = -\frac{3(t - t_c)}{k^{3/2}}. \tag{1.22}$$

Here $q(s, t_1, t_2, \ldots, t_{m-1})$ is a solution to equation (1.2) which has properties (i)-(ii)-(iii) given in Theorem 1.1.

Remark 1.6 In the case of a generic critical point where $m = 2$, Theorem 1.5 describes the main theorem proved in [5]. The result is new for $m > 2$.

Remark 1.7 It is likely that the above result holds also for other equations in the universality class of Dubrovin [16], which contains among others the Camassa-Holm equation, the de-focusing nonlinear Schrödinger equation, and the KdV hierarchy.

1.1.1 Correlation kernels in critical unitary random matrix ensembles

The first conjecture about the existence of real pole-free solutions to the P_I^m equations for even m was posed in random matrix theory [2, 3] in the case where $t_1 = \ldots = t_{m-1} = 0$. However, it leads no doubt that also the general case with non-zero t_j's is of interest when studying critical unitary random matrix ensembles. Consider the space of Hermitian $n \times n$ matrices with a probability distribution of the form

$$\frac{1}{Z_n} \exp(-n \operatorname{tr} V(M)) dM, \tag{1.23}$$

where V is a scalar real analytic function with sufficient growth at $\pm\infty$, for example a polynomial of even degree with positive leading coefficient. The limiting mean eigenvalue distribution for random matrices in such an ensemble has the form [10]

$$d\mu_V(x) = \psi_V(x)dx, \qquad \psi_V(x) = \prod_{j=1}^{k} \sqrt{(b_j - x)(x - a_j)}\, h(x), \quad \text{for } x \in \cup_{j=1}^{k}[a_j, b_j],$$

where h is a real analytic function. The number of intervals in the spectrum, the endpoints a_j, b_j, and $h(x)$ depend on the confining potential $V(x)$ and can be found in terms of the unique solution to an equilibrium problem. Generically h does not vanish on the intervals $[a_j, b_j]$, and in particular not at the endpoints a_j, b_j, so that the limiting mean eigenvalue density vanishes like a square root at the endpoints of the spectrum [36].

The two-point eigenvalue correlation kernel in the model (1.23) is given by

$$K_n(x,y) = \frac{e^{-\frac{n}{2}V(x)} e^{-\frac{n}{2}V(y)}}{x - y} \frac{\kappa_{n-1}}{\kappa_n} (p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)),$$

where the p_j's are polynomials orthonormal with respect to the weight e^{-nV} on the real line; the leading coefficient of p_j is κ_j. Scaling limits of the two-point kernel give rise to well-known limiting kernels such as the sine kernel in the bulk of the spectrum (where ψ_V is positive) and the Airy kernel near an endpoint a_j (or b_j) where $h(a_j) \neq 0$ (or $h(b_j) \neq 0$) [9, 11, 12]. Near points in the spectrum where $h(x) = 0$, more complicated transcendental kernels appear in double scaling limits. Near singular edge points where $h(b_j) = 0, h'(b_j = 0), h''(b_j) \neq 0$, a kernel related to the P_I^2 equation was obtained [7]. Near higher order singular points where $h(x) \sim c(x - b_j)^{m+\frac{1}{2}}$ for m even (the case where m is odd cannot occur in unitary random matrix ensembles of the form (1.23)), it is natural to expect a kernel related to the P_I^m equation. If one takes double scaling limits where V depends on n in a suitable way, the limiting kernel should depend on s and also on the $m-1$ time variables $t_1, \ldots, t_{m-1}$. This is a difference compared to the situation in Theorem 1.5, where only one time variable t_1 is non-zero.

Outline

In Section 2, we prove the results stated in Theorem 1.1 about the equations in the first Painlevé hierarchy. We will construct the pole-free solutions q in terms of a Riemann-Hilbert (RH) problem which depends on $s, t_1, \ldots, t_{m-1}$, and m. We will prove the solvability of this RH problem for m even and $s, t_1, \ldots, t_{m-1} \in \mathbb{R}$. This will imply the absence of real poles for q. Using a Lax pair argument, we will explain the relation between the RH

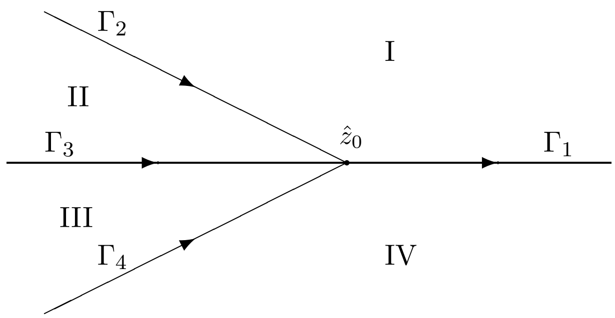

Figure 1: The jump contour Γ.

problem, the equations in the Painlevé I hierarchy, and the KdV hierarchy. An asymptotic analysis of the RH problem will lead to asymptotics for q as $s \to \pm\infty$.

In Section 3, we focus on Theorem 1.5. We will state a well-known RH problem which characterizes solutions to the Cauchy problem for the KdV equation, and rely on the techniques developed in [13, 14, 5] to transform this RH problem to an equivalent one suitable for asymptotic analysis as $\epsilon \to 0$. The main new point is the construction of a local parametrix using the model RH problem associated to the pole-free solution to the P_{I}^{m} equation studied in the first part.

2 Pole-free solutions to the Painlevé I hierarchy

In this section, we will construct the solutions q occurring in Theorem 1.1 in terms of a matrix RH problem. At several points, we will refer to [6] where the proof of Theorem 1.1 has been given in the case $m = 2$, and where a more detailed exposition can be found.

We consider the following RH problem.

RH problem for Ψ:

(a) $\Psi : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2\times 2}$ is analytic, where $\Gamma = \bigcup_{j=1}^{4} \Gamma_j$ is a contour consisting of four straight rays

$$\Gamma_1 : \arg(\zeta - \hat{z}_0) = 0, \qquad \Gamma_2 : \arg(\zeta - \hat{z}_0) = \theta,$$
$$\Gamma_3 : \arg(\zeta - \hat{z}_0) = \pi, \qquad \Gamma_4 : \arg(\zeta - \hat{z}_0) = -\theta,$$

each of them oriented from the left to the right, see Figure 1. Here $\hat{z}_0$ can be any real number, and $\frac{2m+1}{2m+3}\pi < \theta < \pi$.

(b) Ψ has continuous boundary values $\Psi_\pm(\zeta)$ when approaching $\zeta \in \Gamma \setminus \{\hat{z}_0\}$ from the left or right, and they are related by the jump conditions

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{2.1}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2 \cup \Gamma_4, \tag{2.2}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3. \tag{2.3}$$

(c) As $\zeta \to \infty$, Ψ has an expansion of the form

$$\Psi(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} N \left(I + h\sigma_3 \zeta^{-1/2} + \frac{1}{2} \begin{pmatrix} h^2 & -iq \\ iq & h^2 \end{pmatrix} \zeta^{-1} + \mathcal{O}(\zeta^{-3/2}) \right) e^{-\theta(\zeta; s, t_1, \ldots, t_{m-1})\sigma_3}, \tag{2.4}$$

where h, q do not depend on ζ, where

$$N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{1}{4}\pi i \sigma_3}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.5}$$

and

$$\theta(\zeta; s, t_1, \ldots, t_{m-1}) = \frac{4}{2m+3} \zeta^{\frac{2m+3}{2}} + \sum_{j=1}^{m-1} \frac{4}{2j+1} t_j \zeta^{\frac{2j+1}{2}} + s\zeta^{1/2}. \tag{2.6}$$

Remark 2.1 The jump contour $\Gamma = \Gamma(\hat{z}_0, \theta)$ of the RH problem depends on the values of $\hat{z}_0$ and θ, but the RH solution corresponding to $(\hat{z}_0, \theta)$ can be transformed directly to the RH solution for any other value of $(\hat{z}_0', \theta')$, as long as $\frac{2m+1}{2m+3}\pi < \theta' < \pi$. This can be done by analytic continuation of the RH solution Ψ across its jump contour. In this section, we will fix the values of $\hat{z}_0 = 0$ and $\theta = \frac{2m+2}{2m+3}\pi$, but for the asymptotic analysis in Section 2.3, we will need to choose $\hat{z}_0, \theta$ more carefully. Near $\hat{z}_0$, we need to impose that Ψ remains bounded in order to have a unique solution.

