On the integrability of symplectic Monge-Ampére equations

B. Doubrov and E.V. Ferapontov

Department of Mathematical Physics
Faculty of Applied Mathematics
Belarussian State University
Nezavisimosti av. 4, 220030 Minsk, Belarus

and

Department of Mathematical Sciences
Loughborough University
Loughborough, Leicestershire LE11 3TU
United Kingdom

e-mails:

doubrov@islc.org
E.V.Ferapontov@lboro.ac.uk

Abstract

Let u be a function of n independent variables $x^1, ..., x^n$, and $U = (u_{ij})$ the Hessian matrix of u. The symplectic Monge-Ampére equation is defined as a linear relation among all possible minors of U. Particular examples include the equation $\det U = 1$ governing improper affine spheres and the so-called heavenly equation, $u_{13}u_{24} - u_{23}u_{14} = 1$, describing self-dual Ricci-flat 4-manifolds. In this paper we classify integrable symplectic Monge-Ampére equations in four dimensions (for $n = 3$ the integrability of such equations is known to be equivalent to their linearisability). This problem can be reformulated geometrically as the classification of 'maximally singular' hyperplane sections of the Plücker embedding of the Lagrangian Grassmannian. We formulate a conjecture that any integrable equation of the form $F(u_{ij}) = 0$ in more than three dimensions is necessarily of the symplectic Monge-Ampére type.

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1 Introduction

Let us consider a function $u(x^1, ...x^n)$ of n independent variables and introduce the $n \times n$ Hessian matrix $U = (u_{ij})$ of its second order partial derivatives. The symplectic Monge-Ampére equation is a PDE of the form

$$M_n + M_{n-1} + ... + M_1 + M_0 = 0 \tag{1}$$

where M_l is a constant-coefficient linear combination of all $l \times l$ minors of the matrix U, $0 \le l \le n$. Thus, $M_n = \det U = \mathrm{Hess}\, u$, M_0 is a constant, etc. Equivalently, these PDEs can be obtained by equating to zero a constant-coefficient n-form in the $2n$ variables x^i, u_i. Equations of this type belong to the class of completely exceptional Monge-Ampére equations introduced in [6]. Geometric and algebraic aspects of symplectic Monge-Ampére equations have been thoroughly investigated in [23, 3]. We point out that the left hand side of (1), $M(U)$, constitutes the general form of null Lagrangian densities, that is, functionals of the form $\int M(U) d\mathbf{x}$ which generate trivial Euler-Lagrange equations [2]. The class of equations (1) is invariant under the natural contact action of the symplectic group $Sp(2n)$, which is thus the equivalence group of our problem. All subsequent classification results will be formulated modulo this $Sp(2n)$-equivalence.

In the case $n = 2$ one arrives at the standard Monge-Ampére equations,

$$\epsilon(u_{11}u_{22} - u_{12}^2) + \alpha u_{11} + \beta u_{12} + \gamma u_{22} + \delta = 0; \tag{2}$$

these are known to be the only equations of the form $F(u_{11}, u_{12}, u_{22}) = 0$ which are linearisable by a transformation from the equivalence group $Sp(4)$.

The case $n = 3$ is also understood completely: one can show that any non-degenerate symplectic Monge-Ampére equation is either linearizable, or $Sp(6)$-equivalent to either of the canonical forms,

$$\mathrm{Hess}\, u = 1, \qquad \mathrm{Hess}\, u = u_{11} + u_{22} + u_{33}, \qquad \mathrm{Hess}\, u = u_{11} + u_{22} - u_{33}, \tag{3}$$

see [23, 3]; we point out that all three canonical forms are $Sp(6)$-equivalent over $\mathbb{C}$. The first equation arises in the theory of improper affine spheres, while the second describes special Lagrangian 3-folds in $\mathbb{C}^3$ [8, 19]. Here the non-degeneracy is understood as follows: let $F(u_{ij}) = 0$ be a symplectic Monge-Ampére equation. Consider the linearized equation, $\partial F/\partial u_{ij}\ v_{ij} = 0$, obtained by setting $u \to u + \epsilon v$ and keeping terms of the order ϵ. The non-degeneracy means that the corresponding symbol, $\partial F/\partial u_{ij}\ \xi_i \xi_j$, defines an irreducible quadratic form.

The problem of integrability of symplectic Monge-Ampére equations was addressed in [15] based on the method of hydrodynamic reductions [12, 13, 14]. Without going into technical details of this method, let us formulate the main result needed for our purposes:

Theorem 1 *[15]. A non-degenerate three-dimensional symplectic Monge-Ampére equation is integrable by the method of hydrodynamic reductions if and only if it is linearisable.*

In particular, the PDEs (3) are *not* integrable. Although this result is essentially negative, it will be crucial for the classification of integrable equations in higher dimensions (where the situation is far more interesting). Before we proceed to the description of the main results, let us clarify the geometry behind symplectic Monge-Ampére equations (1) and the linearisability/integrability conditions. Let us consider the Lagrangian Grassmannian Λ, which can be (locally) identified with the space of $n \times n$ symmetric matrices U. The minors of U define the Plücker embedding of Λ into projective space P^N (we identify Λ with the image of its projective embedding). Thus, symplectic Monge-Ampére equations correspond to hyperplane sections of Λ. For $n = 3$ we have $\Lambda^6 \subset P^{13}$, and linearisable equations correspond to hyperplanes which are tangential to Λ^6. Therefore, for $n = 3$ the linearisability condition coincides with the equation of the dual variety of Λ^6, which is known to be a hypersurface of degree four in $(P^{13})^*$ (we refer to [21, 22, 26] for a general theory behind this example). In Sect. 2 we provide the following characterisation of linearisable equations in any dimension:

Theorem 2 *For a non-degenerate symplectic Monge-Ampére equation (1) the following conditions are equivalent:*
(1) The equation is linearisable by a transformation from $Sp(2n)$.
(2) The equation is invariant under an n^2*-dimensional subalgebra of* $Sp(2n)$.
(3) The equation corresponds to a hyperplane which contains an osculating subspace O_{n-2} *of the Lagrangian Grassmannian of the order* $n - 2$.

For $n = 3$ the third condition reduces to the requirement of tangency.

