Classical double, R-operators and negative flows of integrable hierarchies.

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Abstract

Using classical double $\mathcal{G}$ of a Lie algebra $\mathfrak{g}$ equipped with the classical R-operator we define two sets of mutually commuting functions with respect to the initial Lie–Poisson bracket on $\mathfrak{g}^*$ and its extensions. We consider in details examples of the Lie algebras $\mathfrak{g}$ with the "Adler–Kostant–Symes" R-operators and the corresponding two sets of mutually commuting functions. Using the constructed commutative hamiltonian flows on different extensions of $\mathfrak{g}$ we obtain zero-curvature equations with $\mathfrak{g}$-valued U-V pairs. Among such the equations are so-called "negative flows" of soliton hierarchies. We illlustrate our approach by examples of abelian and non-abelian Toda field equations.

Keywords: Lie algebras, classical R-operators, classical double, integrable hierarchies.

1 Introduction

The theory of hierarchies of integrable partial differential equations is based on the possibility to represent each of the equations of the hierarchy in the so-called zero-curvature form

$$U_t - V_x + [U, V] = 0,$$

with the corresponding U-V-pair taking values in some infinite-dimensional Lie algebra $\mathfrak{g}$.

There are several approaches to a construction of zero-curvature equations starting from Lie algebras $\mathfrak{g}$. One of the most simple and straightforward of them is the approach of [1], [2] that interprets zero-curvature equations as a compatibility condition of two auxiliary Lax (Euler–Arnold) flows. The commutativity of these Lax flows is guaranteed by the Poisson-commutativity of the corresponding hamiltonians. In this approach the elements U and V from zero-curvature equations coincide with the algebra-valued gradients of the commuting hamiltonians constructed with the help of the Adler–Kostant–Symes scheme. In more details, these hamiltonians coincide with the restrictions of Casimir functions of $\mathfrak{g}$

onto the dual spaces to subalgebras $\mathfrak{g}_\pm$, were $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$. Such an approach permits one to construct two types of integrable equations associated with the Lie algebra $\mathfrak{g}$, namely, integrable equations with the elements U and V belonging to the same Lie subalgebras $\mathfrak{g}_\pm$.

However the approach of [2] does not cover all known integrable equations. In particular, it does not work for integrable equations (sometimes called *negative flows* of integrable hierarchies) possessing U-V-pairs with U-operator belonging to $\mathfrak{g}_+$ and V-operator belonging to $\mathfrak{g}_-$. In the paper [7] such equations were included into the general scheme by showing that the restrictions of Casimir functions of $\mathfrak{g}$ onto the dual spaces to subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ commute not only inside each group but also between the groups. This permits to construct negative flows of integrable hierarchies as a consequence of commutativity of Lax flows generated by "positive" and "negative" hamiltonians. In the papers [8], [9] it was proposed a generalization of the above scheme onto the case of Lie algebras $\mathfrak{g}$ possessing a general classical R operator not always connected with the decomposition $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$ (i.e, not always connected with the Adler–Kostant–Symes scheme). It was shown that the restrictions of Casimir functions of $\mathfrak{g}$ onto the subalgebras $\mathfrak{g}_{R_\pm}$, where $\mathfrak{g}_{R_\pm} = \mathrm{Im} R_\pm$ commute not only inside each group but also between the groups. This observation permits one to obtain two sets of mutually commuting functions on $\mathfrak{g}^*$ and three types of zero-curvature equations, in particular those corresponding to negative flows of soliton hierarchies [9]. Observe that the corresponding commutativity does not follow from the standard R-matrix scheme [3] on $\mathfrak{g}$.

It turns out that the scheme proposed in the papers [8], [9] is still not the most general approach towards a generation of commutative flows on $\mathfrak{g}$ and, hence, not the most general approach to the construction of soliton hierarchies with $\mathfrak{g}$-valued U-V pairs. In particular, it does not include the infinite-component Toda hierarchy and does not produce the corresponding auxiliary Lax equations [11].

In the present paper we propose a more general method of constructing commutative flows and zero-curvature equations with $\mathfrak{g}$-valued U-V pairs. For this purpose we consider commutative flows not on $\mathfrak{g}^*$ but on $\mathcal{G}^*$, where $\mathcal{G}$ is a classical double of $\mathfrak{g}$. We utilize the fact that a classical R-operator on $\mathfrak{g}$ induces a natural R-operator $\mathcal{R}$ on $\mathcal{G} = \mathfrak{g} \oplus \mathfrak{g}$ [5]. This R-operator $\mathcal{R}$ on $\mathcal{G}$ proves to be [5] always of the Adler–Kostant–Symes type, regardless the form of the original operator R on $\mathfrak{g}$. From this it follows that $\mathcal{G}_\mathcal{R} = \mathcal{G}_{\mathcal{R}_+} \ominus \mathcal{G}_{\mathcal{R}_-}$, where algebra $\mathcal{G}_\mathcal{R}$ is a linear space $\mathcal{G}$ equipped with so-called $\mathcal{R}$-bracket [3]. Moreover it turns out that $\mathcal{G}_{\mathcal{R}_+} \simeq \mathfrak{g}$, $\mathcal{G}_{\mathcal{R}_-} \simeq \mathfrak{g}_R$ where the algebra $\mathfrak{g}_R$ is a linear space $\mathfrak{g}$ equipped with the R-bracket [5] and $\mathcal{G}_{\mathcal{R}_\pm} \equiv \mathrm{Im}\mathcal{R}_\pm$.

That is why our first observation is that using the standard R-matrix scheme [3] applied to the Lie algebra $\mathcal{G}$ equipped with the R-operator $\mathcal{R}$ it is possible to obtain a set of commuting flows on extensions of $\mathfrak{g}$ by some Lie algebra $\mathfrak{a}$, where $\mathfrak{a} = \mathfrak{g}_R/J_R$ and J_R is an ideal in $\mathfrak{g}_R$. In the particular case of $J_R = \mathfrak{g}_R$ we rederive in a simple way the result of [8] about commutativity of the restrictions of the Casimir functions of $\mathfrak{g}$ onto the subalgebras $\mathfrak{g}_{R_\pm}$. In such a way we show that the results of [8] fit into the general R-matrix scheme. In the case $J_R = [\mathfrak{g}_R, \mathfrak{g}_R]$ we obtain an important generalization of the above mentioned result, namely, we prove commutativity of the restrictions of the Casimir functions of $\mathfrak{g}$ onto the subalgebras $\mathfrak{g}_{R_\pm}$ shifted with the help of constant elements $c_\mp \in [\mathfrak{g}_{R_\mp}, \mathfrak{g}_{R_\mp}]$ respectively. Let us note that, in this case, the obtained functions commute with respect to the Lie–Poisson bracket on $\mathfrak{g}$ "shifted" by the constant element $c_+ - c_-$.

A consideration of commutative families on more complicated quotients (with non-abelian $\mathfrak{a}$) might also be useful in the theory of soliton equations. Indeed, our second simple observation suggests that, whatever quotient of $\mathcal{G}_{\mathcal{R}}$ one considers, one may choose M-operators from the Lax equations

$$\dot{L} = [L, M]$$

corresponding to the commuting hamiltonians (the Casimir functions restricted onto this quotient) to take the values in $\mathfrak{g}$. From this it follows that one can obtain zero-curvature equations with $\mathfrak{g}$-valued U-V-pairs as a consistency condition of the Lax equations on $\mathfrak{g} \ominus \mathfrak{a}$.

We illustrate the above method by the example of abelian [12], [13] and non-abelian (see [14] and references therein) Toda field equations, that are naturally obtained in the framework of the above scheme if $\mathfrak{g}$ is a loop algebra equipped with various gradings. The corresponding quotient algebra in this case is the simplest non-abelian extension of $\mathfrak{g}$ obtained in the framework of the above construction. In the case if $\mathfrak{g} = gl((\infty))$ equipped with the natural decomposition into a sum of two subalgebras, coinciding with upper triangular and strictly lower triangular matrices, we recover results of [10] about Lie–Poisson structure and Lie-theoretical interpretation of infinite-component Toda field equations, its U-V pair, auxiliary Lax pairs etc. [11].

At the end of the paper for the sake of completeness we also consider the prolongation of the second and third order Poisson structures, existing for the cases of certain R-operators $\mathfrak{g}$ on the classical double. It occurred that the quadratic and cubic structures are always prolonged on $\mathcal{G}$, whenever they exist on $\mathfrak{g}$. Nevertheless their usage in the soliton theory is restricted by the fact that the described above quotient spaces – Poisson spaces of the linear $\mathcal{R}$-bracket on $\mathcal{G}$ are not in general Poisson subspaces of the quadratic and cubic brackets.