The RH problem depends on $s, t_1, \ldots, t_{m-1}$. For fixed real values of those parameters, we will show that there exists a (unique) RH solution $\Psi(\zeta)$ and constants h and q such that (2.4) holds. We have $\Psi = \Psi(\zeta; s, t_1, \ldots, t_{m-1})$, $h = h(s, t_1, \ldots, t_{m-1})$, and $q = q(s, t_1, \ldots, t_{m-1})$. We will show that q satisfies the conditions stated in Theorem 1.1.

Remark 2.2 If $m = 0$ and $s = 0$, this is a well-known RH problem which can be solved explicitly in terms of the Airy function and its derivative. For $m = 2$, $\tilde{\zeta} = 60^{2/7}\zeta$, $T = -4 \cdot 60^{-3/7} t_1$, and $X = 60^{-1/7} s$, we obtain the RH problem studied in [6] related to the pole-free solution to the fourth order equation (1.5) which is equivalent to equation (1.2) for $m = 2$.

2.1 Solvability of the RH problem

In order to prove the solvability of the RH problem for Ψ, we perform the transformation

$$\Phi(\zeta) = \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} \Psi(\zeta) e^{\theta(\zeta)\sigma_3}, \tag{2.7}$$

where we have suppressed the dependence on $s, t_1, \ldots, t_{m-1}$ in our notations. This leads to a slightly modified RH problem for Φ.

RH problem for Φ:

(a) Φ is analytic in $\mathbb{C} \setminus \Gamma$.

(b) We have the jump relations

$$\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 1 & e^{-2\theta(\zeta)} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{2.8}$$

$$\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 1 & 0 \\ e^{2\theta(\zeta)} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2 \cup \Gamma_4, \tag{2.9}$$

$$\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3. \tag{2.10}$$

(c) As $\zeta \to \infty$, Φ has an expansion of the form

$$\Phi(\zeta) = \left(I + \frac{A_1}{\zeta} + \mathcal{O}(\zeta^{-2}) \right) \zeta^{-\frac{1}{4}\sigma_3} N. \tag{2.11}$$

It is straightforward to verify by (2.4) and (2.7) that q is given by

$$q = A_{1,12}^2 - 2A_{1,11}. \tag{2.12}$$

Following the general procedure developed in [24, 23, 11], the solvability of a large class of RH problems is equivalent to the fact that a homogeneous version of the RH problem has no solution except the trivial zero solution. This follows from the observation that a singular integral operator associated to the RH problem is a Fredholm operator of index zero. This procedure is explained in detail in [11, 28] and in [6] for the above RH problem in the case where $m = 2$. All of the arguments work for general m and general real values of the t_j's and we will not repeat them here. In order to have solvability of the RH problem for Φ, it is sufficient to prove the following vanishing lemma.

Lemma 2.3 (Vanishing lemma) *Let $s, t_1, \ldots, t_{m-1} \in \mathbb{R}$, let Φ_0 satisfy conditions (a) and (b) of the RH problem for Φ, and let in addition*

$$\Phi_0(\zeta) = \mathcal{O}(\zeta^{-3/4}), \qquad \text{as } \zeta \to \infty. \tag{2.13}$$

Then $\Phi_0 \equiv 0$.

Proof. The proof of the vanishing lemma is almost the same as in [6, 11] and goes as follows. We first collapse the jump contour Γ in the RH problem for Φ to the real line. We do this by defining

$$A(\zeta) = \Phi_0(\zeta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \text{for } 0 < \arg\zeta < \theta,$$

$$A(\zeta) = \Phi_0(\zeta) \begin{pmatrix} 1 & 0 \\ e^{2\theta(\zeta)} & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \text{for } \theta < \arg\zeta < \pi,$$

$$A(\zeta) = \Phi_0(\zeta) \begin{pmatrix} 1 & 0 \\ -e^{2\theta(\zeta)} & 1 \end{pmatrix}, \qquad \text{for } -\pi < \arg\zeta < -\theta,$$

$$A(\zeta) = \Phi_0(\zeta), \qquad \text{for } -\theta < \arg\zeta < 0.$$

Then one verifies using the jump relations for Φ_0 that A is analytic in $\mathbb{C} \setminus \mathbb{R}$, and that it has the jump properties

$$A_+(\zeta) = A_-(\zeta) \begin{pmatrix} 1 & -e^{2\theta_+(\zeta)} \\ e^{2\theta_-(\zeta)} & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \mathbb{R}_-, \tag{2.14}$$

$$A_+(\zeta) = A_-(\zeta) \begin{pmatrix} e^{-2\theta(\zeta)} & -1 \\ 1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \mathbb{R}_+. \tag{2.15}$$

As $\zeta \to \infty$, the behavior of A follows from the asymptotics for Φ_0 and the fact that the exponentials in the definition of A are uniformly bounded. We have

$$A(\zeta) = \mathcal{O}(\zeta^{-3/4}), \qquad \text{as } \zeta \to \infty, \tag{2.16}$$

uniformly for $\zeta \in \mathbb{C} \setminus \mathbb{R}$. Then we define $Q(\zeta) = A(\zeta)A^H(\bar{\zeta})$, where A^H is the Hermitian conjugate of the matrix A. Q is clearly analytic in the upper half plane, because A is analytic in the upper and lower half plane, and it is continuous up to $\mathbb{R}$. As $\zeta \to \infty$, we have $Q(\zeta) = \mathcal{O}(\zeta^{-3/2})$, which implies that

$$\int_{\mathbb{R}} Q_+(\xi) d\xi = 0 \tag{2.17}$$

by Cauchy's theorem, and using (2.14)-(2.15) we obtain

$$\int_{\mathbb{R}^-} A_-(\xi) \begin{pmatrix} 1 & -e^{2\theta_+(\xi;s,t_1,\ldots,t_{m-1})} \\ e^{2\theta_-(\xi;s,t_1,\ldots,t_{m-1})} & 0 \end{pmatrix} A_-^H(\xi) d\xi + \int_{\mathbb{R}^+} A_-(\xi) \begin{pmatrix} e^{-2\theta(\xi;s,t_1,\ldots,t_{m-1})} & -1 \\ 1 & 0 \end{pmatrix} A_-^H(\xi) d\xi = 0. \tag{2.18}$$

For real values of $s, t_1, \ldots, t_{m-1}$, we have

$$\overline{\theta_+(\xi; s, t_1, \ldots, t_{m-1})} = \theta_-(\xi; s, t_1, \ldots, t_{m-1}), \qquad \text{for } \xi < 0,$$

and adding (2.18) to its Hermitian conjugate, we obtain

$$\int_{\mathbb{R}^-} A_-(\xi) \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} A_-^H(\xi) d\xi + \int_{\mathbb{R}^+} A_-(\xi) \begin{pmatrix} 2e^{-2\theta(\xi;s,t_1,\dots,t_{m-1})} & 0 \\ 0 & 0 \end{pmatrix} A_-^H(\xi) d\xi = 0. \quad (2.19)$$

Since $e^{-2\theta(\xi;s,t_1,\dots,t_{m-1})} > 0$, this implies that the first column of A_- is identically zero (because it is continuous). The jump conditions (2.14)-(2.15) can be used to prove that the second column of A_+ vanishes as well.

Writing out the jump relations for the entries of A for which we have not yet proved that they vanish, the RH problem for A decouples into two scalar RH problems. The same argument as in [11, Step 3 of Section 5.3] shows that those scalar RH problems have only the zero solution, and that $A \equiv 0$. Consequently we have that $\Phi_0 \equiv 0$, which proves the vanishing lemma. □

As a consequence of the vanishing lemma, the RH problem for Φ, and thus also for Ψ (one can invert the transformation defined by (2.7)), has a solution for real values of the parameters $s, t_1, \dots, t_{m-1}$.