In Sect. 3 we address the problem of integrability of symplectic Monge-Ampére equations in four dimensions, $n = 4$. Among the best known four-dimensional integrable examples one should primarily mention the 'heavenly' equation,

$$u_{13}u_{24} - u_{23}u_{14} = 1, \tag{4}$$

[29] which is descriptive of Ricci-flat self-dual 4-manifolds [1]. It was demonstrated in [13] that this equation is integrable by the method of hydrodynamic reductions. Although, in principle, the method of hydrodynamic reductions can be applied in any dimension, it leads to a quite complicated analysis. One way to bypass lengthy calculations is based on the following simple idea: suppose our aim is the classification of four-dimensional integrable equations of the form (1) for a function $u(x^1, x^2, x^3, x^4)$. Let us look for traveling wave solutions in the form

$$u = u(x^1 + \alpha x^4,\ x^2 + \beta x^4,\ x^3 + \gamma x^4),$$

or, more generally,

$$u = u(x^1 + \alpha x^4,\ x^2 + \beta x^4,\ x^3 + \gamma x^4) + Q(x, x),$$

where Q is an arbitrary quadratic form in the variables x^1, x^2, x^3, x^4. The substitution of this ansatz into (1) leads to a three-dimensional symplectic Monge-Ampére equation which must be integrable for *any* values of constants α, β, γ, and an arbitrary quadratic form Q. Since, in three dimensions, the integrability conditions are explicitly known (and are equivalent to the linearisability), this provides strong restrictions on the original four-dimensional equation which are therefore *necessary* for the integrability. In fact, in the present context they turn out to be sufficient: if all three-dimensional equations obtained from a given four-dimensional PDE by traveling wave reductions are linearisable, then the PDE is integrable. The philosophy of this approach is well familiar from the soliton theory: symmetry reductions of integrable systems must be themselves integrable. Thus, for the first heavenly equation, traveling wave solutions are governed by

$$\alpha(u_{12}u_{13} - u_{11}u_{23}) + \beta(u_{13}u_{22} - u_{12}u_{23}) = 1,$$

which is a three-dimensional symplectic Monge-Ampére equation. One can show that it is indeed linearisable for any values of constants. This approach suggests a simple geometric characterisation of integrable equations in four dimensions. Let us first point out that for $n = 4$ the Lagrangian Grassmannian $\Lambda^{10} \subset P^{41}$ is foliated by a 7-parameter family of Λ^6 where each Λ^6 corresponds to a collection of Lagrangian planes in the symplectic space V^8 which pass through a fixed vector.

Theorem 3 *A non-degenerate four-dimensional symplectic Monge-Ampére equation is integrable if and only if the corresponding hyperplane is tangential to the Lagrangian Grassmannian Λ^{10} along a 4-dimensional subvariety which meets all $\Lambda^6 \subset \Lambda^{10}$.*

Thus, integrable equations correspond to maximally singular hyperplane sections. The classification of such hyperplanes leads to a complete list of integrable equations in four dimensions:

Theorem 4 *Over the field of complex numbers any integrable non-degenerate symplectic Monge-Ampére equation is $Sp(8)$-equivalent to one of the following normal forms:*

1. $u_{11} - u_{22} - u_{33} - u_{44} = 0$ *(linear wave equation);*
2. $u_{13} + u_{24} + u_{11}u_{22} - u_{12}^2 = 0$ *(second heavenly equation);*
3. $u_{13} = u_{12}u_{44} - u_{14}u_{24}$ *(modified heavenly equation);*
4. $u_{13}u_{24} - u_{14}u_{23} = 1$ *(first heavenly equation);*
5. $u_{11} + u_{22} + u_{13}u_{24} - u_{14}u_{23} = 0$ *(Husain equation);*
6. $\alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{14}u_{23} = 0$ *(general heavenly equation), $\alpha + \beta + \gamma = 0$.*

Eqs. 2-6 are non-linearisable, and contact non-equivalent. They have appeared in different contexts in [29, 18, 23]. To the best of our knowledge, the integrability of Eqs. 3 and 6 was not recorded before. We refer to [34, 35, 33, 16, 10, 11, 27, 31, 4, 5] for the Hamiltonian, twistorial and symmetry aspects of heavenly-type equations.

Symplectic Monge–Ampére equations play a distiguished role in the classification of all integrable PDEs of the 'dispersionless Hirota' type,

$$F(u_{ij}) = 0, \tag{5}$$

which constitute a single relation among second order partial derivatives of a function $u(x^1, ..., x^n)$. For $n = 3$ integrable equations of this type were studied in [15, 28, 32], revealing a remarkable correspondence with the theory of hypersurfaces of the Lagrangian Grassmannian, generalised hypergeometric functions and $GL(2, \mathbb{R})$-structures. Among them, only linearisable equations belong to the class (1). On the contrary, for $n \geq 4$ all known integrable examples belong to the class (1). This suggest the following conjecture:

Conjecture. *For $n \geq 4$ any non-degenerate integrable equation of the form (5) is necessarily of the symplectic Monge-Ampére type.*

Provided this conjecture is true, Theorem 4 gives a complete list of integrable equations of the form (5) in four dimensions.

2 Characterisation of linearisable equations: proof of Theorem 2

The results of this section are valid for an arbitrary number of independent variables n. We begin with a few general remarks. The Lagrangian Grassmannian Λ is a variety of dimension $n(n+1)/2$ which can be locally identified with $n \times n$ symmetric matrices $U = (u_{ij})$ so that u_{ij} can be viewed as local coordinates on Λ. The action of $Sp(2n)$ is defined by the infinitesimal operators

$$X_{ij} = \frac{\partial}{\partial u_{ij}},$$

$$L_{ij} = \sum_s u_{js} \frac{\partial}{\partial u_{is}} + u_{ij} \frac{\partial}{\partial u_{ii}},$$

$$P_{ij} = 2 \sum_s u_{is} u_{js} \frac{\partial}{\partial u_{ss}} + \sum_{s \neq k} u_{is} u_{jk} \frac{\partial}{\partial u_{sk}};$$

notice that $X_{ij} = X_{ji}$ and $P_{ij} = P_{ji}$, while $L_{ij} \neq L_{ji}$. Thus, we have $n(n+1)/2$ operators X_{ij}, n^2 operators L_{ij} and $n(n+1)/2$ operators P_{ij}. Altogether, they generate a Lie algebra $Sp(2n)$ of dimension $n(2n+1)$. We will work with the affine Plücker embedding of the Lagrangian Grassmannian Λ in the projective space P^{N-1} specified by the position vector

$$\mathbf{r} = (M_1, M_2, ..., M_n);$$

here M_l denotes a collection of all $l \times l$ minors of U, and $N = C_{2n}^n - C_{2n}^{n+2}$ is the total number of minors of U (equivalently, N is the dimension of the vector space of effective n-forms). Thus, in the case $n = 3$ one has

$$\dim M_1 = 6, \ \dim M_2 = 6, \ \dim M_3 = 1, \qquad \Lambda^6 \subset P^{13}.$$

Similarly, for $n = 4$ one has

$$\dim M_1 = 10, \ \dim M_2 = 20, \ \dim M_3 = 10, \ \dim M_4 = 1, \qquad \Lambda^{10} \subset P^{41}.$$

With each point of the Plücker embedding of Λ one can associate a sequence of osculating subspaces

$$O_1 \subset O_2 \subset O_3 \subset ... \subset O_n = P^{N-1}$$

where O_1 is the tangent space, O_2 is the second osculating subspace, etc. At the point corresponding to $u_{ij} = 0$ the subspace O_1 is spanned by vectors from M_1, the subspace O_2 is spanned by vectors from $M_1 \cup M_2$, etc. A symplectic Monge-Ampére equation is naturally identified with a hyperplane section of Λ. Below we present three equivalent characterisations of linearisable equations.