The structure of the present paper is the following: in the second section we introduce main definition and notations. In the third section we use classical double in order to obtain commutative families on $\mathfrak{g}^*$ and its extensions. In the fourth section we utilize the obtained results in order to construct zero-curvature equations with $\mathfrak{g}$-valued U-V pairs and illustrate this approach on the examples of abelian and non-abelian Toda field equations. Finally in the fifth section we consider the prolongation of the second and third order Poisson structures on the double.

2 Definitions and notations

2.1 Lie algebras and classical R-operators

Let $\mathfrak{g}$ be a Lie algebra (finite or infinite-dimensional) with a Lie bracket $[\ ,\ ]$, $R : \mathfrak{g} \to \mathfrak{g}$ be a linear operator. The operator R is called a classical R-operator if it satisfies the modified Yang-Baxter equation:

$$R([R(X), Y] + [X, R(Y)]) - [R(X), R(Y)] = [X, Y], \quad \forall X, Y \in \mathfrak{g}.$$

Using classical R-operator it is possible to define another bracket on $\mathfrak{g}$ by the formula [3]:

$$[X,Y]_R = [R(X),Y] + [X,R(Y)], \quad X,Y \in \mathfrak{g}. \tag{1}$$

We will denote $\mathfrak{g}_R$ the linear space $\mathfrak{g}$ equipped with the Lie bracket $[\ ,\ ]_R$.

We will also use hereafter the following notations: $R_\pm \equiv R \pm \mathrm{Id}$.

It is known [3] that the images $\mathfrak{g}_{R_\pm} = \mathrm{Im}R_\pm$ of the maps $R_\pm$ define Lie subalgebras $\mathfrak{g}_{R_\pm} \subset \mathfrak{g}$. As it is easy to see from their definition $\mathfrak{g}_{R_+} + \mathfrak{g}_{R_-} = \mathfrak{g}$, but, in general, this sum is not a direct sum of vector spaces, i.e., in the general case, $\mathfrak{g}_{R_+} \cap \mathfrak{g}_{R_-} \neq 0$.

Remark 1. The situation is much simpler in the case of a Lie algebra $\mathfrak{g}$ with the so-called "Adler–Kostant–Symes" (AKS) decomposition into a direct sum of two Lie subalgebras: $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$. Indeed, if $P_\pm$ are the projection operators onto the subalgebras $\mathfrak{g}_\pm$ then [3] $R = P_+ - P_-$ is a classical R-matrix. It is easy to see that in this case $R_+ = \mathrm{Id} + R = 2P_+$, $R_- = R - \mathrm{Id} = -2P_-$, are proportional to the projection operators onto the subalgebras $\mathfrak{g}_\pm$. It also follows that $\mathfrak{g}_{R_\pm} \equiv \mathfrak{g}_\pm$ and $\mathfrak{g}_{R_+} \cap \mathfrak{g}_{R_-} = 0$. It is also known [3] that in this case

$$\mathfrak{g}_R = \mathfrak{g}_+ \ominus \mathfrak{g}_-.$$

2.2 Classical double

Let us now consider the "double" of the Lie algebra $\mathfrak{g}$, namely the direct sum algebra $\mathcal{G} = \mathfrak{g} \oplus \mathfrak{g}$. Let us identify the elements of $\mathcal{X} \in \mathcal{G}$ with vector columns $\mathcal{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, where $X_i \in \mathfrak{g}$. The bracket of two elements $\mathcal{X}, \mathcal{Y} \in \mathcal{G}$ is given by the standard formula $[\mathcal{X},\mathcal{Y}] = \begin{pmatrix} [X_1,Y_1] \\ [X_2,Y_2] \end{pmatrix}$. The following construction has been developed in [5].

Theorem 2.1 *(i) Given an arbitrary classical R-operator on $\mathfrak{g}$, the operator defined on the double by the formula*

$$\mathcal{R} = \begin{pmatrix} R & -R_- \\ R_+ & -R \end{pmatrix},$$

will be a classical R-operator on $\mathcal{G}$. (ii) The corresponding R-bracket $[\ ,\ ]_{\mathcal{R}}$ on $\mathcal{G}$ has the form:

$$[\mathcal{X},\mathcal{Y}]_{\mathcal{R}} = \begin{pmatrix} [X_1,Y_1]_R - ([X_1,R_-(Y_2)] + [R_-(X_2),Y_1]) \\ -[X_2,Y_2]_R + ([X_2,R_+(Y_1)] + [R_+(X_1),Y_2]) \end{pmatrix}. \tag{2}$$

(iii) The R-matrix $\mathcal{R}$ is of the Adler–Kostant–Symes type.

For convenience of the reader we give a sketch of the proof of the third item of the Theorem. Let us consider the following operators:

$$\mathcal{R}_+ = \mathcal{R} + \mathrm{id} = \begin{pmatrix} R_+ & -R_- \\ R_+ & -R_- \end{pmatrix}, \ \mathcal{R}_- = \mathcal{R} - \mathrm{id} = \begin{pmatrix} R_- & -R_- \\ R_+ & -R_+ \end{pmatrix}.$$

Denote $\mathcal{G}_{\mathcal{R}_\pm} = \mathrm{Im} R_\pm$ the corresponding Lie subalgebras. It is easy to see that

$$\mathcal{R}_+(\mathcal{X}) = \begin{pmatrix} R_+(X_1) - R_-(X_2) \\ R_+(X_1) - R_-(X_2) \end{pmatrix}, \ \mathcal{R}_-(\mathcal{X}) = \begin{pmatrix} R_-(X_1 - X_2) \\ R_+(X_1 - X_2) \end{pmatrix}.$$

From this it follows that $\mathcal{G}_{\mathcal{R}_+} \equiv \mathcal{G}_d \simeq \mathfrak{g}$, $\mathcal{G}_{\mathcal{R}_-} \simeq \mathfrak{g}_R$ where

$$\mathcal{G}_d = \left\{ \left(\begin{array}{c} X \\ X \end{array} \right) | X \in \mathfrak{g} \right\}, \ \mathcal{G}_{\mathcal{R}_-} = \left\{ \left(\begin{array}{c} R_-(X) \\ R_+(X) \end{array} \right) | X \in \mathfrak{g} \right\}.$$

It is easy to see that $\mathrm{Ker}\mathcal{R}_+ = \mathrm{Im}\mathcal{R}_-$ and $\mathrm{Ker}\mathcal{R}_- = \mathrm{Im}\mathcal{R}_+$. Hence the decomposition $\mathcal{G} = \mathcal{G}_{\mathcal{R}_+} + \mathcal{G}_{\mathcal{R}_-}$ is a decomposition into a direct sum of vector spaces and operator $\mathcal{R}$ is of the AKS type, i.e. $\mathcal{R} = (\mathcal{P}_{\mathcal{G}_{\mathcal{R}_+}} - \mathcal{P}_{\mathcal{G}_{R_-}})$. This fact also follows from the easily proved identities $\mathcal{R}_\pm^2 = 2\mathcal{R}_\pm$, $\mathcal{R}_+\mathcal{R}_- = 0$ implying that $\mathcal{R}_\pm$ are proportional to projection operators and their images do not intersect. That is why we have that

$$\mathcal{G}_\mathcal{R} = \mathcal{G}_{\mathcal{R}_+} \ominus \mathcal{G}_{R_-}.$$

For the $\mathcal{R}$-bracket on the double this identity means that

$$[\mathcal{X}, \mathcal{Y}]_\mathcal{R} = 2([\mathcal{X}_+, \mathcal{Y}_+] - [\mathcal{X}_-, \mathcal{Y}_-]),$$

where $\mathcal{X}_+ \equiv \mathcal{R}_+(\mathcal{X})$, $\mathcal{X}_- \equiv \mathcal{R}_-(\mathcal{X})$ etc. or, more explicitly

$$[\mathcal{X}, \mathcal{Y}]_\mathcal{R} = \begin{pmatrix} [(R_+(X_1) - R_-(X_2)), (R_+(Y_1) - R_-(Y_2))] \\ [(R_+(X_1) - R_-(X_2)), (R_+(Y_1) - R_-(Y_2))] \end{pmatrix} - \begin{pmatrix} [(R_-(X_1 - X_2)), R_-(Y_1 - Y_2)] \\ [(R_+(X_1 - X_2)), R_+(Y_1 - Y_2)] \end{pmatrix}.$$

Remark 2. In the case of the Adler–Kostant–Symes R-operators all the formulas of this subsection are substantially simplified. In particular, the action of the R-operator $\mathcal{R}$ on the element $\mathcal{X}$ is given by the formula: $\mathcal{R}(\mathcal{X}) = \left(\begin{array}{c} X_1^+ - X_1^- + 2X_2^- \\ 2X_1^+ - X_2^+ + X_2^- \end{array} \right)$, the R-bracket (1) is written as follows:

$$[\mathcal{X}, \mathcal{Y}]_\mathcal{R} = \begin{pmatrix} [X_1^+, Y_1^+] - [X_1^-, Y_1^-] - ([X_1, Y_2^-] + [X_2^-, Y_1]) \\ -[X_2^+, Y_2^+] + [X_2^-, Y_2^-] + ([X_2, Y_1^+] + [X_1^+, Y_2]) \end{pmatrix},$$

where $X_i = X_i^+ + X_i^-$, $Y_i = Y_i^+ + Y_i^-$ and $X_i^\pm = P_\pm(X)$, $Y_i^\pm = P_\pm(Y)$.