It follows from the general theory of RH problems that the subset of $\mathbb{C}^m$ of values $(s, t_1, \dots, t_{m-1})$ where the RH problem is solvable, is an open set, and this implies in particular that, for any $(s, t_1, \dots, t_{m-1}) \in \mathbb{R}^m$, there exists a neighborhood in $\mathbb{C}^m$ such that the RH problem is solvable also in this neighborhood. Moreover if this neighborhood is chosen sufficiently small, condition (2.4) is valid uniformly, see [6], and Ψ, q, and h depend analytically on each of the variables $s, t_1, \dots, t_{m-1}$. Values of $(s, t_1, \dots, t_{m-1})$ where the RH problem is not solvable, correspond to poles of q.

2.2 Relation between the RH problem, the Painlevé I hierarchy, and the KdV hierarchy

We will show that the function q appearing in the asymptotic expansion (2.4) solves equation $P_{\rm I}^m$, and that it also solves the equations in the KdV hierarchy. This follows from Lax pair arguments which are rather standard, see e.g. [21, 22, 29] in general and [35, 43] for the Lax pair corresponding to the first Painlevé hierarchy. We recall the arguments briefly for the reader's convenience.

2.2.1 Lax pair in s, ζ, and t_k

Since Ψ is differentiable in s, we can define

$$L(\zeta) = \Psi_s(\zeta)\Psi(\zeta)^{-1}, \qquad A(\zeta) = \Psi_\zeta(\zeta)\Psi(\zeta)^{-1}. \quad (2.20)$$

The jump matrices for Ψ do not depend on ζ and s, so L and A are entire functions in the complex plane, and because of the asymptotics (2.4), they are polynomials in ζ. For L, one deduces directly from (2.4) that

$$L(\zeta) = \begin{pmatrix} 0 & 1 \\ \zeta + (h_s + q) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \zeta + 2q & 0 \end{pmatrix}, \quad (2.21)$$

where it was used in the latter equation that $h_s = q$. This follows by developing $\Psi_s \Psi^{-1}$ as $\zeta \to \infty$ and imposing that the 12-entry of the ζ^{-1}-term is zero. For the ζ-derivative, we use (2.4) to conclude that A is a polynomial of degree $m+1$,

$$A(\zeta) = \sum_{j=0}^{m+1} A_j \zeta^j, \tag{2.22}$$

where the A_j's depend on $s, t_1, \ldots, t_{m-1}$ but not on ζ. Let us take a closer look at the 12-entry $\beta(\zeta) := A_{12}(\zeta)$: by (2.4) and (2.20), it has the form

$$\beta(\zeta) = \beta^{(m+1)}(\zeta) + \sum_{k=1}^{m-1} t_k \beta^{(k)}(\zeta), \tag{2.23}$$

where $\beta^{(k)}$ is a polynomial of degree $k-1$ which can be written in the form

$$\beta^{(k)}(\zeta) = \sum_{j=0}^{k-1} \frac{\mathcal{L}_{k-j-2}}{2} \zeta^j, \tag{2.24}$$

where $\mathcal{L}_j$ is independent of the value of k in the above formula, and $\mathcal{L}_{-1} = 4, \mathcal{L}_0 = -4q$. Similar formulas can be obtained for the other entries but are not needed.

Now, as in [43], write

$$A(\zeta) = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \tag{2.25}$$

(it is easily verified that $\operatorname{Tr} A \equiv 0$). Then $\Psi_{\zeta s} = \Psi_{s\zeta}$ implies the compatibility condition

$$L_\zeta - A_s + [L, A] = 0, \tag{2.26}$$

and if we write this down entry-wise, we find

$$\alpha_s + \beta(\zeta + 2q) - \gamma = 0, \tag{2.27}$$

$$\beta_s + 2\alpha = 0, \tag{2.28}$$

$$\gamma_s - 1 - 2\alpha(\zeta + 2q) = 0. \tag{2.29}$$

We solve the second equation for α and then the first for γ, and we substitute their values in the third equation. This gives

$$\frac{1}{2}\beta_{sss} - 2\beta_s(\zeta + 2q) - 2q_s\beta + 1 = 0. \tag{2.30}$$

The degree $m+1$ term in (2.30) is trivial, and the degree m term reads $4q_s + \mathcal{L}_{0,s} = 0$, which we knew already since $\mathcal{L}_0 = -4q$. From the term of degree j with $1 \le j \le m-1$, we get

$$\frac{1}{2}\mathcal{L}_{m-j-1,sss} - 2\mathcal{L}_{m-j,s} - 4q\mathcal{L}_{m-j-1,s} - 2q_s\mathcal{L}_{m-j-1} = 0, \tag{2.31}$$

and the constant term in (2.30) gives

$$1 + \frac{1}{4}\mathcal{L}_{m-1,sss} - 2q\mathcal{L}'_{m-1} - q_s\mathcal{L}_{m-1} + \sum_{j=1}^{m-1} t_j \left(\frac{1}{4}\mathcal{L}_{j-2,sss} - 2q\mathcal{L}'_{j-2} - q_s\mathcal{L}_{j-2}\right) = 0, \tag{2.32}$$

or

$$1+\frac{d}{ds}\mathcal{L}_m+\sum_{j=1}^{m-1}t_j\frac{d}{ds}\mathcal{L}_{j-1}=0. \tag{2.33}$$

Integrating this equation gives the P_I^m equation (1.2). One shows using the asymptotic behavior of solutions to the Schrödinger equation $\Psi_s = L\Psi$ that the constants of integration when integrating (2.33) and (2.31) are zero. This proves that q solves the P_I^m equation (1.2).

The jump matrices for Ψ are also independent of $t_1,\ldots,t_{m-1}$, and consequently $B^{(k)} = \Psi_{t_k}\Psi^{-1}$ is a polynomial of degree $k+1$ in ζ. Exploiting the compatibility of the t_k-derivative with the s-derivative, an analogous argument as before leads to the time flow

$$q_{t_k}+\frac{1}{2k+1}\frac{d}{ds}\mathcal{L}_k=0. \tag{2.34}$$

2.3 Asymptotics for q

We will now analyze the RH problem for Ψ asymptotically as $s\to\pm\infty$. We will use the Deift/Zhou steepest descent method to obtain asymptotics for Ψ. For $m=2$, this analysis has been done in [6], see also [30, 31]. For $m>2$, the general approach remains the same, but in particular the construction of the g-function and the contour deformation are more delicate.

2.3.1 Re-scaling of Ψ

Until now, we have always considered the RH problem for Ψ corresponding to $\hat{z}_0 = 0$ and $\theta = \frac{2m+2}{2m+3}\pi$, see Remark 2.1. For the asymptotic analysis of the RH problem, we will need other values of $\hat{z}_0$ and θ which we will specify later.

Define

$$Y(\zeta)=\begin{pmatrix}1 & 0\\ h & 0\end{pmatrix}\Psi(|s|^{\frac{1}{m+1}}\zeta;s,t_1,\ldots,t_{m-1}). \tag{2.35}$$

RH problem for Y:

(a) Y is analytic in $\mathbb{C}\setminus\Gamma(z_0,\theta)$, where $z_0 = |s|^{-\frac{1}{m+1}}\hat{z}_0$.

(b) We have the jump conditions

$$Y_+(\zeta)=Y_-(\zeta)\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \qquad \text{for } \zeta\in\Gamma_1, \tag{2.36}$$

$$Y_+(\zeta)=Y_-(\zeta)\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \qquad \text{for } \zeta\in\Gamma_2\cup\Gamma_4, \tag{2.37}$$

$$Y_+(\zeta)=Y_-(\zeta)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad \text{for } \zeta\in\Gamma_3. \tag{2.38}$$

(c) As $\zeta \to \infty$, we have

$$Y(\zeta) = \left(I + |s|^{-\frac{1}{m+1}} A_1 \zeta^{-1} + \mathcal{O}(\zeta^{-2})\right) |s|^{-\frac{1}{4m+4}\sigma_3} \zeta^{-\frac{\sigma_3}{4}} N e^{-|s|^{\frac{2m+3}{2m+2}} \hat{\theta}(\zeta; s, t_1, \ldots, t_{m-1})\sigma_3}. \tag{2.39}$$

where

$$\hat{\theta}(\zeta; s, t_1, \ldots, t_{m-1}) = |s|^{-\frac{2m+3}{2m+2}} \theta(|s|^{\frac{1}{m+1}}\zeta; s, t_1, \ldots, t_{m-1}) \tag{2.40}$$

$$= \frac{4}{2m+3}\zeta^{\frac{2m+3}{2}} + \mathrm{sgn}(s)\zeta^{1/2} + \sum_{j=1}^{m-1} \frac{4}{2j+1} t_j |s|^{\frac{j-m-1}{m+1}} \zeta^{\frac{2j+1}{2}}. \tag{2.41}$$

The matrix A_1 in (2.39) is the same as in (2.11), so (2.12) holds.