Theorem 2. *For a non-degenerate symplectic Monge-Ampére equation in n dimensions the following conditions are equivalent:*
(1) The equation is linearisable by a transformation from $Sp(2n)$.
(2) The equation is invariant under an n^2-dimensional subalgebra of $Sp(2n)$.
(3) The equation corresponds to a hyperplane which contains an osculating space O_{n-2}.

Proof:

equivalence (1) $\iff$ (2): since all n-dimensional linear equations are equivalent under the (complexified) action of $Sp(2n)$, let us consider the Laplace equation,

$$u_{11} + \cdots + u_{nn} = 0.$$

A direct calculation shows that it is invariant under the n^2-dimensional subalgebra of $Sp(2n)$ generated by the operators X_{ij} $(i \neq j)$, $X_{ii} - X_{jj}$, $L_{ij} - L_{ji}$ $(i \neq j)$, and $\sum L_{ii}$. Since this property is manifestly $Sp(2n)$-invariant, any non-degenerate linearisable equation is invariant under a subalgebra of $Sp(2n)$ of dimension n^2. The converse is true for any (not necessarily Monge-Ampére) non-degenerate equation of the form $F(u_{ij}) = 0$. Indeed, let G be a symmetry group of such equation. We can always assume that the point 0, specified by $u_{ij} = 0$, belongs to the hypersurface in the Lagrangian Grassmannian Λ corresponding to our equation. Let G_0 be the stabilizer of this point in G. Note that $\dim G - \dim G_0 \leq \dim \Lambda - 1 = n(n+1)/2 - 1$, as G takes the equation to itself. The stabilizer of the point $0 \in \Lambda$ under the action of $Sp(2n)$ has the form:

$$P = \left\{ \begin{pmatrix} A & AB \\ 0 & {}^t\!A^{-1} \end{pmatrix} \;\middle|\; A \in GL(n), B = {}^t\!B \right\}.$$

Since the equation is non-degenerate, we can always bring it to the canonical form:

$$F(u_{ij}) = u_{11} + \cdots + u_{nn} + o(u_{ij}) = 0. \tag{6}$$

This form (together with the point 0) is stabilized by the elements of P given by the condition $A \in CO(n)$, $B = 0$. Hence, $\dim G_0 \leq \dim CO(n)$ so that $\dim G \leq \dim \Lambda - 1 + \dim CO(n) = n^2$. The equality holds only if $G_0 = CO(n)$. But the subgroup of scalar matrices in $CO(n)$ acts by non-trivial scalings on the terms of the order 2 and higher in (6). Hence, if $G_0 = CO(n)$, all higher order terms vanish identically, and our equation coincides with the Laplace equation.

equivalence (1) $\Longleftrightarrow$ (3): Let us fix a point on Λ corresponding to the parameter values $u_{ij} = 0$ (since $Sp(2n)$ acts transitively on Λ, all points are equivalent). The osculating space O_{n-2} at this point is specified by the equations $M_{n-1} = 0,\ M_n = 0$. Any hyperplane containing this osculating space corresponds to an equation which is a linear combination of minors of U of the orders $n-1$ and n. Any such equation linearises under the Legendre transformation. This finishes the proof.

Remark. Linearisable equations of Monge-Ampére type are of interest in their own, for instance, the equation

$$u_{tt}(1 + u_{xx} + u_{yy}) - u_{xt}^2 - u_{yt}^2 = \epsilon$$

governs the evolution of 'Kähler potentials' [9]. It corresponds to the Lagrangian

$$\int \left[u_t^2(1 + u_{xx} + u_{yy}) + 2\epsilon u\right] dxdydt,$$

and linearises under the partial Legendre transformation, $\tilde{x} = x,\ \tilde{y} = y,\ \tilde{t} = u_t,\ \tilde{u} = tu_t - u,\ \tilde{u}_{\tilde{x}} = -u_x,\ \tilde{u}_{\tilde{y}} = -u_y,\ \tilde{u}_{\tilde{t}} = t$, taking the form

$$1 - \tilde{u}_{\tilde{x}\tilde{x}} - \tilde{u}_{\tilde{y}\tilde{y}} = \epsilon \tilde{u}_{\tilde{t}\tilde{t}}.$$

Notice that all symplectic Monge-Ampére equations are Lagrangian. This is a general fact which does not require the linearisability/integrability.

3 Classification of integrable equations

In this section we classify integrable four-dimensional symplectic Monge-Ampére equations up to the action of the equivalence group $Sp(8)$. In Sect. 3.1 we begin with a geometric discussion which establishes a correspondence between integrable equations and 'maximally singular' hyperplane sections of the Lagrangian Grassmannian $\Lambda^{10} \subset P^{41}$. This leads to the main classification result proved in Sect. 3.2. Lax pairs, symmetry algebras and singular varieties associated with integrable equations arising in the classification are discussed in Sect. 3.3 – 3.5.

3.1 Geometric preliminaries: proof of Theorem 3

Let $v \in V$ be a vector in the 8-dimensional linear symplectic space V. Lagrangian subspaces passing through v form a sub-Grassmannian $\Lambda^6 \subset \Lambda^{10} \subset P^{41}$. Thus, the Lagrangian Grassmannian Λ^{10} is foliated by a 7-parameter family of Λ^6. Each Λ^6 lies in some $P^{13} \subset P^{41}$. Given a four-dimensional symplectic Monge-Ampére equation (1), let π be the corresponding hyperplane in P^{41}. The condition that all traveling wave reductions of the four-dimensional equation are linearisable is equivalent to the requirement that the corresponding hyperplane π is tangential to all $\Lambda^6 \subset \Lambda^{10}$. This can only happen if either of the following conditions is met:
(1) The 7-parameter family of sub-Grassmannians Λ^6 possesses an envelope $E \subset \Lambda^{10}$ which is contained in π;

(2) The hyperplane π is tangential to Λ^{10} along a 4-dimensional subvariety $X^4 \subset \Lambda^{10}$ which meets each Λ^6 (we point out that no hyperplane can be tangential to Λ^{10} along a subvariety of more than four dimensions).
The first case can be ruled out straightaway: the 7-parameter family of sub-Grassmannians Λ^6 is 'too big' to possess a non-trivial envelope (e.g., a one-parameter family of lines in the plane always possesses an envelope, however, the full two-parameter family of lines doesn't). Thus, we are left with the second possibility, which establishes the necessity part of Theorem 3. The sufficiency will follow from the classification of all non-degenerate equations with a four-dimensional variety of tangency provided in Sect. 3.2: the requirement of the existence of such X^4 proves to be very strong indeed, and leads to a finite list of examples all of which turn out to be integrable.