2.3 Dual spaces, Lie–Poisson brackets and invariant functions

Let $\mathfrak{g}^*$ be the dual space to $\mathfrak{g}$ and $\langle\ ,\ \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$ be the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Let $\{X_i | i \in I\}$ be a basis in the Lie algebra $\mathfrak{g}$, where the set I is finite for the case of finite dimensional Lie algebra and countable in the infinite-dimensional case. Let $\{X_i^* | i \in I\}$, $\langle X_j^*, X_i \rangle = \delta_{ij}$, be a basis in the dual space $\mathfrak{g}^*$. Let $L = \sum\limits_i L_i X_i^* \in \mathfrak{g}^*$ be a generic element of

$\mathfrak{g}^*$, L_i be the coordinate functions on $\mathfrak{g}^*$. Let us consider the standard Lie–Poisson bracket between $F_1, F_2 \in C^\infty(\mathfrak{g}^*)$:

$$\{F_1(L), F_2(L)\} = \langle L, [\nabla F_1, \nabla F_2]\rangle,$$

where $\nabla F_k(L) = \sum\limits_{i\in I} \dfrac{\partial F_k(L)}{\partial L_i} X_i$ is a so-called algebra-valued gradient of F_k. Like in [3] an R-operator provides us with the so-called "R-bracket" on $\mathfrak{g}^*$ in the following way

$$\{F_1(L), F_2(L)\}_R = \langle L, [\nabla F_1, \nabla F_2]_R\rangle, \tag{3}$$

Let us consider the dual space $\mathcal{G}^*$ to the double. We identify its elements $\mathcal{L} \in \mathfrak{g}^* \oplus \mathfrak{g}^*$ with the vector columns: $\mathcal{L} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$ where $L_1,\ L_2 \in \mathfrak{g}^*$ and the pairing between $\mathcal{L} \in \mathcal{G}^*$ and $\mathcal{X} \in \mathcal{G}$ is defined as follows:

$$\langle \mathcal{L}, \mathcal{X}\rangle = \langle L_1, X_1\rangle + \langle L_2, X_2\rangle.$$

It is also defined the original Lie–Poisson bracket on $\mathcal{G}^*$

$$\{F_1(\mathcal{L}), F_2(\mathcal{L})\} = \langle \mathcal{L}, [\widetilde{\nabla} F_1, \widetilde{\nabla} F_2]\rangle, \tag{4}$$

and R-bracket on $\mathcal{G}^*$ corresponding to the R-operator $\mathcal{R}$

$$\{F_1(\mathcal{L}), F_2(\mathcal{L})\}_{\mathcal{R}} = \langle \mathcal{L}, [\widetilde{\nabla} F_1, \widetilde{\nabla} F_2]_{\mathcal{R}}\rangle \tag{5}$$

where

$$\widetilde{\nabla} F = \begin{pmatrix} \nabla_1 F \\ \nabla_2 F \end{pmatrix}$$

and $\nabla_{1,2} F$ is the algebra-valued gradient with respect to the variable $L_{1,2}$.

Denote R^* the adjoint operator to R,

$$R^* : \mathfrak{g}^* \to \mathfrak{g}^*, \quad \langle R^*(L), X\rangle \equiv \langle L, R(X)\rangle.$$

It is easy to see that the adjoint operators to $\mathcal{R}_\pm$ have the following form:

$$\mathcal{R}_+^*(\mathcal{L}) = \begin{pmatrix} R_+^*(L_1) + R_+^*(L_2) \\ -(R_-^*(L_1) + R_-^*(L_2)) \end{pmatrix}, \quad \mathcal{R}_-^*(\mathcal{L}) = \begin{pmatrix} R_-^*(L_1) + R_+^*(L_2) \\ -(R_-^*(L_1) + R_+^*(L_2)) \end{pmatrix} \tag{6}$$

We will need these explicit formulas below while constructing Poisson-commuting integrals.

In the subsequent considerations we will also need to know the explicit form of Casimir functions of $\mathcal{G}$. In more details, let $I(L) \in I^G(\mathfrak{g}^*)$ be a Casimir function of $\mathfrak{g}$, i.e.

$$\{I(L), F(L)\} = 0 \quad \forall F(L) \in S(\mathfrak{g}^*).$$

Denote $\{I_k(L)\}_{k\in K}$ the set of generators of the ring of Casimirs of $\mathfrak{g}^*$. Here the set of labels K is infinite if the Lie algebra is infinite-dimensional.

Lemma 2.1 *The ring of Casimir functions of $\mathcal{G}$ is generated by the following functions*

$$I_{k,1}(\mathcal{L}) \equiv I_k(L_1),\ I_{k,2}(\mathcal{L}) \equiv I_k(L_2), \quad k \in K. \tag{7}$$

The proof is straightforward.

Let us say few words about quadratic Casimirs and commuting functions that are obtained with their help. Let (,) be an invariant form on $\mathfrak{g}$. With the help of the latter we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Then we have the following obvious second-order Casimir functions (or generating functions of formal Casimirs in the case of loop algebras)

$$I_{2,1} = \frac{1}{2}(L_1, L_1),\ I_{2,2} = \frac{1}{2}(L_2, L_2).$$

3 Classical double and commuting flows.

In order to pass to the main construction of the present article let us remind the following theorem that can be obtained from the general theory of R-brackets [3] applied to the case of the classical double.

Theorem 3.1 *(i) The Casimir functions $I_{k,\epsilon}(\mathcal{L})$ of the Lie–Poisson brackets of $\mathcal{G}$ commute with respect to the brackets $\{\ ,\ \}_{\mathcal{R}}$ on $\mathcal{G}^*$.*
(ii) The hamiltonian flows

$$\frac{d}{dt_k^{\epsilon}} F(\mathcal{L}) = \{F(\mathcal{L}), I_{k,\epsilon}\}_{\mathcal{R}}\,, \quad k \in K, \quad \epsilon = 1, 2$$

generated by the Casimir invariants $I_{k,\epsilon}$ are written in the Lax-type form:

$$\frac{d\mathcal{L}}{dt_k^{\epsilon}} = ad^*_{\mathcal{R}_+ \tilde{\nabla} I_{k,\epsilon}} \mathcal{L}. \tag{8}$$

Now, let us briefly consider commuting flows on $\mathfrak{g}$ and its extensions that can be obtained using the theory of classical double. For this purpose let us remind that, using the fact that the projection onto the quotient algebra is a canonical homomorphism, one can deduce the following

Corollary 3.1 *Let J be an ideal in $\mathcal{G}_{\mathcal{R}}$. Denote $\pi : \mathcal{G} \to \mathcal{G}/J$ the projection onto the quotient algebra. Let $\pi^* : (\mathcal{G}/J)^* \to \mathcal{G}^*$ be the dual map. Then:*

(i) The functions $I_{k,\epsilon}(\pi^*(\mathcal{L}))$ of $\mathcal{G}$ commute with respect to the brackets $\{\ ,\ \}_{\mathcal{R}}$ on $(\mathcal{G}/J)^*$.

(ii) The hamiltonian flows corresponding to functions $I_{k,\epsilon}(\pi^*(\mathcal{L}))$ are written in the Lax-type form:

$$\pi^*(\frac{d\mathcal{L}}{dt_k^{\epsilon}}) = ad^*_{M_{k,\epsilon}} \pi^*(\mathcal{L}), \quad k \in K, \quad \epsilon = 1,\, 2 \tag{9}$$
$$M_{k,\epsilon} = \mathcal{R}_+(\pi \widetilde{\nabla} I_{k,\epsilon}(\pi^*(\mathcal{L})).$$

Let us now assume that there exist non-trivial ideals $J_{R_\pm} \subset \mathfrak{g}_{R_\pm}$ such that the quotients $\mathfrak{g}_{R_\pm}/J_{R_\pm}$ are finite-dimensional. In such a case, applying the above Corollary 3.1 and taking the quotient over the ideal $J = J_{R_+} + J_{R_-}$ we will obtain a Poisson-commuting set of functions on the dual space to the finite-dimensional extensions of $\mathfrak{g}$. Indeed, we have

$$\mathcal{G}_{\mathcal{R}}/J = (\mathcal{G}_{\mathcal{R}_+} \ominus \mathcal{G}_{\mathcal{R}_-})/J \simeq \mathfrak{g} \ominus \mathfrak{a}, \text{ where } \mathfrak{a} \simeq \mathcal{G}_{\mathcal{R}_-}/J.$$

Remark 3. For the quotient algebras described above the M-operators from the Lax equations (9) have the following form:

$$M_{k,1} = \begin{pmatrix} R_+(\nabla I_{k,1}(\pi^*(\mathcal{L})) \\ R_+(\nabla I_{k,1}(\pi^*(\mathcal{L})) \end{pmatrix}, \quad M_{k,2} = -\begin{pmatrix} R_-(\nabla I_{k,2}(\pi^*(\mathcal{L})) \\ R_-(\nabla I_{k,2}(\pi^*(\mathcal{L})) \end{pmatrix},$$

i.e. they belong to the diagonal subalgebra and may be identified with the elements of $\mathfrak{g}$.