2.3.2 Construction of the g-function

We proceed with our analysis in the case where $t_1, \ldots, t_{m-1} = 0$. Straightforward modifications described in Remark 2.4 allow us to treat also the general case where the t_j's are fixed. We search a g-function $g = g(\zeta; s)$ of the form

$$g(\zeta) = (\zeta - z_0)^{3/2} p\left(\frac{\zeta}{z_0}\right), \tag{2.42}$$

where $p(z)$ is a polynomial of degree m. Its coefficients and z_0 are uniquely determined by the condition

$$g(\zeta) = \hat{\theta}(\zeta) + \mathcal{O}(\zeta^{-1/2}), \qquad \text{as } \zeta \to \infty. \tag{2.43}$$

We have

$$\hat{\theta}(\zeta) = \frac{4}{2m+3}\zeta^{\frac{2m+3}{2}} + \mathrm{sgn}(s)\zeta^{1/2}, \tag{2.44}$$

and because

$$(\zeta - z_0)^{-3/2} = \zeta^{-3/2}\left(1 + \sum_{j=1}^{\infty} \frac{(2j+1)!!}{2^j . j!} z_0^j \zeta^{-j}\right), \qquad \text{as } \zeta \to \infty, \tag{2.45}$$

(2.43) requires us to take p of the form

$$p(z) = \frac{4}{2m+3} z_0^m \sum_{j=0}^{m} c_j z^{m-j}, \qquad c_j = \frac{(2j+1)!!}{2^j . j!}. \tag{2.46}$$

The missing condition in order to have (2.43) is

$$z_0 = -\mathrm{sgn}(s)\left(\frac{2^{m-1}(m+1)!}{(2m+1)!!}\right)^{\frac{1}{m+1}}. \tag{2.47}$$

With this choice of g, we have

$$e^{|s|^{\frac{2m+3}{2m+2}}(g(\zeta)-\hat{\theta}(\zeta))\sigma_3} = I + \sum_{k=1}^{\infty} d_k \sigma_3^k \zeta^{-k/2}, \qquad \text{as } \zeta \to \infty, \tag{2.48}$$

where the coefficients d_k can be calculated but are not important. Since the determinant of the left hand side of (2.48) is 1, we have

$$d_2 = \frac{1}{2} d_1^2. \tag{2.49}$$

Remark 2.4 If the t_j's do not vanish, we need to modify p and z_0, but (2.43) still determines the coefficients c_j of the polynomial

$$p(z) = \frac{4}{2m+3} z_0^m \sum_{j=0}^{m} c_j z^{m-j} \tag{2.50}$$

uniquely. We don't need their explicit form, it is sufficient to know that

$$c_j = \frac{(2j+1)!!}{2^j.j!} + \mathcal{O}(|s|^{-\frac{2}{m+1}}), \qquad \text{as } |s| \to \infty, \tag{2.51}$$

$$z_0 = -\operatorname{sgn}(s) \left(\frac{2^{m-1}(m+1)!}{(2m+1)!!} \right)^{\frac{1}{m+1}} + \mathcal{O}(|s|^{-\frac{2}{m+1}}), \qquad \text{as } |s| \to \infty. \tag{2.52}$$

To prove Theorem 1.1, we only deal with the case where $t_1 = \ldots = t_{m-1} = 0$, but the entire analysis done below can be easily generalized as long as the t_j's remain bounded. Formula (1.9) will then follow from (2.74) together with (2.52).

Proposition 2.5 *We have*

$$g(\zeta) > 0, \qquad \textit{for } \zeta > z_0, \tag{2.53}$$

$$\operatorname{Im} g'_+(\zeta) > 0, \qquad \textit{for } \zeta < z_0. \tag{2.54}$$

Proof. For the first equality, we first prove that $p(z) > 0$ for $z \in (-\infty, -3/2] \cup [-1, +\infty)$. Since the coefficients of p are positive, this is clear for z positive. For $-1 < z < 0$, we have

$$p(z) = \frac{4}{2m+3} z_0^m \left((c_m - c_{m-1}|z|) + z^2(c_{m-2} - c_{m-3}|z|) + \ldots + z^{m-2}(c_2 - c_1|z|) + c_0 z^m \right),$$

and all the terms in this expression are positive since the c_j's increase with j. For $z < -3/2$, a similar argument shows that $p(z) > 0$ (the terms in the sum are alternating and their absolute value increases with the degree). This implies that $g(\zeta) > 0$ for $\frac{\zeta}{z_0} \in (-\infty, -3/2] \cup [-1, +\infty]$. In the case where $z_0 > 0$ (or $s < 0$), this implies (2.53), for $z_0 < 0$, we still need an estimate for $\frac{\zeta}{z_0} \in (-3/2, -1)$, or $\zeta \in (-z_0, -\frac{3}{2}z_0)$. We already know that g is positive at both endpoints of the interval, so it suffices to prove that $g(\zeta)$ is monotonic on $(-z_0, -\frac{3}{2}z_0)$.

Write $g'(\zeta) = (\zeta - z_0)^{1/2} q(\frac{\zeta}{z_0})$ with

$$q(z) = 2z_0^m \left(z^m + \sum_{j=1}^{m} b_j z^{m-j} \right), \qquad b_j = \frac{(2j-1)!!}{2^j j!}.$$

We will show that $q(z)$ is strictly positive for all $z \in \mathbb{R}$, which implies (2.54), and also the fact that $g(\zeta)$ is monotonic on $(-z_0, -\frac{3}{2}z_0)$, which completes the proof of the first inequality.

For $z \in (-\infty, -1] \cup [-\frac{1}{2}, +\infty)$, it is not difficult to see that $q(z)$ is positive. Indeed, the terms in the sum are either positive or alternating and monotonic with increasing degree. For $-1 < z < -\frac{1}{2}$, write

$$\frac{z_0^{-m}}{2} z^{2-m} q(z) = \left(z^2 + \frac{1}{2}z + \frac{3}{8}\right) + \sum_{k=2}^{\frac{m}{2}} \left(b_{2k-1} z^{-2k+3} + b_{2k} z^{-2k+2}\right). \quad (2.55)$$

The first part is bigger than $\frac{5}{16}$, and each part of the sum reaches its minimal value on $[-1, -1/2]$ at -1. Since the b_j's are decreasing, rearranging the alternating terms, we obtain the estimate

$$\frac{z_0^{-m}}{2} z^{2-m} q(z) \geq \frac{5}{16} + \sum_{k=2}^{\frac{m}{2}} (-b_{2k-1} + b_{2k}) \geq \frac{5}{16} - \frac{5}{16} + b_m > 0.$$

□

The above result enables us to apply the Cauchy-Riemann conditions, which leads to the following corollary.

Corollary 2.6 *There exists a $\theta > 0$ such that $\operatorname{Re} g(\zeta) < 0$ if $\arg(\zeta - z_0) = \pi \pm \theta$.*

We still have the freedom to choose the values of θ and z_0 that determine the jump contour $\Gamma(z_0, \theta)$ for Y. We take z_0 as in (2.47) and θ such that the above corollary holds.

2.3.3 Normalization of the RH problem

Define

$$S(\zeta) = \begin{pmatrix} 1 & 0 \\ d_1 |s|^{\frac{1}{2m+2}} & 1 \end{pmatrix} Y(\zeta) e^{|s|^{\frac{2m+3}{2m+2}} g(\zeta) \sigma_3}, \quad (2.56)$$

so that we have

RH problem for S:

(a) S is analytic in $\mathbb{C} \setminus \Gamma(z_0, \theta)$.