In the following we will call the variety of tangency X^4 *the singular variety* of the symplectic Monge-Ampére equation defined by the hyperplane section π. Indeed, X^4 coincides with the set of singular points of the intersection of π with Λ^{10}.

The condition that the singular variety X^4 meets *each* Λ^6 is essential for the integrability. In more detail, consider the incidence variety $Y \subset \Lambda^{10} \times PV$,

$$Y = \{(l, \langle v \rangle) \mid v \in l\},$$

along with the two projections $\pi_1 \colon Y \to \Lambda^{10}$ and $\pi_2 \colon Y \to PV$. Then $\pi_1^{-1}(X^4)$ is a 7-dimensional closed algebraic variety, and the projection $\pi_2(\pi_1^{-1}(X^4))$ is a closed algebraic subvariety in PV. The condition that X^4 meets all sub-Grassmannians Λ^6 is equivalent to $\pi_2(\pi_1^{-1}(X^4)) = PV$. This allows us to check this condition using affine charts on Λ^{10} which have non-empty intersections with X^4.

Example 1. Let us describe the singular variety x^4 for the non-degenerate linear equation $u_{12} = u_{34}$. As shown above, the corresponding hyperplane has a contact of order 2 with the Lagrangian Grassmannian Λ^{10} at the Lagrangian subspace l_∞ given by $x^i = 0$. There is also a natural conformal structure on l_∞ associated with this equation. Explicitly, it is given by $du_1 du_2 - du_3 du_4 = 0$. A direct computation in various affine charts of Λ^{10} shows that the set of points where the corresponding hyperplane is tangential to Λ^{10} consists of all Lagrangian subspaces l such that one of the two following cases holds:

- l has a 3-dimensional intersection with l_∞;
- l has a 2-dimensional intersection with l_∞ which is isotropic with respect to the above conformal structure.

Altogether, these points form a 4-dimensional closed algebraic variety with three irreducible components: one for subspaces with a 3-dimensional intersection, and two for subspaces with 2-dimensional isotropic intersections.

It is easy to see that the first component meets all Λ^6, that is for any $v \in V$ there exists a Lagrangian subspace $l \subset V$ such that $v \in l$ and $\dim l \cap l_\infty = 3$. Moreover, if v does not lie in l_∞, such Lagrangian subspace l is unique.

Example 2. Consider the equation $\operatorname{Hess} u = 1$, which is known to be non-integrable. Let us show that its singular variery X^4 misses most of the Λ^6. Partial Legendre transform brings it to $u_{11}u_{22} - u_{12}^2 = u_{33}u_{44} - u_{34}^2$. The set of points where the corresponding hyperplane is tangential to Λ^{10} is given by the equations $u_{11} = u_{22} = u_{12} = 0$ and $u_{33} = u_{44} = u_{34} = 0$. In other words, the intersection X_0^4 of X^4 with an affine chart of Λ^{10} defined by subspaces $x^i = 0$ and $u_j = 0$ is represented as the following set of symmetric matrices,

$$\begin{pmatrix} 0 & 0 & u_{13} & u_{14} \\ 0 & 0 & u_{23} & u_{24} \\ u_{13} & u_{23} & 0 & 0 \\ u_{14} & u_{24} & 0 & 0 \end{pmatrix}.$$

It is easy to check that $\pi_2(\pi_1^{-1}(X_0^4))$ lies in a quadric in PV and, hence, $\pi_2(\pi_1^{-1}(X_0^4)) \neq PV$. This confirms that the equation $\mathrm{Hess}\, u = 1$ is not integrable, even though the corresponding hyperplane is tangential to Λ^{10} along a 4-dimensional variety.

As we have seen in Example 1, the variety of tangency X^4 may consist of several irreducible components with different geometric properties. However, if the equation under study is non-degenerate and non-linearisable, any two generic Lagrangian subspaces from X^4 must have trivial intersections. This follows from the two Lemmas which analyse the cases when generic pairs of Lagrangian subspaces of X^4 have two- or one-dimensional intersections.

Lemma 1. Let X^4 be a four-dimensional subvariety of Λ^{10} such that any two generic Lagrangian subspaces from X^4 have two-dimensional intersections. Then there exists a fixed Lagrangian subspace L which has three-dimensional intersections with all subspaces from X^4. This situation corresponds to linearisable equations.

Proof:

Let l_1, l_2, l_3 be any three generic subspaces from X^4 such that $\dim(l_i \cap l_j) = 2$. Then $\dim(l_1 \cap l_2 \cap l_3) = 1$, and we can introduce the Lagrangian subspace $l = \mathrm{span}\{l_1 \cap l_2, l_1 \cap l_3, l_2 \cap l_3\}$. Any other generic Lagrangian subspace from X^4 will have one-dimensional intersections with $l_1 \cap l_2$, $l_1 \cap l_3$ and $l_2 \cap l_3$. Thus, it will have a three-dimensional intersection with l. Any hyperplane π which is tangential to Λ^{10} along X^4 will automatically contain the osculating subspace to Λ^{10} at l, so that the corresponding equation will be linearisable.

Lemma 2. There exists no non-degenerate symplectic Monge-Ampére equation for which any two generic Lagrangian subspaces in the singular variety X^4 have one-dimensional intersections.

Proof:

Let l_1 and l_2 be two Lagrangian subspaces from X^4 such that $\dim(l_1 \cap l_2) = 1$. Let v be a vector in the intersection. Consider the Lagrangian Grassmannian $\Lambda^6 \subset P^{13}$ generated by v. Then the travelling wave reduction in the direction of v produces a three-dimensional linearisable Monge-Ampére equation such that the corresponding hyperplane in P^{13} is tangential to Λ^6 at two distinct points, $\tilde{l}_1 = l_1/\langle v\rangle$ and $\tilde{l}_2 = l_2/\langle v\rangle$. The points $\tilde{l}_1$ and $\tilde{l}_2$ correspond to three-dimensional Lagrangian subspaces in V^6 which do not intersect by construction. This situation is, however, not possible for linearisable equations in three dimensions (even degenerate). Thus, the travelling wave reduction in the direction of v must produce the 'zero' equation, that is, $\Lambda^6 \subset \pi \cap \Lambda^{10}$ where π is the hyperplane corresponding to our Monge-Ampére equation. When l_1 and l_2 are allowed to vary within X^4, one obtains at least a four-parameter variety of pairwise intersections v (as explained below). Thus, we obtain a four-parameter family of Λ^6 which belong to the nine-dimensional hyperplane section $\pi \cap \Lambda^{10}$. This, however, is not possible since the union of any four-parameter family of Λ^6 must be essentially ten-dimensional.