This fact will be used when constructing zero-curvature equations with values in $\mathfrak{g}$. Note that the corresponding Lax equations (9), dynamical variables, Poisson brackets etc. are written on the double of $\mathfrak{g}$. Note also that the absence of the operator π in the above formulas for the M-operators is explained by the fact that the ideal J in this case was chosen in such a way that $\mathcal{R}_+\pi = \mathcal{R}_+$.

Example 1. Let us consider the case of Adler–Kostant–Symes R-operators: $R = P_+ - P_-$. Let us describe more explicitly the quotient algebras $\mathcal{G}_{\mathcal{R}}/J$ and the corresponding dual spaces. We have $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, $\mathfrak{g}_{R_\pm} = \mathfrak{g}_\pm$ and $J_{R_\pm} \equiv J_\pm$ are ideals in $\mathfrak{g}_\pm$. Elements of the corresponding quotient $\mathcal{G}_{\mathcal{R}}/J$, where and $J = J_+ + J_-$, have the form: $\mathcal{X} = \begin{pmatrix} X_1^+ + X_1^{-'} \\ X_2^- + X_2^{+'} \end{pmatrix}$, where $X_1^{-'} \in \mathfrak{g}_-/J_-$, $X_2^{+'} \in \mathfrak{g}_+/J_+$. The corresponding dual space consists of the elements $\mathcal{L} = \begin{pmatrix} L_1^+ + L_1^{-'} \\ L_2^- + L_2^{+'} \end{pmatrix}$, where $L_1^{-'} \in (\mathfrak{g}_-/J_-)^*$, $L_1^+ \in (\mathfrak{g}_+)^*$, $L_2^- \in (\mathfrak{g}_-)^*$, $L_2^{+'} \in (\mathfrak{g}_+/J_+)^*$.

In the next subsections we will consider in more details the cases of $\mathfrak{a} \simeq \mathcal{G}_{\mathcal{R}_-}/J = 0$ and of an abelian $\mathfrak{a}$ that lead to the commutative algebras of integrals on $\mathfrak{g}^*$ itself.

3.1 "Dual" R-matrix commutativity.

Let us at first consider the consequences of the general Theorem 3.1 to a construction of Poisson-commuting sets on $\mathfrak{g}^*$ with respect to the standard Lie–Poisson brackets $\{\ ,\ \}$. The following Theorem holds true:

Theorem 3.2 *(i) The functions $I_k(R^*_\pm(L))$ on $\mathfrak{g}^*$ generate an abelian subalgebra in $C^\infty(\mathfrak{g}^*)$ with respect to the Lie–Poisson brackets $\{\ ,\}$ on $\mathfrak{g}^*$:*

$$\{I_k(R^*_+(L)), I_l(R^*_+(L))\} = 0,\ \{I_k(R^*_-(L)), I_l(R^*_-(L))\} = 0,\ \{I_k(R^*_+L), I_l(R^*_-(L))\} = 0.$$

(ii) The hamiltonian equations corresponding to the hamiltonians $I_k^{R_\pm}(L)$ *are written in the Lax-type form:*

$$\frac{dL}{dt_k^\pm} = ad^*_{M_k^\pm} L \qquad (10)$$
$$M_k^\pm = \nabla I_k(R^*_\pm(L)).$$

Proof. In order to prove the item (i) of the theorem let us project the functions $I_{k,\epsilon}(\mathcal{L})$, $\epsilon \in \overline{1,2}$ onto the dual space to the quotient algebra $\mathcal{G}_{\mathcal{R}}/\mathcal{G}_{R_-}$ isomorphic to the subalgebra $\mathcal{G}_{R_+}$. Due to the explicit formulas (6), (7) we obtain the following expressions for the projected Casimirs:

$$I_{k,1}(P^*_{\mathcal{G}_{R_+}}(\mathcal{L})) = I_k(R^*_+(L_1+L_2)),\ \ I_{k,2}(P^*_{\mathcal{G}_{R_+}}(\mathcal{L})) = I_k(R^*_-(L_1+L_2)),$$

where we have used that $P^*_{\mathcal{G}_{R_+}} = \frac{1}{2}\mathcal{R}^*_+$ and assumed that I_k are homogeneous function of L.

Let us observe that, due to the fact that $\mathcal{R}_+$ is projection operator and the corresponding R-operator $\mathcal{R}$ is of the Adler–Kostant–Symes type we have:

$$\{F(\mathcal{R}^*_+(\mathcal{L})), G(\mathcal{R}^*_+(\mathcal{L}))\} = \{F(\mathcal{R}^*_+(\mathcal{L})), G(\mathcal{R}^*_+(\mathcal{L}))\}_{\mathcal{R}}.$$

Using the fact that the projection onto the quotient algebra is a canonical homomorphism we obtain:

$$\{F(\mathcal{R}^*_\pm(\mathcal{L})), G(\mathcal{R}^*_\pm(\mathcal{L}))\}_{\mathcal{R}} = (\{F(\mathcal{L}), G(\mathcal{L})\}_{\mathcal{R}})\,|_{\mathcal{L}=\mathcal{R}^*_\pm(\mathcal{L})}.$$

Hence, putting $F = I_{k,\epsilon}$, $G = I_{l,\epsilon'}$, we will have:

$$\{I_{k,\epsilon}(\mathcal{R}^*_+(\mathcal{L})), I_{l,\epsilon'}(\mathcal{R}^*_+(\mathcal{L}))\} = (\{I_{k,\epsilon}(\mathcal{L}), I_{l,\epsilon'}(\mathcal{L})\}_{\mathcal{R}})|_{\mathcal{L}=\mathcal{R}^*_+(\mathcal{L})},$$

where $\epsilon, \epsilon' \in \overline{1,2}$. On the other hand, $\{I_{k,\epsilon}(\mathcal{L}), I_{l,\epsilon'}(\mathcal{L})\}_{\mathcal{R}} = 0$ by virtue of the Theorem 3.1.

Taking into account the explicit form of the functions $I_{k,\epsilon}(\mathcal{L})$ we finally obtain

$$\{I_k(R^*_+(L_1+L_2)), I_l(R^*_+(L_1+L_2))\} = 0, \{I_k(R^*_+(L_1+L_2)), I_l(R^*_-(L_1+L_2))\} = 0,$$

$$\{I_k(R^*_-(L_1+L_2)), I_l(R^*_-(L_1+L_2))\} = 0.$$

Now, in order to prove the item (i) of the theorem it remains to observe that elements of the form $L \equiv L_1 + L_2$ belong to the subspace $(\mathcal{G}_d)^*$ and the corresponding coordinate functions constitute Lie algebra isomorphic to $(\mathfrak{g}, \{\ ,\ \})$ with respect to the initial Lie–Poisson brackets on $\mathcal{G}$.

Item (ii) of the Theorem can be proven using the part (ii) of the Theorem 3.1. It can also be proven by noticing that any hamiltonian equation on $\mathfrak{g}^*$ is re-written in the Euler–Arnold form.

Theorem is proven.

Remark 4. In the paper [7] the above theorem was proven directly without any appeal to the classical double. Nevertheless the proof using the classical double is more simple and makes the Theorem 3.2 fit into the general R-matrix scheme.

Example 2. Let us now consider in more details the case $R = P_+ - P_-$ and more explicitly describe Lie algebra $\mathcal{G}_{\mathcal{R}_+}$ realized as a quotient algebra. We have $\mathrm{Ker}\mathcal{R}_+ = \mathrm{Im}\mathcal{R}_-$ and $\mathrm{Im}\mathcal{R}_- = \begin{pmatrix} X_1^- \\ X_2^+ \end{pmatrix}$. The corresponding quotient algebra $\mathcal{G}_{\mathcal{R}}/\mathrm{Im}\mathcal{R}_-$ can be identified with the linear space consisting of the following elements

$$\mathcal{X} = \begin{pmatrix} X_1^+ \\ X_2^- \end{pmatrix}.$$

Such the space is isomorphic to $\mathcal{G}_{\mathcal{R}_+}$ and the isomorphism is established by the map $\mathcal{R}_+$:

$$\begin{pmatrix} X_1^+ \\ X_2^- \end{pmatrix} \rightarrow \begin{pmatrix} X_1^+ - X_2^- \\ X_1^+ - X_2^- \end{pmatrix}$$

The corresponding dual space consists of the elements $\mathcal{L} = \begin{pmatrix} L_1^+ \\ L_2^- \end{pmatrix}$. Observe that such elements can be identified with the elements $L = L_1^+ + L_2^-$ of the linear space $\mathfrak{g}^*$. Moreover, the corresponding Lie–Poisson brackets of these elements on $\mathcal{G}_{\mathcal{R}}/\mathrm{Im}\mathcal{R}_-$ coincide with the standard Lie–Poisson bracket of the element $L = L^+ + L^-$ on $\mathfrak{g}^*$.