(b) For $\zeta \in \Gamma(z_0, \theta)$,

$$S_+(\zeta) = S_-(\zeta) \begin{pmatrix} 1 & e^{-2|s|^{\frac{2m+3}{2m+2}} g(\zeta)} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \quad (2.57)$$

$$S_+(\zeta) = S_-(\zeta) \begin{pmatrix} 1 & 0 \\ e^{2|s|^{\frac{2m+3}{2m+2}} g(\zeta)} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2 \cup \Gamma_4, \quad (2.58)$$

$$S_+(\zeta) = S_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3. \quad (2.59)$$

(c) As $\zeta \to \infty$, S behaves like

$$S(\zeta) = \left[I + B_1\zeta^{-1} + \mathcal{O}(\zeta^{-2})\right] |s|^{-\frac{\sigma_3}{4m+4}} \zeta^{-\frac{\sigma_3}{4}} N, \tag{2.60}$$

and q is given by

$$q = |s|^{\frac{2}{m+1}} B_{1,12}^2 - 2|s|^{\frac{1}{m+1}} B_{1,11}. \tag{2.61}$$

As $s \to \pm\infty$, the jump matrices for S tend to constant matrices except near z_0: indeed on Γ_1, Γ_2, and Γ_4 we have exponentially fast convergence to the identity matrix by Proposition 2.5 and Corollary 2.6, and on Γ_3, the jump matrix is identically equal to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

2.3.4 Global parametrix

In the limit $s \to \pm\infty$, if we ignore a small neighborhood of z_0, the RH problem for S reduces to a RH problem with a jump only on $(-\infty, z_0)$. We can explicitly construct a solution $P^{(\infty)}$ to this RH problem.

RH problem for $P^{(\infty)}$:

(a) $P^{(\infty)}$ is analytic in $\mathbb{C} \setminus (-\infty, z_0]$.

(b) We have

$$P_+^{(\infty)}(\zeta) = P_-^{(\infty)}(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in (-\infty, z_0). \tag{2.62}$$

(c) As $\zeta \to \infty$,

$$P^{(\infty)}(\zeta) = \left(I + \mathcal{O}(\zeta^{-1})\right) |s|^{-\frac{1}{4m+4}\sigma_3} \zeta^{-\frac{\sigma_3}{4}} N. \tag{2.63}$$

This RH problem can easily be solved explicitly: if we take

$$P^{(\infty)}(\zeta) = |s|^{-\frac{\sigma_3}{4m+4}} (\zeta - z_0)^{-\frac{\sigma_3}{4}} N, \tag{2.64}$$

one verifies that $P^{(\infty)}$ satisfies the required conditions. The asymptotic condition (2.63) can be specified as

$$P^{(\infty)}(\zeta) = \left(I + \frac{z_0}{4\zeta}\sigma_3 + \mathcal{O}(\zeta^{-2})\right) |s|^{-\frac{\sigma_3}{4m+4}} \zeta^{-\frac{\sigma_3}{4}} N. \tag{2.65}$$

We will show that $P^{(\infty)}$ determines the leading order asymptotics of S and thus indirectly of the P_I^m solution q, by (2.61). Therefore we first need to know that there exists a local parametrix near z_0 which matches with the global parametrix.

2.3.5 Local parametrix near z_0

Let us fix a small neighborhood U of z_0, for example a small disk. Given m and $\operatorname{sgn}(s)$, we take U fixed for $|s|$ sufficiently large. Near z_0, the g-function vanishes like $c(\zeta - z_0)^{3/2}$. Following a well understood procedure, one can explicitly construct a local parametrix in U in terms of the Airy function and its derivative. We refer to [6] for the explicit construction (and to [9, 11, 12] for similar constructions). The only thing that we need here, is the existence of a local parametrix which satisfies the RH problem

RH problem for P:

(a) P is analytic in $\overline{U} \setminus \Gamma(z_0, \theta)$,

(b) for $\zeta \in \Gamma(z_0, \theta) \cap U$, P satisfies exactly the same jump conditions than S (see (2.57)-(2.59)),

(c) for $\zeta \in \partial U$, we have

$$P(\zeta)P^{(\infty)}(\zeta)^{-1} = I + \mathcal{O}(|s|^{-1}), \qquad \text{as } s \to \pm\infty. \tag{2.66}$$

2.3.6 Final transformation

Define

$$R(\zeta) = \begin{cases} S(\zeta)P(\zeta)^{-1}, & \text{for } \zeta \in U, \\ S(\zeta)P^{(\infty)}(\zeta)^{-1}, & \text{for } \zeta \in \mathbb{C} \setminus \overline{U}. \end{cases} \tag{2.67}$$

Then R is analytic in the interior of U and across $(-\infty, z_0)$ because the jumps of S cancel against the jumps of the parametrices P and $P^{(\infty)}$. On the boundary of U, the matching of the local parametrix with the global parametrix, see (2.66), implies that R has a jump that is $I + \mathcal{O}(|s|^{-1})$ as $s \to \pm\infty$.

RH problem for R:

(a) R is analytic in $\mathbb{C} \setminus \Sigma_R$, with $\Sigma_R = (\Gamma \setminus U) \cup \partial U$,

(b) $R_+(\zeta) = R_-(\zeta)v_R(\zeta)$ for $\zeta \in \Sigma_R$, where

$$v_R(\zeta) = I + \mathcal{O}(|s|^{-1}), \qquad \text{as } s \to \pm\infty, \text{ for } \zeta \in \partial U, \tag{2.68}$$

$$v_R(\zeta) = I + \mathcal{O}(e^{-c|s|(|\zeta|+1)}), \qquad \text{as } s \to \pm\infty, \text{ for } \zeta \in \Sigma_R \setminus \partial U, \tag{2.69}$$

(c) There exists a matrix $R_1 = R_1(s)$ such that $R(\zeta) = I + R_1\zeta^{-1} + \mathcal{O}(\zeta^{-2})$ as $\zeta \to \infty$.

It is a standard fact that the solution to a RH problem of this form (with small jump matrices and normalized at infinity) is close to the identity matrix [12]: we have

$$R(\zeta) = I + \mathcal{O}(|s|^{-1}), \qquad \text{as } s \to \pm\infty, \tag{2.70}$$

and for the residue matrix at infinity we have

$$R_1(s) = \mathcal{O}(|s|^{-1}), \qquad \text{as } s \to \pm\infty. \tag{2.71}$$

Using (2.65) and (2.67), one derives the identity

$$R_1 = B_1 - \frac{z_0}{4}\sigma_3, \tag{2.72}$$

and this implies that

$$B_{1,11} = \frac{z_0}{4} + \mathcal{O}(|s|^{-1}), \qquad B_{1,12} = \mathcal{O}(|s|^{-1}), \qquad \text{as } s \to \pm\infty. \tag{2.73}$$

By (2.61) we obtain

$$q(s) = -\frac{z_0}{2}|s|^{\frac{1}{m+1}} + \mathcal{O}(|s|^{-\frac{m}{m+1}}), \qquad \text{as } s \to \pm\infty, \tag{2.74}$$

Now (1.7) follows from (2.52), and the more general asymptotic formula (1.9) follows from (2.52).

3 Critical behavior of solutions to the KdV equation

In this section we will prove Theorem 1.5 and show that the pole-free solutions to the even members of the Painlevé I hierarchy describe the critical behavior of solutions to the KdV equation. A RH procedure to obtain asymptotics for KdV solutions was developed in [13, 14]. In [5] the method was used to prove Theorem 1.5 in the generic case where $m = 2$. For the sake of brevity and because many of the arguments are valid also for $m > 2$, we will refer to this paper at several points. In the RH analysis of the KdV RH problem, we will construct auxiliary matrix functions S, $P^{(\infty)}$, P, and R. They are not the same functions as in the previous section, we hope this does not cause any confusion.

3.1 RH problem for the KdV equation

Given initial data $u_0(x)$ satisfying the conditions specified in the introduction (i.e. $u_0(x)$ is negative, real analytic, has a single negative hump, and decays sufficiently fast at $\pm\infty$), we are interested in the solution $u(x,t,\epsilon)$ to the Cauchy problem for the KdV equation (1.10). The following RH problem characterizes $u(x,t,\epsilon)$ at any time $t > 0$.