It remains to prove that we have at least a four-dimensional variety of pairwise intersections v. Let us fix l_1 and consider the set of all intersections $Y(l_1) = \{l_1 \cap l_2, \ l_2 \in X^4\}$. Let s be the dimension of $Y(l_1)$. Fixing v in $Y(l_1)$ we obtain a $(4-s)$-parametric family of Lagrangian subspaces l in X^4 which contain v. Each of them contains an s-parameter subvariety $Y(l)$. Taking a union of $Y(l)$ over all l passing through v, one obtains a four-parameter family of pairwise intersections v. This finishes the proof of Lemma 2.

Let us now return to the general case when any two generic Lagrangian subspaces of X^4 have trivial intersections. Take any two such planes, say l_1 and l_2, and introduce canonical coordinates $x^i, \ u_i$ in V such that l_1, l_2 become the coordinate planes $x^i = 0$ and $u_i = 0$, respectively. Suppose now that a symplectic Monge-Ampére equation corresponds to a hyperplane π which is tangential to the Lagrangian Grassmannian Λ^{10} at the points corresponding to l_1 and l_2. In coordinates x^i, u_i, any such equation takes a 'purely quadratic' form generated by a linear combination of 2×2 minors of the matrix U only. Notice that such form is invariant under $GL(4) \subset Sp(8)$. Let us choose a third Lagrangian plane l_3 which does not intersect l_1 and l_2, and bring it to a normal form using the remaining $GL(4)$-freedom. The requirement that π is tangential to Λ^{10} at the point corresponding to l_3 imposes further constraints on the quadratic part of the equation. Further details of the classification are provided in the next Section.

3.2 Classification results: proof of Theorem 4

Theorem 4. *Over the field of complex numbers any integrable non-degenerate symplectic Monge-Ampére equation is $Sp(8)$-equivalent to one of the following normal forms:*

1. $u_{11} - u_{22} - u_{33} - u_{44} = 0$ *(linear wave equation);*
2. $u_{13} + u_{24} + u_{11}u_{22} - u_{12}^2 = 0$ *(second heavenly equation);*
3. $u_{13} = u_{12}u_{44} - u_{14}u_{24}$ *(modified heavenly equation);*
4. $u_{13}u_{24} - u_{14}u_{23} = 1$ *(first heavenly equation).*
5. $u_{11} + u_{22} + u_{13}u_{24} - u_{14}u_{23} = 0$ *(Husain equation);*
6. $\alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{14}u_{23} = 0$ *(general heavenly equation),* $\alpha + \beta + \gamma = 0$.

Proof:

As shown above, we can always assume that the Monge-Ampére equation contains only quadratic terms, or, in other words, the corresponding hyperplane $\pi \subset P^{41}$ is tangential to the Lagrangian Grassmannian Λ^{10} at the points $l_1 = \infty$ (specified by $x^i = 0$) and $l_2 = 0$ (specified by $u_i = 0$). Any other Lagrangian plane l_3 which has trivial intersections with l_1 and l_2 is defined by a non-degenerate symmetric 4×4 matrix u_{ij}. It can be viewed up to the natural action of the subgroup $GL(4) \subset Sp(8)$ which stabilizes l_1 and l_2. Over the field of complex numbers all such symmetric matrices are equivalent to each other. Fix l_3 such that $u_{14} = 1$, $u_{23} = -1$ and all other u_{ij} equal 0. Assume that, in addition to l_1 and l_2, our hyperplane π is also tangential to Λ^{10} at the point l_3.

The set of all quadratic Monge-Ampére equations such that the corresponding hyperplanes are tangential to the Lagrangian Grassmannian Λ^{10} at the points l_1, l_2, l_3 forms a 10-dimensional vector space spanned by the following expressions:

$$\begin{aligned}
E_0 &= u_{11}u_{22} - u_{12}^2,\\
E_1 &= 1/2(u_{11}u_{24} - u_{12}u_{14} + u_{22}u_{13} - u_{12}u_{23}),\\
E_2 &= 1/6(u_{11}u_{44} - u_{14}^2 + u_{22}u_{33} - u_{23}^2) + 1/3(2u_{13}u_{24} - u_{14}u_{23} - u_{12}u_{34}),\\
E_3 &= 1/2(u_{33}u_{24} - u_{23}u_{34} + u_{44}u_{13} - u_{14}u_{34}),\\
E_4 &= u_{33}u_{44} - u_{34}^2;
\end{aligned}$$

and

$$\begin{aligned}
F_0 &= u_{11}u_{33} - u_{13}^2,\\
F_1 &= 1/2(u_{11}u_{34} - u_{13}u_{14} + u_{33}u_{12} - u_{13}u_{23}),\\
F_2 &= 1/6(u_{11}u_{44} - u_{14}^2 + u_{22}u_{33} - u_{23}^2) + 1/3(2u_{12}u_{34} - u_{14}u_{23} - u_{13}u_{24}),\\
F_3 &= 1/2(u_{22}u_{34} - u_{23}u_{24} + u_{44}u_{12} - u_{14}u_{24}),\\
F_4 &= u_{22}u_{44} - u_{24}^2.
\end{aligned}$$

Denote by E and F the subspaces spanned by E_i and F_j, respectively. Note that permutations of indices $(1,2)$ and $(3,4)$ leave both subspaces invariant, while permutations $(1,4)$ and $(2,3)$ interchange them. Both subspaces are also invariant under the natural action of $SO(4,\mathbb{C})$ given as the set of all linear symplectic transformations stabilizing l_1, l_2 and l_3. Moreover, it is possible to show that under the identification of $SO(4,\mathbb{C})$ with $(SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))/(\pm E_2, \pm E_2)$, each of the copies of $SL(2,\mathbb{C})$ preserves the decomposition $E \oplus F$, acting irreducibly on one summand and trivially on the other.

The standard model for an irreducible 5-dimensional $SL(2,\mathbb{C})$-action is given as the following action on degree 4 polynomials:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} p(t) = (ct+d)^4 p\left(\frac{at+b}{ct+d}\right).$$

The vectors E_i and F_j are chosen in such a way that the correspondence $E_i \leftrightarrow t^i$ and $F_i \leftrightarrow t^i$, $i = 0,\dots,4$, establishes the isomorphisms of $SL(2,\mathbb{C})$-actions on subspaces E and F with the standard model. Polynomials of degree 4 are easily classified with respect to the $SL(2,\mathbb{C})$ action and the multiplication by a non-zero scalar. We have the following set of canonical representatives according to the roots of the polynomial:

(0) 0;

(1) 1 (all four roots of $p(t)$ coincide and are equal to ∞);

(2) t ($p(t)$ has one triple root ∞ and one simple root 0);

(3) t^2 ($p(t)$ has two double roots 0 and ∞);

(4) t^2-1 ($p(t)$ has one double root ∞ and two simple roots ± 1);

(5) $(t^2-1)(t-a)$, $a \neq \pm 1$ ($p(t)$ has four simple roots ∞, ± 1, a).