The Casimir functions $I_{k,1}(\mathcal{L})$ and $I_{k,2}(\mathcal{L})$ restricted to the dual space of the quotient algebra are simply the functions $I_k(L_1^+)$ and $I_k(L_2^-)$. After the above identification they reduce to the functions $I_k(L^+)$ and $I_k(L^-)$ on $\mathfrak{g}^*$.

3.2 "Shift of the argument" and commutative algebras.

Let us now consider commutative subalgebras of functions on $\mathfrak{g}^*$ generalizing the commutative algebras constructed in the previous subsection that depend on additional parameters and can be obtained using the theory of classical double.

The method that allows one to introduce additional parameters into commutative subalgebras is the so-called shift of the argument. The following theorem holds true.

Theorem 3.3 *Let $c_\pm$ be constant elements of $\mathfrak{g}^*_{R_\pm}$ such that $c_\pm \perp ([\mathfrak{g}_{R_+}, \mathfrak{g}_{R_+}] \bigcup [\mathfrak{g}_{R_-}, \mathfrak{g}_{R_-}])$ and $I_k(L)$, $I_l(L)$ be Casimir functions of $\mathfrak{g}$. Then (i)*

$$\{I_k(R_+^*(L) + c_-), I_l(R_+^*(L) + c_-)\}_c = 0, \{I_k(R_-^*(L) + c_+), I_l(R_-^*(L) + c_+)\}_c = 0, \quad (11)$$

$$\{I_k(R_+^*(L) + c_-), I_l(R_-^*(L) + c_+)\}_c = 0, \quad (12)$$

where $\{\ ,\ \}_c$ is the shifted bracket:

$$\{F_1(L), F_2(L)\}_c = \langle L, [\nabla F_1, \nabla F_2]\rangle + \langle c_- - c_+, [\nabla F_1, \nabla F_2]\rangle \quad (13)$$

(ii) The corresponding hamiltonian equations are written in the Euler-Arnold form:

$$\frac{dL}{dt_k^\pm} = ad^*_{\nabla I_k(R_\pm^*(L)+c_\mp)}(L + c_- - c_+). \quad (14)$$

Proof. In order to prove item (i) of this theorem let us take into account that $\mathcal{G}_{\mathcal{R}} = \mathcal{G}_{\mathcal{R}_+} \ominus \mathcal{G}_{\mathcal{R}_-}$. Hence, $[\mathcal{G}_{\mathcal{R}}, \mathcal{G}_{\mathcal{R}}] = [\mathcal{G}_{\mathcal{R}_+}, \mathcal{G}_{\mathcal{R}_+}] \ominus [\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$. Let us explicitly describe the ideal $[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$. We have $\mathcal{G}_{\mathcal{R}_-} = \{\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mid \text{where } X_1 \in \mathfrak{g}_{R_-}, X_2 \in \mathfrak{g}_{R_+}\}$. From this it follows that element $C = (c_1, c_2) \in \mathcal{G}^*$ is orthogonal to $[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$ if $c_1 \perp [\mathfrak{g}_{R_-}, \mathfrak{g}_{R_-}]$, $c_2 \perp [\mathfrak{g}_{R_+}, \mathfrak{g}_{R_+}]$. On the other hand, as it follows from the explicit form of the elements of $\mathcal{G}^*_{\mathcal{R}_-}$, $c_2 = -c_1 = -c$. Hence $C = (c, -c)$ is an element of the dual space to the Lie subalgebra $\mathcal{G}_{\mathcal{R}_-}$. It is orthogonal to $[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$ if $c \perp ([\mathfrak{g}_{R_+}, \mathfrak{g}_{R_+}] \bigcup [\mathfrak{g}_{R_-}, \mathfrak{g}_{R_-}])$.

That is why, factorizing the Lie algebra $\mathcal{G}_{\mathcal{R}}$ over the ideal $[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$ and taking into account that projection onto the quotient algebra is a canonical homomorphism, applying the Theorem 3.1 to the Casimir functions $I_{k,\epsilon}$, $I_{l,\epsilon'}$ we obtain that

$$\{I_k(R^*_+(L_1 + L_2) + c), I_l(R^*_+(L_1 + L_2) + c)\} = \{I_k(R^*_-(L_1 + L_2) + c), I_l(R^*_-(L_1 + L_2) + c)\} = 0,$$
$$\{I_k(R^*_+(L_1 + L_2) + c), I_l(R^*_-(L_1 + L_2) + c)\} = 0.$$

In such a way we have obtain a commutative subalgebra with a shift element c entering symmetrically in both "positive" and "negative" integrals, i.e. c has components belonging both to $\mathfrak{g}^*_{R_-}$ and $\mathfrak{g}^*_{R_+}$. Let us also note, that shift of the parts of the Lax matrices belonging to $\mathfrak{g}^*_{R_\pm}$ by the constant element of the very same $\mathfrak{g}^*_{R_\pm}$ can be eliminated by changing the variables. But this will lead to a redefinition of the Poisson brackets. Making such a shift and putting $c_\pm = \pm R^*_\pm(c)$, $L = L_1 + L_2$ we obtain item (i) of the Proposition.

Item (ii) is proved in an analogous way to the item (ii) of the previous Theorem.

Theorem is proven.

Example 3. Let us, like in the previous Examples, consider the case of Adler-Kostant-Symes R-operators: $R = P_+ - P_-$. Let us more explicitly describe the quotient algebras $\mathcal{G}_{\mathcal{R}}/[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$ and the corresponding dual spaces. We have $[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}] = [\mathfrak{g}_+, \mathfrak{g}_+] \ominus [\mathfrak{g}_-, \mathfrak{g}_-]$. Elements of the corresponding quotients $\mathcal{G}_{\mathcal{R}}/[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$ have the form: $\mathcal{X} = \begin{pmatrix} X_1^+ + X_1^{-'} \\ X_2^- + X_2^{+'} \end{pmatrix}$ where $X_1^{-'} \in \mathfrak{g}_-/[\mathfrak{g}_-, \mathfrak{g}_-]$, $X_1^+ \in \mathfrak{g}_+$, $X_2^- \in \mathfrak{g}_-$, $X_2^{+'} \in \mathfrak{g}_+/[\mathfrak{g}_+, \mathfrak{g}_+]$. The corresponding dual space consists of the elements $\mathcal{L} = \begin{pmatrix} L_1^+ + L_1^{-'} \\ L_2^- + L_2^{+'} \end{pmatrix}$, where $L_1^{-'} \in (\mathfrak{g}_-/[\mathfrak{g}_-, \mathfrak{g}_-])^*$, $L_2^{+'} \in (\mathfrak{g}_+/[\mathfrak{g}_+, \mathfrak{g}_+])^*$. The elements $L_1^{-'}$, $L_2^{+'}$ are constant with respect to the Poisson brackets on $\mathcal{G}_{\mathcal{R}}/[\mathcal{G}_{\mathcal{R}_-}, \mathcal{G}_{\mathcal{R}_-}]$ and we can put

$$c_- = L_1^{-'}, \quad c_+ = L_2^{+'}.$$

The Casimir functions restricted to the dual space of the quotient algebra are functions $I_k(L_1^+ + L_1^{-'})$ and $I_l(L_2^- + L_2^{+'})$. Using the same arguments as in the Example 2 and the making the same identification we can write them as $I_k(L^+ + c_-)$ and $I_l(L^- + c_+)$ where L is a generic element of $\mathfrak{g}^*$. By virtue of the above theorem they commute with respect to the brackets $\{\ ,\ \}_c$ on $\mathfrak{g}^*$:

$$\{I_k(L^+ + c_-), I_l(L^+ + c_-)\}_c = 0, \quad \{I_k(L^- + c_+), I_l(L^- + c_+)\}_c = 0,$$
$$\{I_k(L^+ + c_-), I_l(L^- + c_+)\}_c = 0,$$

where the shifted bracket $\{\ ,\ \}_c$ is defined with the help of the formula (13).

4 Integrable hierarchies and negative flows

4.1 Zero curvature equations

Let us remind one of the Lie algebraic approaches to the theory of soliton equations [2]. It is based on the zero-curvature conditions and its interpretation as a consistency condition of two commuting Lax flows.