RH problem for M:

(a) $M : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) M has continuous boundary values $M_+(\lambda)$ and $M_-(\lambda)$ when approaching $\lambda \in \mathbb{R}\backslash\{0\}$ from above and below, and

$$M_+(\lambda) = M_-(\lambda)\begin{pmatrix} 1 & r(\lambda;\epsilon)e^{2i\alpha(\lambda;x,t)/\epsilon} \\ -\bar{r}(\lambda;\epsilon)e^{-2i\alpha(\lambda;x,t)/\epsilon} & 1-|r(\lambda;\epsilon)|^2 \end{pmatrix}, \qquad \text{for } \lambda < 0,$$

$$M_+(\lambda) = M_-(\lambda)\sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{for } \lambda > 0,$$

with α given by

$$\alpha(\lambda;x,t) = 4t(-\lambda)^{3/2} + x(-\lambda)^{1/2}. \tag{3.1}$$

The branches of $(-\lambda)^{3/2}$ and $(-\lambda)^{1/2}$ are analytic in $\mathbb{C}\setminus[0,+\infty)$ and positive for $\lambda < 0$.

(c) As $\lambda \to \infty$,

$$M(\lambda) = \left(I + \mathcal{O}(\lambda^{-1})\right)\begin{pmatrix} 1 & 1 \\ i\sqrt{-\lambda} & -i\sqrt{-\lambda} \end{pmatrix}. \tag{3.2}$$

The solution $M = M(\lambda;x,t,\epsilon)$ depends on x, t, ϵ. If $r(\lambda;\epsilon)$ is the reflection coefficient from the left for the Schrödinger equation $\epsilon^2\frac{d^2}{dx^2}f + u_0(x)f = \lambda f$ with potential u_0, then it is known that

$$u(x,t,\epsilon) = -2i\epsilon\frac{\partial}{\partial x}\lim_{\lambda\to\infty}\left(\sqrt{-\lambda}[M_{11}(\lambda;x,t,\epsilon) - 1]\right) \tag{3.3}$$

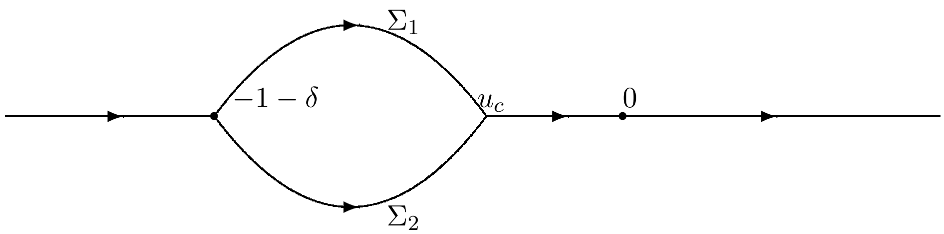

Figure 2: The jump contour Σ_S after the transformation $M \mapsto S$

is the solution to the KdV equation with initial data $u_0(x)$ at time $t \geq 0$. Using certain smoothness and asymptotic (as $\epsilon \to 0$) properties of the reflection coefficient, the RH problem for M can be transformed to a RH problem with modified jump matrices. We refer to [5] for the explicit construction of the function S which satisfies the RH problem stated below, with jumps on a deformed jump contour, see Figure 2: lenses are opened along an interval $(-1-\delta, u_c)$ for some small $\delta > 0$. The point u_c is the Hopf solution $u(x,t)$ evaluated at the point x_c and time t_c of gradient catastrophe.

RH problem for S:

(a) S is analytic in $\mathbb{C} \setminus \Sigma_S$,

(b) $S_+(\lambda) = S_-(\lambda) v_S(\lambda)$ for $\lambda \in \Sigma_S$, with

$$v_S(\lambda) = \begin{cases} \begin{pmatrix} 1 & i\kappa(\lambda;\epsilon) e^{\frac{2i}{\epsilon}\phi(\lambda)} \\ 0 & 1 \end{pmatrix}, & \text{on } \Sigma_1, \\ \begin{pmatrix} 1 & 0 \\ i\bar{\kappa}(\bar{\lambda};\epsilon) e^{-\frac{2i}{\epsilon}\phi(\lambda;x)} & 1 \end{pmatrix}, & \text{on } \Sigma_2 = \overline{\Sigma_1}, \\ \begin{pmatrix} e^{\frac{-2i}{\epsilon}\phi_+(\lambda;x)} & i\kappa(\lambda;\epsilon) \\ i\bar{\kappa}(\lambda;\epsilon) & (1-|r(\lambda)|^2) e^{\frac{2i}{\epsilon}\phi_+(\lambda;x)} \end{pmatrix}, & \text{as } \lambda \in (u_c, 0), \\ \sigma_1, & \text{as } \lambda \in (0,+\infty), \end{cases} \tag{3.4}$$

and for $\lambda \in (-\infty, -1-\delta)$, we have

$$v_S(\lambda) = I + \mathcal{O}(e^{-\frac{c}{\epsilon}(|\lambda|+1)}), \qquad \text{as } \epsilon \to 0,\ c > 0, \tag{3.5}$$

uniformly in λ for x, t sufficiently close to x_c, t_c.

(c) $S(\lambda) = \left(I + \mathcal{O}(\lambda^{-1})\right) \begin{pmatrix} 1 & 1 \\ i\sqrt{-\lambda} & -i\sqrt{-\lambda} \end{pmatrix}$ as $\lambda \to \infty$.

S can be expressed explicitly in terms of M [5], and it follows from this explicit expression that

$$u(x,t,\epsilon) = u_c - 2i\epsilon \frac{\partial}{\partial x} S_{1,11}(x,t,\epsilon), \tag{3.6}$$

where

$$S_{11}(\lambda;x,t,\epsilon) = 1 + \frac{S_{1,11}(x,t,\epsilon)}{\sqrt{-\lambda}} + \mathcal{O}(\lambda^{-1}), \qquad \text{as } \lambda \to \infty. \tag{3.7}$$

The function κ can be expressed in terms of the (analytic continuation of the) reflection coefficient and satisfies the important asymptotic property

$$\kappa(\lambda;\epsilon) = 1 + \mathcal{O}(\epsilon), \qquad \text{for } \lambda \in \Sigma_1 \cup [u,0], \qquad \text{as } \epsilon \to 0. \tag{3.8}$$

Furthermore ϕ depends explicitly on the initial data:

$$\phi(\lambda;x,t) = \sqrt{u_c-\lambda}(x - x_c - 6u_c(t-t_c)) + 4(u_c-\lambda)^{3/2}(t-t_c) \\ + \int_\lambda^{u_c} (f_L'(\xi) + 6t_c)\sqrt{\xi-\lambda}\,d\xi. \tag{3.9}$$

If

$$f_L^{(2)}(u_c) = f_L^{(3)}(u_c) = f_L^{(4)}(u_c) = \ldots = f_L^{(m)}(u_c) = 0, \qquad f_L^{(m+1)}(u_c) \neq 0, \tag{3.10}$$

repeated integration by parts gives (since $f_L'(u_c) + 6t_c = 0$)

$$\phi(\lambda;x,t) = \sqrt{u_c-\lambda}(x - x_c - 6u_c(t-t_c)) + 4(u_c-\lambda)^{3/2}(t-t_c) \\ + \frac{2^m}{(2m+1)!!}\int_\lambda^{u_c} f_L^{(m+1)}(\xi)(\xi-\lambda)^{\frac{2m+1}{2}}d\xi. \tag{3.11}$$

For any fixed neighborhood $\mathcal{U}$ of u_c, it was also proved in [5] that there exists $\delta > 0$ such that

$$v_S(\lambda) = \begin{cases} I + \mathcal{O}(e^{-\frac{c}{\epsilon}}), & \text{for } \lambda \in \Sigma_S \setminus (\mathcal{U} \cup (u_c,+\infty)), \\ i\sigma_1 + \mathcal{O}(\epsilon), & \text{for } \lambda \in (u_c,0) \setminus \mathcal{U}, \end{cases} \qquad \text{as } \epsilon \to 0, \tag{3.12}$$

if $|x - x_c| < \delta$ and $|t - t_c| < \delta$.