This classification gives us an easy way to classify vectors in $E \oplus F$ viewed up to the action of $SO(4,\mathbb{C})$, permutation of E and F and the multiplication by a non-zero scalar. Namely, decompose any non-zero vector v in $E \oplus F$ into the difference $v_e - v_f$, $v_e \in E, v_f \in F$ and match each of these two vectors with the corresponding polynomial in the standard model. This gives a pair of polynomials $(p(t), q(t))$, which fully describe the vector v. Next, we classify this pair up to independent actions of $SL(2,\mathbb{C})$ on each of them, permutations $p \leftrightarrow q$ and a multiplication by a non-zero constant. We easily get the set of normal forms $v_{i,j} = (p_i(t), cq_j(t))$, $i, j = 0, \ldots, 5$, $i \geq j$, $c \neq 0$, where p_i and q_j are polynomials listed above under the i-th and j-th items. We can assume $c \neq 0$ since the case $c = 0$ is included in the subcase $q(t) = q_0(t) = 0$. Next, we consider each of these normal forms case-by-case and check for which of the corresponding Monge-Ampére equations the variety of points where the corresponding hyperplane π is tangential to the Lagrangian Grassmannian Λ^{10} has dimension 4. The complete list of all such cases is given below:

1. $p(t) = q(t) = (t^2-1)(t-a), \;\; a \neq \pm 1$;
2. $p(t) = q(t) = t^2 - 1$;
3. $p(t) = t^2 - 1, \; q(t) = t^2$;
4. $p(t) = q(t) = t^2$;
5. $p(t) = q(t) = t$;
6. $p(t) = t, \; q(t) = 1$;
7. $p(t) = q(t) = 1$;
8. $p(t) = t^3 - t, \; q(t) = 0$;
9. $p(t) = t, \; q(t) = 0$;
10. $p(t) = 1, \; q(t) = 0$.

The cases 4, 7 and 10 correspond to the degenerate equations $u_{13}u_{24} - u_{12}u_{34} = 0$, $u_{11}u_{22} - u_{12}^2 = u_{11}u_{33} - u_{13}^2$ and $u_{11}u_{22} - u_{12}^2 = 0$, respectively. In fact, the last two equations are equivalent to the degenerate linear equations $u_{22} = u_{33}$ and $u_{22} = 0$, respectively, via the partial Legendre transform of (x^1, u_1): $(x^1, u_1) \mapsto (-u_1, x^1)$.

The case 9 is given by

$$u_{11}u_{24} - u_{12}u_{14} + u_{22}u_{13} - u_{12}u_{23} = 0,$$

and is equivalent to a non-degenerate linear equation (and, hence, to the linear wave equation) via the partial Legendre transforms of (x^1, u_1) and (x^2, u_2).

Similarly, the case 8 is equivalent to $\mathrm{Hess}\, u = 1$. Indeed, the polynomial $t^3 - t$ is equivalent to $t^4 - 1$ under the action of $SL(2, \mathbb{C})$ since in both cases all 4 roots are distinct and their cross-ratio is equal to -1. The pair of polynomials $p(t) = t^4 - 1$, $q(t) = 0$ corresponds to the equation $u_{11}u_{22} - u_{12}^2 = u_{33}u_{44} - u_{34}^2$ which can be brought to $\mathrm{Hess}\, u = 1$ via the partial Legendre transform of (x^1, u_1) and (x^2, u_2). As we have seen in Example 2, this equation is not integrable.

Other cases can be considered in a similar manner. It appears that they correspond to the remaining non-linear integrable Monge-Ampére equations listed in Theorem 4. Namely, equation 5 can be brought to the modified heavenly equation via the change of variables $(x^1, x^2, x^3, x^4) \mapsto (x^1, x^2 + x^3, x^2 - x^3, x^4)$, partial Legendre transform of (x^1, u_1) and a permutation of indices.

Equations from cases 2 and 3 are brought to the Husain and the first heavenly equations, respectively, via the partial Legendre transform of (x^1, u_1).

The case 6 is brought to the second heavenly equation via the partial Legendre transforms of (x^1, u_1) and (x^2, u_2), and a permutation of indices.

Finally, to establish the correspondence between equation 1 and the general heavenly equation one needs to transform the canonical form $(t^2 - 1)(t - a)$ to $t^4 + 2bt^2 + 1$ under the action of $SL(2, \mathbb{C})$ and then use the following change of variables: $(x^1, x^2, x^3, x^4) \mapsto (x^1 + x^4, x^2 + x^3, x^2 - x^3, x^1 - x^4)$.

The obtained correspondence is summarized in the following table:

Case	$p(t)$	$q(t)$	Equation
1	$(t^2-1)(t-a)$	$(t^2-1)(t-a)$	*general heavenly*
2	t^2-1	t^2-1	*Husain equation*
3	t^2-1	t^2	*first heavenly*
4	t^2	t^2	degenerate equation
5	t^2	t	*modified heavenly*
6	t	1	*second heavenly*
7	1	1	degenerate equation
8	$t(t^2-1)$	0	$\mathrm{Hess}\, u = 1$ (non-integrable)
9	t	0	*linear wave equation*
10	1	0	degenerate equation

It remains to point out that all equations appearing in Theorem 4 are indeed integrable: they possess the required number of hydrodynamic reductions (for the first and second heavenly equations this has been demonstrated in [13, 14]), along with the Lax pairs depending on an auxiliary spectral parameter. This completes the proof of Theorem 4.

Remark. Although all equations listed in Theorem 4 are not $Sp(8)$-equivalent (in fact, not even contact equivalent), there may exist more complicated connections among them.