Theorem 4.1 *Let $\mathfrak{g}$ be an infinite-dimensional Lie algebra of $\mathfrak{a}$-valued meromorphic functions of one complex variable and $\mathfrak{a}$ be a simple Lie algebra. Let H_i be Poisson-commuting polynomial functions on $\mathfrak{g}^*$*

$$\{H_1, H_2\} = 0$$

where $\{\ ,\ \}$ is a standard Lie–Poisson brackets on $\mathfrak{g}^*$. Then their $\mathfrak{g}$-valued gradients satisfy zero-curvature equation:

$$\frac{\partial \nabla H_1}{\partial t_2} - \frac{\partial \nabla H_2}{\partial t_1} + [\nabla H_1, \nabla H_2] = 0 \tag{15}$$

and t_i are parameters along the trajectories of hamiltonian equations

$$\frac{\partial}{\partial t_i} = \{\ \cdot\ , H_i\}, \quad i = 1, 2$$

generated by the hamiltonians H_i.

Proof. In order to prove the Theorem let us first observe that the hamiltonian equations on $\mathfrak{g}^*$ corresponding to the hamiltonians H_i and the standard Lie–Poisson brackets are always written in the Euler–Arnold (generalized Lax) form:

$$\frac{\partial L}{\partial t_i} = ad^*_{\nabla H_i} L,$$

where ∇H_s is an algebra-valued gradient of H_s, i.e. $\nabla H_s = \sum_i \frac{\partial H_s}{\partial L_i} X_i$, and $L = \sum_i L_i X_i^*$ is a generic element of the dual space $\mathfrak{g}^*$. Using the commutativity of the time flows corresponding to $s = 1, 2$ one derives the following identity:

$$ad^*_{\left(\frac{\partial \nabla H_1}{\partial t_2} - \frac{\partial \nabla H_2}{\partial t_1} + [\nabla H_1, \nabla H_2]\right)} L = 0.$$

From this follows that

$$\frac{\partial \nabla H_1}{\partial t_2} - \frac{\partial \nabla H_2}{\partial t_1} + [\nabla H_1, \nabla H_2] = k \nabla I, \tag{16}$$

where I is a Casimir function and k is some constant. Hence the algebra-valued gradients ∇H_i satisfy the "modified" zero-curvature equations (16). On the other hand, it is not difficult to show, for the case of the Lie algebras $\mathfrak{g}$ described in the Proposition the algebra-valued gradients of the Casimir functions are formal power series. In particular, if the coadjoint representation is equivalent to the adjoint one, they are proportional to a power of the generic element of the dual space L, which in such a case is an infinite linear combinations of the basic elements of $\mathfrak{g}$. On the other hand, due to the assumption that all H_i are finite polynomials, their algebra-valued gradients are *finite* linear combinations of the basic elements of the Lie algebra $\mathfrak{g}$. Hence the corresponding modified zero-curvature equations are satisfied if and only if the corresponding coefficient k in these equations is equal to zero, i.e. when they pass to the usual zero-curvature conditions. This proves the Theorem.

Remark 5. Using the same arguments it is possible to also show that a similar theorem holds true for more complicated infinite-dimensional Lie algebras, e.g for the algebras of the type A_∞, C_∞, D_∞ etc. In this case instead of the condition that H_i are finite polynomials one may require less rigid condition.

4.2 Doubles, R-operators and negative flows of soliton hierarchies

Now, using the results of previous section it is possible to construct hierarchy of integrable equations in partial derivatives admitting zero-curvature representations.

Indeed, we have obtained using classical double $\mathcal{G}$, a large commutative algebra on the quotients of $\mathcal{G}^*$. On the other hand, due to the Theorem 4.1 in order to obtain zero-curvature conditions as a consequence of the corresponding commutative flows we will require that $\mathfrak{g}$ be infinite-dimensional and possesses infinitely many Casimir invariants that will produce infinitely many commuting flows. In such a case we will obtain, as a consequence of the general Theorem 4.1, the following Proposition:

Proposition 4.1 *Let algebra $\mathfrak{g}$ be infinite-dimensional. Denote $\mathcal{G}$ its double. Let $\mathcal{G}^*$ be the corresponding dual space. Let J be ideal in $\mathcal{G}_{\mathcal{R}_-}$. Denote $\pi : \mathcal{G} \to \mathcal{G}/J$ the natural projection onto the quotient algebra. Let $\pi^* : (\mathcal{G}/J)^* \to \mathcal{G}^*$ be the dual map. If the functions $I_{k,\epsilon}(\pi^*(\mathcal{L}))$ on $\mathcal{G}^*$ are finite polynomials then the $\mathfrak{g}$-valued functions*

$$V_{k,+} = R_+ \nabla I_{k,1}(\pi^*(\mathcal{L})),\ V_{l,-} = R_- \nabla I_{l,2}(\pi^*(\mathcal{L}))$$

satisfy zero-curvature equation with values in $\mathfrak{g}$:

$$\frac{\partial V_{k,\pm}}{\partial t_l^\pm} - \frac{\partial V_{l,\pm}}{\partial t_k^\pm} + [V_{k,\pm}, V_{l,\pm}] = 0, \tag{17}$$

$$\frac{\partial V_{k,\pm}}{\partial t_l^\mp} - \frac{\partial V_{l,\mp}}{\partial t_k^\pm} + [V_{k,\pm}, V_{l,\mp}] = 0. \tag{18}$$

Remark 6. Note, that equations (17)–(18) define three types of integrable hierarchies: two "small" hierarchies associated with the Lie subalgebras $\mathfrak{g}_{R_\pm}$ defined by equations (17) and one "large" hierarchy associated with the whole Lie algebra $\mathfrak{g}$, that include both types of equations (17) and (18). Equations (18) contain U-V pair with the U-operators taking their values in $\mathfrak{g}_{R_+}$ and V-operator taking the values in $\mathfrak{g}_{R_-}$. They have an interpretation of the "negative flows" of the integrable hierarchy associated with $\mathfrak{g}_{R_\pm}$.

4.3 Case of graded Lie algebras.

In this subsection we will demonstrate how the described above general scheme of production of U-V pairs satisfying zero-curvature equations works for the concrete Lie algebras. We will concentrate on the simplest possible examples associated with the graded Lie algebras.

4.3.1 Quotients of double and invariant functions.

Let us now consider the example of $\mathbb{Z}$-graded algebras and the quotient algebras of the corresponding double. By the definition of graded Lie algebras we have that

$$\mathfrak{g} = \sum_{j\in\mathbb{Z}} \mathfrak{g}_j, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

From this one obtains a decomposition $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, where $\mathfrak{g}_+ = \sum\limits_{j\geq 0} \mathfrak{g}_j$, $\mathfrak{g}_- = \sum\limits_{j<0} \mathfrak{g}_j$ are Lie subalgebras. Denote $P_\pm$ the projection operators onto the Lie subalgebras $\mathfrak{g}_\pm$. Hence $R = P_+ - P_-$ is a classical R-operator [4] . In a standard way [4] one obtains that $J_{+k} = \sum\limits_{j>k} \mathfrak{g}_j$ and $J_{-l} = \sum\limits_{j>l} \mathfrak{g}_{-l}$ are ideals in $\mathfrak{g}_\pm$ and in $\mathfrak{g}_R = \mathfrak{g}_+ \ominus \mathfrak{g}_-$. Hence one can consider the quotient algebra $\mathfrak{g}_R/(J_{+k} \ominus J_{-l})$ and the quotient algebra of the corresponding "double": $\mathcal{G}_{\mathcal{R}}/(J_{+k} \ominus J_{-l})$. The elements of this quotient algebra have the following form: $\begin{pmatrix} X_1^+ + X_1^{-'} \\ X_2^- + X_2^{+'} \end{pmatrix}$ where $X_1^+ \in \mathfrak{g}_+$, $X_1^{-'} \in \sum\limits_{j=1}^{l} \mathfrak{g}_{-j}$, $X_2^- \in \mathfrak{g}_-$, $X_2^{+'} \in \sum\limits_{j=0}^{k} \mathfrak{g}_j$.

The corresponding elements of the dual space have the following explicit form: $\mathcal{L} = \begin{pmatrix} L_1^+ + L_1^{-'} \\ L_2^- + L_2^{+'} \end{pmatrix}$, where $L_1^+ \in \mathfrak{g}_+^*$, $L_1^{-'} \in \sum\limits_{j=1}^{l} \mathfrak{g}_{-j}^*$, $L_2^- \in \mathfrak{g}_-^*$, $L_2^{+'} \in \sum\limits_{j=0}^{k} \mathfrak{g}_j^*$. Let us notice once more that the corresponding components of the Lax operator $L_1 = L_1^+ + L_1^{-'}$ and $L_2 = L_2^- + L_2^{+'}$ are semi-infinite (i.e., infinite only in one direction).