3.2 Construction of the global parametrix

If we ignore the jump matrices that are small as $\epsilon \to 0$ and the jumps in a fixed sufficiently small neighborhood $\mathcal{U}$ of u_c, we obtain the following RH problem:

RH problem for $P^{(\infty)}$:

(a) $P^{(\infty)} : \mathbb{C} \setminus [u_c,+\infty) \to \mathbb{C}^{2\times 2}$ is analytic,

(b) $P^{(\infty)}$ satisfies the jump conditions

$$P_+^{(\infty)} = P_-^{(\infty)}\sigma_1, \qquad \text{on } (0,+\infty), \tag{3.13}$$

$$P_+^{(\infty)} = iP_-^{(\infty)}\sigma_1, \qquad \text{on } (u_c,0), \tag{3.14}$$

(c) $P^{(\infty)}$ has the following behavior as $\lambda \to \infty$,

$$P^{(\infty)}(\lambda) = (I + \mathcal{O}(\lambda^{-1}))\begin{pmatrix} 1 & 1 \\ i(-\lambda)^{1/2} & -i(-\lambda)^{1/2} \end{pmatrix}. \tag{3.15}$$

This RH problem is solved by

$$P^{(\infty)}(\lambda) = (-\lambda)^{1/4}(u_c-\lambda)^{-\sigma_3/4}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \tag{3.16}$$

3.3 Construction of the local parametrix

We need to construct a local parametrix in a neighborhood $\mathcal{U}$ of u_c. As $\epsilon \to 0$, we have $\kappa(\lambda) = 1+\mathcal{O}(\epsilon)$, and we will construct a function P, defined in $\mathcal{U}$, which satisfies the same jump relations as S, but in the limiting case where κ is set to 1.

RH problem for P:

(a) $P : \overline{\mathcal{U}} \setminus \Sigma_S \to \mathbb{C}^{2\times 2}$ is analytic,

(b) P satisfies the following jump condition on $\mathcal{U} \cap \Sigma_S$,

$$P_+(\lambda) = P_-(\lambda) v_P(\lambda), \tag{3.17}$$

with v_P given by

$$v_P(\lambda) = \begin{cases} \begin{pmatrix} 1 & ie^{\frac{2i}{\epsilon}\phi(\lambda;x,t)} \\ 0 & 1 \end{pmatrix}, & \text{as } \lambda \in \Sigma_1, \\ \begin{pmatrix} 1 & 0 \\ ie^{-\frac{2i}{\epsilon}\phi(\lambda;x,t)} & 1 \end{pmatrix}, & \text{as } \lambda \in \Sigma_2, \\ \begin{pmatrix} e^{-\frac{2i}{\epsilon}\phi_+(\lambda;x,t)} & i \\ i & 0 \end{pmatrix}, & \text{as } \lambda \in (u_c, 0), \end{cases} \tag{3.18}$$

(c) in the double scaling limit where $\epsilon \to 0$ and simultaneously $x \to x_c$, $t \to t_c$ in such a way that

$$\lim \frac{x - x_c - 6u_c(t-t_c)}{\epsilon^{\frac{2m+2}{2m+3}} k^{1/2}} = \tau_0, \quad \lim \frac{-3(t-t_c)}{\epsilon^{\frac{2m}{2m+3}} k^{3/2}} = \tau_1, \quad \tau_0, \tau_1 \in \mathbb{R}, \tag{3.19}$$

with k given by (1.20), we have the matching

$$P(\lambda) P^{(\infty)}(\lambda)^{-1} \to I, \qquad \text{for } \lambda \in \partial\mathcal{U}. \tag{3.20}$$

We will use the RH solution $\Psi = \Psi^{(m)}$ studied in Section 2 to construct the local parametrix P. First we transform the RH problem for Ψ to a RH problem for Φ which models the jumps needed for P in an appropriate way.

3.3.1 Modified model RH problem

Define

$$\Phi(\zeta; s, t_1) = e^{\frac{-\pi i}{4}\sigma_3} \Psi(\zeta; s, t_1, 0, \dots, 0) e^{\theta(\zeta; s, t_1, 0, \dots, 0)\sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e^{\frac{\pi i}{4}\sigma_3} \tag{3.21}$$

for $\operatorname{Im} \zeta > 0$, and

$$\Phi(\zeta; s, t_1) = e^{-\frac{\pi i}{4}\sigma_3} \Psi(\zeta; s, t_1, 0, \dots, 0) e^{\theta(\zeta; s, t_1, 0, \dots, 0)\sigma_3} e^{\frac{\pi i}{4}\sigma_3} \tag{3.22}$$

for $\operatorname{Im}\zeta < 0$. We also write

$$\widetilde{\theta}(\zeta; s, t_1) = -\frac{4}{2m+3}(-\zeta)^{\frac{2m+3}{2}} - \frac{4}{3}t_1(-\zeta)^{3/2} + s(-\zeta)^{1/2}, \tag{3.23}$$

which is related to θ in the case where $t_2 = \ldots = t_{m-1} = 0$, but with its branch cut on $(0, +\infty)$. One has the identities

$$\theta = i\widetilde{\theta}_+, \quad \text{on } (0, +\infty), \qquad \theta = i\widetilde{\theta}, \quad \text{on } \Gamma_2, \qquad \theta = -i\widetilde{\theta}, \quad \text{on } \Gamma_4. \tag{3.24}$$

Then it is straightforward to verify that Φ solves the RH problem

RH problem for Φ:

(a) Φ is analytic for $\zeta \in \mathbb{C} \setminus \widehat{\Gamma}$, with $\widehat{\Gamma} = \Gamma_1 \cup \Gamma_2 \cap \Gamma_4$.

(b) Φ satisfies the following jump relations on $\widehat{\Gamma}$,

$$\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} e^{-2i\widetilde{\theta}_+(\zeta;s,t_1)} & i \\ i & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{3.25}$$

$$\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 1 & ie^{2i\widetilde{\theta}(\zeta;s,t_1)} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2. \tag{3.26}$$

$$\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 1 & 0 \\ ie^{-2i\widetilde{\theta}(\zeta;s,t_1)} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_4. \tag{3.27}$$

(c) Φ has the following behavior at infinity,

$$\Phi(\zeta) = \frac{1}{\sqrt{2}}(-\zeta)^{-\frac{1}{4}\sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \left(I + ih\sigma_3(-\zeta)^{-1/2} + \mathcal{O}(\zeta^{-1}) \right), \tag{3.28}$$

with the branches positive for $\zeta < 0$ and analytic off $[0, +\infty)$.

We search for a parametrix P of the form

$$P(\lambda) = E(\lambda; \epsilon)\Phi(\epsilon^{-\frac{2}{2m+3}} f(\lambda); \epsilon^{-\frac{2m+2}{2m+3}} \tau_0(\lambda; x, t), \epsilon^{-\frac{2m}{2m+3}} \tau_1(\lambda; t)), \tag{3.29}$$

where E, f, τ_0, τ_1 are analytic in $\mathcal{U}$. So we evaluate $\Phi(\zeta; s, t_1)$ at the values

$$\zeta = \epsilon^{-\frac{2}{2m+3}} f(\lambda), \qquad s = \epsilon^{-\frac{2m+2}{2m+3}} \tau_0(\lambda; x, t), \tag{3.30}$$

$$t_1 = \epsilon^{-\frac{2m}{2m+3}} \tau_1(\lambda; t), \tag{3.31}$$

and we will construct f, τ_0, and τ_1 in such a way that

$$\widetilde{\theta}(\epsilon^{-\frac{2}{2m+3}} f(\lambda); \epsilon^{-\frac{2m+2}{2m+3}} \tau_0(\lambda; x, t), \epsilon^{-\frac{2m}{2m+3}} \tau_1(\lambda; t)) = \frac{1}{\epsilon}\phi(\lambda; x, t). \tag{3.32}$$

This condition is satisfied if we define f by

$$-\frac{4}{2m+3}(-f(\lambda))^{\frac{2m+3}{2}} = \frac{2^m}{(2m+1)!!} \int_\lambda^{u_c} f_L^{(m+1)}(\xi)(\xi - \lambda)^{\frac{2m+1}{2}} d\xi, \tag{3.33}$$

τ_1 by

$$-\frac{4}{3}\tau_1(\lambda;t)(-f(\lambda))^{\frac{3}{2}} = 4(t-t_c)(u_c-\lambda)^{3/2}, \tag{3.34}$$

and τ_0 by

$$\tau_0(\lambda;x,t)(-f(\lambda))^{\frac{1}{2}} = \sqrt{u_c-\lambda}(x-x_c-6u_c(t-t_c)). \tag{3.35}$$