Below we describe the construction of [29] which links the first and second heavenly equations. Let us take the second heavenly equation in the form $\theta_{1\tilde{1}}+\theta_{2\tilde{2}}+\theta_{\tilde{1}\tilde{1}}\theta_{\tilde{2}\tilde{2}}-\theta_{\tilde{1}\tilde{2}}^2=0$, here θ is a function of the four independent variables $x^1, x^2, \tilde{x}^1, \tilde{x}^2$. Introducing the two-form

$$\Omega=(d\tilde{x}^1-\theta_{\tilde{2}\tilde{2}}dx^1+\theta_{\tilde{1}\tilde{2}}dx^2)\wedge(d\tilde{x}^2+\theta_{\tilde{1}\tilde{2}}dx^1-\theta_{\tilde{1}\tilde{1}}dx^2),$$

one can verify the relations $d\Omega=0,\ \Omega\wedge\Omega=0$. Thus, by Darboux's theorem, there exist coordinates x^3, x^4 such that $\Omega=dx^3\wedge dx^4$. This implies the expansions

$$d\tilde{x}^1-\theta_{\tilde{2}\tilde{2}}dx^1+\theta_{\tilde{1}\tilde{2}}dx^2=u_{13}dx^3+u_{14}dx^4, \qquad d\tilde{x}^2+\theta_{\tilde{1}\tilde{2}}dx^1-\theta_{\tilde{1}\tilde{1}}dx^2=u_{23}dx^3+u_{24}dx^4$$

where u_{ij} are so far arbitrary coefficients satisfying the relation $u_{13}u_{24}-u_{14}u_{23}=1$. Rewriting these relations in the form

$$d\tilde{x}^1=\theta_{\tilde{2}\tilde{2}}dx^1-\theta_{\tilde{1}\tilde{2}}dx^2+u_{13}dx^3+u_{14}dx^4, \qquad d\tilde{x}^2=-\theta_{\tilde{1}\tilde{2}}dx^1+\theta_{\tilde{1}\tilde{1}}dx^2+u_{23}dx^3+u_{24}dx^4,$$

one can see that, by virtue of the relation $\partial\tilde{x}^1/\partial x^2=\partial\tilde{x}^2/\partial x^1$, there exists a function $u(x^1,x^2,x^3,x^4)$ such that $\tilde{x}^1=\partial u/\partial x^1,\ \tilde{x}^2=\partial u/\partial x^2$. In this case coefficients u_{ij} become second order partial derivatives of u, and the relation $u_{13}u_{24}-u_{14}u_{23}=1$ becomes the second heavenly equation. The passage from $x^1, x^2, \tilde{x}^1, \tilde{x}^2$ to the new independent variables x^1, x^2, x^3, x^4 can be viewed as a multi-dimensional analogue of reciprocal transformations familiar from the $(1+1)$-dimensional theory. This link is highly nonlocal, and in any case not a contact equivalence. The only 'local' relations one can write here are the following:

$$\tilde{x}^1=u_1,\ \tilde{x}^2=u_2,\ \theta_{\tilde{2}\tilde{2}}=u_{11},\ \theta_{\tilde{1}\tilde{2}}=-u_{12},\ \theta_{\tilde{1}\tilde{1}}=u_{22}.$$

One can expect similar non-local links between other equations listed in Theorem 4.

3.3 Lax pairs

In this section we present Lax pairs for integrable equations from Theorem 4. Some of them readily follow from the Lax pair for the six-dimensional version of the second heavenly equation [34, 30],

$$u_{15}+u_{26}+u_{13}u_{24}-u_{14}u_{23}=0,$$

which is given by a pair of vectors fields

$$X_1=\partial_6+u_{13}\partial_4-u_{14}\partial_3+\lambda\partial_1, \quad X_2=\partial_5-u_{23}\partial_4+u_{24}\partial_3-\lambda\partial_2,$$

λ =const, such that $[X_1,X_2]=0$ modulo the equation. Straightforward dimensional reductions provide Lax pairs for the first heavenly, second heavenly and Husain equations (we point out that the general heavenly equation *does not* arise as a travelling wave reduction of this six-dimensional equation).

Second heavenly equation $u_{13}+u_{24}+u_{11}u_{22}-u_{12}^2=0$:

$$X_1=\partial_4+u_{11}\partial_2-u_{12}\partial_1+\lambda\partial_1, \quad X_2=\partial_3-u_{12}\partial_2+u_{22}\partial_1-\lambda\partial_2.$$

Modified heavenly equation $u_{13} = u_{12}u_{44} - u_{14}u_{24}$:

$$X_1 = u_{14}\partial_2 - u_{12}\partial_4 + \lambda\partial_1, \quad X_2 = -\partial_3 + u_{44}\partial_2 - u_{24}\partial_4 + \lambda\partial_4.$$

First heavenly equation $u_{13}u_{24} - u_{14}u_{23} = 1$:

$$X_1 = u_{13}\partial_4 - u_{14}\partial_3 + \lambda\partial_1, \quad X_2 = -u_{23}\partial_4 + u_{24}\partial_3 - \lambda\partial_2.$$

Husain equation $u_{11} + u_{22} + u_{13}u_{24} - u_{14}u_{23} = 0$:

$$X_1 = \partial_2 + u_{13}\partial_4 - u_{14}\partial_3 + \lambda\partial_1, \quad X_2 = \partial_1 - u_{23}\partial_4 + u_{24}\partial_3 - \lambda\partial_2.$$

General heavenly equation $\alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{14}u_{23} = 0$:

$$X_1 = u_{34}\partial_1 - u_{13}\partial_4 + \gamma\lambda(u_{34}\partial_1 - u_{14}\partial_3), \quad X_2 = u_{23}\partial_4 - u_{34}\partial_2 + \beta\lambda(u_{34}\partial_2 - u_{24}\partial_3).$$

Notice that in the last case the commutator of X_1, X_2 is not identically zero on the solutions, this condition is substituted by $[X_1, X_2] = 0 \mod X_1, X_2$. Thus, the 2-dimensional distribution spanned by X_1, X_2 is integrable. Modifications of the inverse scattering transform and the $\overline{\partial}$-dressing method for Lax pairs of this type were developed in [24, 25, 5].

3.4 Symmetry algebras

In this section we describe symmetry algebras for equations from Theorem 4. To be precise, we consider only those symmetries which belong to the equivalence group $Sp(8)$ or, equivalently, stabilisers of the corresponding effective forms. It should be emphasized that the full contact symmetry algebras of these equations are infinite dimensional. We adopt the notation of Sect. 2.