Let us assume that on $\mathfrak{g}$ there is an invariant bilinear form $(\ ,\ )$ such that $(\mathfrak{g}_i, \mathfrak{g}_j) \sim \delta_{i+j,0}$. In this case one can identify the spaces $\mathfrak{g}^*$ and $\mathfrak{g}$ and construct the second order Casimir functions by the following formula:

$$I^0_{2,1} = \frac{1}{2}\sum_{i\in\mathbb{Z}} (L_1^{(i)}, L_1^{(-i)}),\ \ I^0_{2,2} = \frac{1}{2}\sum_{i\in\mathbb{Z}} (L_2^{(i)}, L_2^{(-i)}),$$

where $L_{1,2}^{(\pm i)} \subset \mathfrak{g}_{\mp i}$. Note that on the quotient algebra described above all these expressions are finite polynomials if the space $\mathfrak{g}_i$ is finite-dimensional.

Let us consider several Examples.

4.3.2 General non-abelian Toda systems and graded Lie algebras

Let us consider the above construction in the case $k = 1$, $l = 1$. In this case we have:

$$I^0_{2,1} = \frac{1}{2}(L^{(0)}_1, L^{(0)}_1) + (L^{(1)}_1, L^{(-1)}_1),\ \ I^0_{2,2} = \frac{1}{2}(L^{(0)}_2, L^{(0)}_2) + (L^{(1)}_2, L^{(-1)}_2),$$

and the Lax matrix is: $\mathcal{L} = \begin{pmatrix} L^+_1 + L^{(-1)}_1 \\ L^-_2 + L^{(0)}_2 + L^{(1)}_2 \end{pmatrix}$, where $L^+_1 = \sum\limits_{i=0}^{\infty} L^{(i)}_1$, $L^-_2 = \sum\limits_{i=1}^{\infty} L^{(-i)}_1$. Let us also note, that $L^{(-1)}_1$ is a central element due to the fact that $L^{(-1)}_1 \in (\mathfrak{g}_-/[\mathfrak{g}_-, \mathfrak{g}_-])^*$.

The M-operators corresponding to the above integrals $I^0_{2,1}$, $I^0_{2,2}$ have the following form:

$$M^0_{2,1} = \mathcal{R}_+\nabla I^0_{2,1} = \begin{pmatrix} \bar{L}^{(0)}_1 + \bar{L}^{(1)}_1 \\ \bar{L}^{(0)}_1 + \bar{L}^{(1)}_1 \end{pmatrix}, \quad M^0_{2,2} = \mathcal{R}_+\nabla I^0_{2,2} = -\begin{pmatrix} \bar{L}^{(-1)}_2 \\ \bar{L}^{(-1)}_2 \end{pmatrix}, \text{ where}$$

$\bar{L}^{(0)}_1 = \frac{1}{2}\nabla(L^{(0)}_1, L^{(0)}_1) \in \mathfrak{g}_0$, $\bar{L}^{(1)}_1 = P_+\nabla(L^{(1)}_1, L^{(-1)}_1) \in \mathfrak{g}_1$, $\bar{L}^{(-1)}_2 = P_-\nabla(L^{(1)}_2, L^{(-1)}_2) \in \mathfrak{g}_{-1}$.

Their components, namely, the operators

$$U = \bar{L}^{(0)}_1 + \bar{L}^{(1)}_1, \quad V = \bar{L}^{(-1)}_2 \tag{19}$$

are U-V pairs of abelian and non-abelian Toda field equations, as we will show in a moment. Let us also note, that $\bar{L}^{(1)}_1$ is a constant element because $L^{(-1)}_1$ is a central element that can be identified with a constant.

Let us consider the corresponding zero-curvature equation in the graded Lie algebra

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0.$$

It yields the following equations for the homogeneous components

$$\frac{\partial \bar{L}^{(1)}_1}{\partial t} = 0,\ \frac{\partial \bar{L}^{(0)}_1}{\partial t} = -[\bar{L}^{(1)}_1, \bar{L}^{(-1)}_2],\ \frac{\partial \bar{L}^{(-1)}_2}{\partial x} = [\bar{L}^{(0)}_1, \bar{L}^{(-1)}_2]. \tag{20}$$

The first of these equations is satisfied automatically, using the fact that $\bar{L}^{(1)}_1$ is obtained from the central element $L^{(-1)}_1$ and we can put $\bar{L}^{(1)}_1 = C^{(1)} = const$. Let us solve the last equation of (20). Due to the grading it is easy to see that $\mathfrak{g}_0 \subset \mathfrak{g}$ is a subalgebra. Denote G_0 the corresponding Lie group. Let $g_0 \in G_0$. By direct verification one can show that the substitution

$$\bar{L}^{(-1)}_2 = g_0 C^{(-1)} g_0^{-1},\ \bar{L}^{(0)}_1 = (\partial_x g_0) g_0^{-1}, \quad g_0 = g_0(x, t)$$

where $C^{(-1)}$ is a constant element of the space $\mathfrak{g}_{-1}$, solves the last of the three equations (20). The second of the equations (20) takes after the substitution of this solution the following form:

$$\partial_t \left((\partial_x g_0) g_0^{-1}\right) = -[C^{(1)}, g_0 C^{(-1)} g_0^{-1}]. \tag{21}$$

Equation (21) is the so-called non-abelian Toda field equations [14].

4.3.3 Loop algebras and the standard Toda system

The main example of the above construction is connected with loop algebras. Let $\mathfrak{a}$ be the Lie algebra of a simple Lie group G. Denote $\mathfrak{g} = \mathfrak{a} \otimes Pol(\lambda, \lambda^{-1})$ the loop algebra. Assume that $\mathfrak{a}$ is equipped with an automorphism $\sigma : \mathfrak{a} \to \mathfrak{a}$ of order p. One has a natural decomposition [6]

$$\mathfrak{a} = \sum_{i=0}^{p-1} \mathfrak{a}_i$$

such that

$$\mathfrak{a}_k = \{X \in \mathfrak{a} | \sigma(X) = e^{\frac{2\pi i k}{p}} X\}.$$

In particular $\mathfrak{a}_0$ is the subalgebra stable under the action of automorphism σ.

Let us extend this grading to the loop space $\mathfrak{g}$ prescribing by the definition

$$\begin{aligned} &\deg \lambda = p \\ &\deg X \otimes q(\lambda) = \deg X + \deg q(\lambda). \end{aligned}$$

In this case we obtain

$$\mathfrak{g}_j = \{X(\lambda) \in \mathfrak{a} \otimes Pol(\lambda, \lambda^{-1}) \,|\, \deg X(\lambda) = j\}.$$

In particular $\mathfrak{g}_0 = \mathfrak{a}_0$. Hence the corresponding group $G_0 \subset G$ coincides with the Lie group of the Lie subalgebra $\mathfrak{a}_0$. The equation (21) is written for the generic element of this group.

Let us consider the most interesting example of such a situation corresponding to the case of abelian G_0. Let $\mathfrak{g}$ be a loop algebra with the principal grading. In more details, $\sigma : \mathfrak{a} \to \mathfrak{a}$ is a Coxeter automorphism. Denote h the Coxeter number of $\mathfrak{a}$. We have the corresponding $\mathbb{Z}_h$-grading of $\mathfrak{a}$

$$\mathfrak{a} = \sum_{i=0}^{h-1} \mathfrak{a}_i.$$

The subalgebra $\mathfrak{a}_0$ coincides with the Cartan subalgebra

$$\mathfrak{h} = \mathrm{Span}_{\mathbb{C}}\{H_{\alpha_i} | i \in 1, \ldots, \mathrm{rank}\, \mathfrak{g}\}.$$

Moreover

$$\mathfrak{a}_k = \mathrm{Span}_{\mathbb{C}}\{X_\alpha | \alpha \in \Delta, |\alpha| = k \bmod h\}.$$

Here H_{α_i}, X_α is a Cartan-Weil basis of $\mathfrak{a}$, Δ is the set of all roots, and $|\alpha|$ stands for the height of root. In particular $H_{\alpha_i} = [X_{\alpha_i}, X_{-\alpha_i}]$, where α_i are simple roots.