Indeed, summing (3.33)-(3.35) gives (3.32) by (3.11). This defines f, τ_0, τ_1 analytically near u_c, and we have

$$f(u_c) = 0, \qquad f'(u_c) = \left(-\frac{2^{m-1}}{(2m+1)!!}f_L^{(m+1)}(u_c)\right)^{\frac{2}{2m+3}} = k > 0, \tag{3.36}$$

$$\tau_1(u_c) = -\frac{3(t-t_c)}{k^{3/2}}, \tag{3.37}$$

$$\tau_0(u_c) = \frac{x-x_c-6u_c(t-t_c)}{k^{1/2}}. \tag{3.38}$$

Since f is a conformal mapping from a neighborhood of u_c to a neighborhood of 0, we can choose the lenses of the jump contour for S in such a way that $f(\Sigma_S \cap \mathcal{U}) \subset \widehat{\Gamma}$. Then for any analytic function E near u_c, P satisfies the required jump conditions on $\Sigma_S \cap \mathcal{U}$ (see (3.18)), but we also need the matching (3.20), which has to be valid in the double scaling limit where $\epsilon \to 0$, $x \to x_c$, $t \to t_c$ in such a way that (3.19) holds, or in other words

$$\lim \epsilon^{-\frac{2m+2}{2m+3}}\tau_0(u_c;x,t) = \tau_0, \qquad \lim \epsilon^{-\frac{2m}{2m+3}}\tau_1(u_c;t) = \tau_1.$$

If $\mathcal{U}$ is sufficiently small, $(\epsilon^{-\frac{2m+2}{2m+3}}\tau_0(\lambda;x,t), \epsilon^{-\frac{2m}{2m+3}}\tau_1(\lambda;t))$ will lie in a small complex neighborhood of (τ_0,τ_1) for $\lambda \in \partial\mathcal{U}$. By (3.20), we have

$$P(\lambda)P^{(\infty)}(\lambda)^{-1} = \frac{1}{\sqrt{2}}E(\lambda)(-\epsilon^{-\frac{2}{2m+3}}f(\lambda))^{-\frac{1}{4}\sigma_3}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \left(I + ih\sigma_3\epsilon^{\frac{1}{2m+3}}(-f(\lambda))^{-1/2} + \mathcal{O}(\epsilon^{\frac{2}{2m+3}})\right)P^{(\infty)}(\lambda)^{-1}, \tag{3.39}$$

as $\epsilon \to 0$, where $h = h(\epsilon^{-\frac{2m+2}{2m+3}}\tau_0(\lambda;x,t), \epsilon^{-\frac{2m}{2m+3}}\tau_1(\lambda;t))$. If we define

$$E(\lambda) = \frac{1}{\sqrt{2}}P^{(\infty)}(\lambda)\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}(-\epsilon^{-\frac{2}{2m+3}}f(\lambda))^{\frac{1}{4}\sigma_3}, \tag{3.40}$$

it is easily verified that E is analytic in $\mathcal{U}$ and that we have

$$P(\lambda)P^{(\infty)}(\lambda)^{-1} = P^{(\infty)}(\lambda) \left(I + ih\sigma_3\epsilon^{\frac{1}{2m+3}}(-f(\lambda))^{-1/2} + \mathcal{O}(\epsilon^{\frac{2}{2m+3}})\right)P^{(\infty)}(\lambda)^{-1}, \tag{3.41}$$

in the double scaling limit, for $\lambda \in \partial\mathcal{U}$.

3.4 Final RH problem

We define R in such a way that it has jumps that are uniformly $I+\mathcal{O}(\epsilon^{\frac{1}{2m+3}})$ in the double scaling limit: we let

$$R(\lambda;x,t,\epsilon)=\begin{cases}S(\lambda;x,t,\epsilon)P^{(\infty)}(\lambda)^{-1}, & \text{as } \lambda\in\mathbb{C}\setminus\overline{\mathcal{U}},\\ S(\lambda;x,t,\epsilon)P(\lambda;x,t,\epsilon)^{-1}, & \text{as } \lambda\in\mathcal{U}.\end{cases} \tag{3.42}$$

Then, using the fact that

$$v_S(\lambda;x,t,\epsilon)v_P^{-1}(\lambda;x,t,\epsilon)=I+\mathcal{O}(\epsilon), \qquad \text{uniformly for } \lambda\in\mathcal{U}\cap\Sigma_S \text{ as } \epsilon\to 0, \tag{3.43}$$

one can verify that R solves a RH problem of the following form.

RH problem for R:

(a) R is analytic in $\mathbb{C}\setminus(\Sigma_S\cup\partial\mathcal{U})$.

(b) R has the jump condition $R_+(\lambda;x,t,\epsilon)=R_-(\lambda;x,t,\epsilon)v_R(\lambda;x,t,\epsilon)$ for $\lambda\in\Sigma_S\cup\partial\mathcal{U}$, where

$$v_R(\lambda;x,t,\epsilon)=I+\mathcal{O}(e^{-\frac{c}{\epsilon}}), \qquad \text{for } \lambda\in\Sigma_S\setminus\overline{\mathcal{U}}, \tag{3.44}$$

$$v_R(\lambda;x,t,\epsilon)=I+\mathcal{O}(\epsilon), \qquad \text{for } \lambda\in\Sigma_S\cap\mathcal{U}, \tag{3.45}$$

$$v_R(\lambda;x,t,\epsilon)=I+\mathcal{O}(\epsilon^{\frac{1}{2m+3}}), \qquad \text{for } \lambda\in\partial\mathcal{U}, \tag{3.46}$$

in the double scaling limit where $\epsilon\to 0$, $x\to x_c$, $t\to t_c$ and simultaneously $\epsilon^{-\frac{2m+2}{2m+3}}\tau_0(u_c;x,t)\to\tau_0$ and $\epsilon^{-\frac{2m}{2m+3}}\tau_1(u_c;t)\to\tau_1$.

(c) As $\lambda\to\infty$, we have

$$R(\lambda;x,t,\epsilon)=I+\frac{R_1(x,t,\epsilon)}{\lambda}+\mathcal{O}(\lambda^{-2}). \tag{3.47}$$

On $\partial\mathcal{U}$ with clockwise orientation, the jump matrix has the form

$$v_R(\lambda;x,t,\epsilon)=I+v_1(\lambda;x,t)\epsilon^{\frac{1}{2m+3}}+\mathcal{O}(\epsilon^{\frac{2}{2m+3}}), \tag{3.48}$$

with

$$v_1(\lambda;x,t)=ih\cdot(-f(\lambda))^{-1/2}P^{(\infty)}(\lambda)\sigma_3P^{(\infty)}(\lambda)^{-1}. \tag{3.49}$$

This is a meromorphic function in $\mathcal{U}$ with a simple pole at u_c, the residue is given by

$$\mathrm{Res}(v_1;u_c)=-h(\epsilon^{-\frac{2m+2}{2m+3}}\tau_0(u_c;x,t),\epsilon^{-\frac{2m}{2m+3}}\tau_1(u_c;t))k^{-1/2}\begin{pmatrix}0&1\\0&0\end{pmatrix}. \tag{3.50}$$

Then as in [5, Section 4] one can conclude that

$$R_1(x,t,\epsilon)=\epsilon^{\frac{1}{2m+3}}\mathrm{Res}(v_1;u_c)+\mathcal{O}(\epsilon^{\frac{2}{2m+3}}), \tag{3.51}$$

and by (3.6) and (3.38) this leads to

$$\begin{aligned}
u(x,t,\epsilon) &= u_c - 2\epsilon\frac{\partial}{\partial x}R_{1,12}(x,t,\epsilon)\\
&= u_c + 2k^{-1/2}\epsilon^{\frac{2}{2m+3}}\frac{\partial\tau_0(u_c;x,t)}{\partial x}q(\epsilon^{-\frac{2m+2}{2m+3}}\tau_0(u_c;x,t),\epsilon^{-\frac{2m}{2m+3}}\tau_1(u_c;t),0,\ldots,0)+\mathcal{O}(\epsilon^{\frac{4}{2m+3}})\\
&= u_c - 2k^{-1}\epsilon^{\frac{2}{2m+3}}q(\epsilon^{-\frac{2m+2}{2m+3}}\tau_0(u_c;x,t),\epsilon^{-\frac{2m}{2m+3}}\tau_1(u_c;t),0,\ldots,0)+\mathcal{O}(\epsilon^{\frac{4}{2m+3}}),
\end{aligned}$$

which proves Theorem 1.5.

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