Equation	Generators of the symmetry algebra	Dimension
linear wave	$X_{11}+X_{22},\ X_{11}+X_{33},\ X_{11}+X_{44},\ X_{12},\ X_{13},\ X_{14},\ X_{23},\ X_{24},\ X_{34},$ $L_{11}+L_{22}+L_{33}+L_{44},\ L_{12}+L_{21},\ L_{13}+L_{31},\ L_{14}+L_{41},\ L_{23}-L_{32},\ L_{24}-L_{42},\ L_{34}-L_{43}$	16
second heavenly	$X_{11},\ X_{12},\ X_{13}-X_{24},\ X_{14},\ X_{22},\ X_{23},\ X_{33}-L_{24},\ X_{34}+2L_{23},\ X_{44}-L_{13},$ $L_{12}-L_{43},\ L_{21}-L_{34},\ L_{14}-L_{23},\ 2L_{11}+L_{22}+L_{44},\ L_{11}+2L_{22}+L_{33}$	14
modified heavenly	$X_{11},\ X_{22},\ X_{23},\ X_{24}-L_{34},\ X_{33},\ X_{34},\ X_{44}+L_{32},$ $L_{11},\ L_{22}+L_{33},\ 2L_{33}+L_{44},\ L_{24},\ P_{11},\ P_{44}-2L_{23}$	13
first heavenly	$X_{11},\ X_{12},\ X_{22},\ X_{33},\ X_{34},\ X_{44},\ L_{12},$ $L_{21},\ L_{34},\ L_{43},\ L_{11}-L_{22},\ L_{33}-L_{44},\ L_{11}+L_{22}-L_{33}-L_{44}$	13
Husain	$X_{11}-X_{22},\ X_{12},\ X_{33},\ X_{34},\ X_{44},\ L_{11}+L_{22},$ $L_{12}-L_{21},\ L_{33}-L_{44},\ L_{34},\ L_{43},\ P_{11}-P_{22},\ P_{12}$	12
general heavenly	$X_{ii},\ L_{ii},\ P_{ii},\quad i=1,\ldots,4.$	12

The table shows that equations from Theorem 4 are not equivalent under the action of the symplectic group. Indeed, the second heavenly equation is the only one with a 14-dimensional symmetry algebra. Both the modified heavenly and the first heavenly equations have 13-dimensional symmetry algebras, but only the symmetry algebra of the modified heavenly equation has an invariant 2-dimensional non-isotropic subspace in $\mathbb{C}^8$. Similarly, both the general heavenly and the Husain equations have 12-dimensional symmetry algebras, but the symmetry algebra of the general heavenly equation is reductive, while that of the Husain equation is not.

Remark. For any effective n-form ω on the $2n$-dimensional symplectic space V with the symplectic 2-form Ω we can define a bilinear form B_ω [23],

$$B_\omega\colon (X,Y) \mapsto \frac{i_X\omega \wedge i_Y\omega \wedge \Omega}{\Omega^n},$$

which will be symmetric for odd, and skew-symmetric for even values of n. It is known that B_ω is proportional to Ω for all even values of n (we thank Bertrand Banos for pointing this out). As ω is only defined up to a scalar multiple, the form B_ω also makes sense only up to multiplication by a non-zero constant. Hence, if $B_\omega = \lambda\Omega$, we can always assume that either $\lambda = 0$ (i.e., the form B_ω vanishes identically), or $\lambda = 1$ and $B_\omega = \Omega$. It is clear that the vanishing of λ is an invariant condition for the form ω. Below we list the values of λ for integrable Monge-Ampére equations from Theorem 4:

Equation	Coefficient λ
linear wave	0
second heavenly	0
modified heavenly	0
first heavenly	1
Husain	1
general heavenly	1

For instance, this confirms that the modified heavenly equation is not contact equivalent to the first heavenly equation, although both have stabilizers of dimension 13.

3.5 Geometry of singular varieties

We recall that any integrable equation corresponds to a hyperplane which is tangential to the Plücker embedding of the Lagrangian Grassmannian Λ^{10} along a four-dimensional subvariety X^4 which meets all $\Lambda^6 \subset \Lambda^{10}$. The second condition means that the corresponding congruence of Lagrangian subspaces fills the symplectic space V^8. In this section we provide invariant descriptions of X^4 for all examples arising in the classification.

Linear equations: Fix a Lagrangian subspace $L \subset V^8$. Then X^4 consists of all Lagrangian subspaces which have three-dimensional intersections with L. We emphasize that this constitutes only one irreducible component of the singular variety. See Example 1 for more details.

Second heavenly equation: Fix a Lagrangian subspace L and a two-dimensional isotropic plane l such that $l \subset L \subset V^8$. Let k be the P^1 of lines in l, and K the P^1 of three-dimensional subspaces in L which contain l. Fix a projective isomorphism between these two P^1's, $k \leftrightarrow K$. Let c be the union, over k, of two-dimensional isotropic planes which pass through k and are contained in the corresponding K. Similarly, let C be the union, over K, of all Lagrangian subspaces which pass through K. Fix a projective isomorphism $c \leftrightarrow C$ which covers the isomorphism $k \leftrightarrow K$. Then X^4 consists of all Lagrangian subspaces which contain an isotropic plane from c, and have a three-dimensional intersection with the corresponding Lagrangian subspace C.
Modified heavenly equation: Fix a decomposition $V^8 = V^2 + V^6$ into a direct sum of two skew-orthogonal symplectic spaces. Let l be a two-dimensional isotropic plane in V^6, and L the corresponding co-isotropic four-dimensional subspace, $l \subset L \subset V^6$. Let k be the P^1 of lines in l, and K the P^1 of three-dimensional subspaces in L which contain l. Fix a projective isomorphism between these two P^1's, $k \leftrightarrow K$. Finally, let F be the family of Lagrangian subspaces in V^6 which pass through one of the k's, and have a two-dimensional intersection with the corresponding K. Notice that F is three-dimensional. Then X^4 consists of all Lagrangian subspaces which have one-dimensional intersections with V^2, and intersect V^6 along a Lagrangian subspace from F.
First heavenly equation: Fix a decomposition $V^8 = V^4 + V^4$ into a direct sum of two skew-orthogonal symplectic spaces. Fix a two-dimensional Lagrangian plane in each copy of V^4, denote these planes l_1 and l_2, respectively. Let F_1 be the family of all Lagrangian planes in the first copy of V^4 which have one-dimensional intersections with l_1. Similarly, let F_2 be the family of all Lagrangian planes in the second copy of V^4 which have one-dimensional intersections with l_2. Then X^4 consists of all Lagrangian subspaces which intersect each copy of V^4 along two-dimensional Lagrangian planes from F_1 and F_2, respectively.
Husain equation: Fix a decomposition $V^8 = V^2 + V^2 + V^4$ into a direct sum of three skew-orthogonal symplectic spaces. Fix a two-dimensional Lagrangian plane $l \subset V^4$ and consider a family F of all Lagrangian planes in V^4 which have one-dimensional intersections with l. Then X^4 consists of all Lagrangian subspaces which have one-dimensional intersections with each copy of V^2, and intersect V^4 along a Lagrangian plane from F.
General heavenly equation: Fix a decomposition $V^8 = V^2 + V^2 + V^2 + V^2$ into a direct sum of four skew-orthogonal symplectic spaces. Then X^4 consists of all Lagrangian subspaces which have one-dimensional intersections with each copy of V^2 (four-secant subspaces).

One can show that congruences of Lagrangian subspaces in V^8 corresponding to X^4 are of class one, that is, a generic vector in V^8 lies in a unique Lagrangian subspace from X^4.

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