Let us describe more explicitly the subspaces $\mathfrak{g}_i$. By definition we have:

$$\mathfrak{g}_0 = \mathfrak{h}, \ \mathfrak{g}_k = \sum_{|\alpha|=k} \mathfrak{a}_\alpha + \lambda \sum_{|\alpha|=h-k} \mathfrak{a}_{-\alpha}, \quad k \in \overline{1, h-1},$$

where $\mathfrak{a}_\alpha = Span_{\mathbb{C}}\{X_\alpha\}$. The other graded subspaces are:

$$\mathfrak{g}_{k+nh} = \lambda^n \mathfrak{g}_k, \text{ where } k \in \overline{1, h-1}.$$

Let us consider the corresponding U-V-pair (19)

$$U = (\partial_x g_0) g_0^{-1} + C^{(1)}, \quad V = g_0 C^{(-1)} g_0^{-1}.$$

In this case the group G_0 is abelian and coincides with the Cartan subgroup. Because of this it is easy to parametrize the element g_0 in the following way: $g_0 = \exp \sum_{i=1}^{\mathrm{rank}\mathfrak{a}} \phi_i H_{\alpha_i}$ and obtain that

$$U = \sum_{i=1}^{\mathrm{rank}\mathfrak{a}} \partial_x \phi_i H_{\alpha_i} + \sum_{\alpha_i \in P} c_{\alpha_i}^{(1)} X_{\alpha_i} + \lambda c_{-\theta}^{(1)} X_{-\theta}$$
$$V = \sum_{\alpha_i \in P} c_{\alpha_i}^{(-1)} e^{-\alpha_i(\phi)} X_{-\alpha_i} + \lambda^{-1} c_\theta^{(-1)} e^{-\theta(\phi)} X_\theta,$$

where P is the set of simple roots, θ is the highest root and H_{α_i} is the basic element in the Cartan subalgebra corresponding to the simple root α_i.

It is easy to recognize in this U-V pair the U-V pair of finite-component Toda field equation [13]. The corresponding equations (21) have the following form:

$$\partial_t \partial_x \phi_i = c_{\alpha_i}^{(1)} c_{-\alpha_i}^{(-1)} e^{-\alpha_i(\phi)} + a_i c_{-\theta}^{(1)} c_\theta^{(-1)} e^{\theta(\phi)}, \tag{22}$$

where $\phi = \sum_{i=1}^{\mathrm{rank}\mathfrak{a}} \phi_i H_{\alpha_i}$ and the constants a_i are defined from the decomposition of $H_\theta = [X_\theta, X_{-\theta}]$

$$H_\theta = \sum_{i=1}^{\mathrm{rank}\mathfrak{a}} a_i H_{\alpha_i}$$

It is easy to see from the very form of the equations (22) that the coefficients $c_{-\alpha_i}^{(-1)}$, $c_{\alpha_i}^{(1)}$, $c_\theta^{(-1)}$, $c_{-\theta}^{(1)}$ are redundant and, if non-zero, they can be eliminated from the equations by a rescaling.

4.3.4 Infinite-component Toda system

Let now $\mathfrak{g} = gl((\infty))$. Recall that this is the Lie algebra of infinite matrices

$$M = (M_{ij})_{i,j \in \mathbb{Z}}, \quad M_{ij} = 0 \quad \text{for} \quad |i-j| >> 1.$$

This situation can be considered as the $n \to \infty$ limit of the case $\mathfrak{g} = gl(n)$ of the previous section. However it deserves more careful considerations.

The basis in the algebra $gl((\infty))$ consists of the elements X_{ij}, $i, j \in \mathbb{Z}$ with the standard commutation relations:

$$[X_{ij}, X_{kl}] = \delta_{kj} X_{il} - \delta_{il} X_{kj}.$$

In terms of this basis we have the following graded subspaces of the natural $\mathbb{Z}$-grading:

$$\mathfrak{g}_k = Span_{\mathbb{C}}\{X_{ij} \,|\, j - i = k\}.$$

On $gl((\infty))$ there exists a natural invariant bilinear form $(\ ,\ )$ such that

$$(X_{ij}, X_{kl}) = \delta_{kj}\delta_{il}.$$

Using this form one identifies $\mathfrak{g}^*$ with $\mathfrak{g}$ so that $\mathfrak{g}_k^* = \mathfrak{g}_{-k}$.

Let us consider the classical double of $gl((\infty))$. We apply the construction of Section 4.3.1 to the corresponding dual space and its quotient spaces with respect to the ideals of the form J_{+k} and J_{-l}, k, l are given positive integers. The elements of the dual spaces to these quotients

$$\mathcal{L} = \begin{pmatrix} L_1^+ + L_1^{-'} \\ L_2^- + L_2^{+'} \end{pmatrix} \in [\mathcal{G}_{\mathcal{R}}/(J_{+k} \ominus J_{-l})]^*$$

have the following explicit form

$$L_1^+ = \sum_{i=0}^{\infty} L_1^{(i)}, \quad L_2^- = \sum_{i=1}^{\infty} L_2^{(-i)}, \quad L_1^{-'} = \sum_{j=1}^{l} L_1^{(-i)}, \quad L_2^{+'} = \sum_{j=0}^{k} L_1^{(i)},$$

$L_s^{(i)} \in gl((\infty))_{-i}$ $(s = 1, 2)$. In particular the Lax matrix $\mathcal{L}$ of the infinite-component Toda system will correspond to the case $k = l = 1$. So we will consider only this case in sequel.

We will denote the natural basis in the dual space to $gl((\infty))$ by the same symbols X_{ij}. In this basis the Lax operator $\mathcal{L}$ can be described by the coordinates $l_1^{(m)}(i)$, $l_2^{(m)}(i)$ in the following manner

$$L_s^{(m)} = \sum_{i \in \mathbb{Z}} l_s^{(m)}(i - m) X_{i,i-m}, \quad s = 1,\, 2.$$

As above the coordinates $l_1^{(-1)}(i)$ are Casimirs of the Lie–Poisson bracket on the dual to the quotient $\mathcal{G}_{\mathcal{R}}/(J_{+1} \ominus J_{-1})$. So we can put them to be equal to constants, $l_1^{(-1)}(i) = c_i$. Thus

$$L_1^{(-1)} = \sum_{i \in \mathbb{Z}} c_i X_{i-1,i}.$$

Using the invariant form $(\ ,\ )$ on $gl((\infty))$ we obtain two quadratic Hamiltonians $I_{2,1}^0$, $I_{2,2}^0$ on the double of $gl((\infty))$ having the following explicit form on the quotient space under consideration

$$I_{2,s}^0 = \frac{1}{2} \sum_{i \in \mathbb{Z}} \left(l_s^{(0)}(i)\right)^2 + \sum_{i \in \mathbb{Z}} l_s^{(1)}(i) l_s^{(-1)}(i), \;\; s \in 1, 2.$$

The flows generated by these Hamiltonians are written in the form

$$\frac{\partial \mathcal{L}}{\partial t_s} = \left[\tilde{M}^0_{2,s}, \mathcal{L}\right], \quad s = 1,\, 2$$

where the M-operators with values in the double of $gl((\infty))$ are

$$\widetilde{M}^0_{2,1} = \mathcal{R}_+ \widetilde{\nabla} I^0_{2,1} = \begin{pmatrix} L_1^{(0)} + L_1^{(-1)} \\ L_1^{(0)} + L_1^{(-1)} \end{pmatrix}, \ \widetilde{M}^0_{2,2} = \mathcal{R}_+ \widetilde{\nabla} I^0_{2,2} = \begin{pmatrix} L_2^{(1)} \\ L_2^{(1)} \end{pmatrix}.$$

Despite of the fact that in this case $\mathfrak{g}$ is not a loop algebra and functions $I^0_{2,s}$ are not finite polynomials, one can prove, in a similar way to the proof of the Theorem 4.1, that the corresponding M-operators satisfy zero-curvature condition. Hence, one can write the following $gl((\infty))$-valued U-V-pair satisfying zero-curvature equation:

$$U = L_1^{(0)} + L_1^{(-1)}, \quad V = L_2^{(1)}, \text{ where } L_s^{(i)} \in gl((\infty))_{-i}, \quad s = 1,\, 2.$$

It yields the following equations:

$$\partial_x v_i = v_i(u_{i+1} - u_i),\ \partial_t u_i = c_{i-1} v_{i-1} - c_i v_i,\ i \in \mathbb{Z}, \tag{23}$$

where

$$u_i \equiv l_1^{(0)}(i), \quad v_i \equiv l_2^{(1)}(i), \quad x = t_1, \quad t = t_2.$$

By the substitution $u_i = \partial_x \phi_i$, $v_i = e^{\phi_{i+1} - \phi_i}$ the equations (23) reduce to the usual infinite-component Toda equations [11]:

$$\partial^2_{xt} \phi_i = c_{i-1} e^{\phi_i - \phi_{i-1}} - c_i e^{\phi_{i+1} - \phi_i},\ i \in \mathbb{Z}. \tag{24}$$

As above the constants c_i, if non-zero, can be eliminated by a rescaling.

Remark 7. Note that the equation (24) does not coincide in its form with the $gl(n)$-equations (22) because in this subsection we have worked with another basis in Cartan subalgebra: $H_i \equiv X_{ii}$ instead of $H_{\alpha_i} = X_{ii} - X_{i-1i-1}$.

4.3.5 Lie–Poisson bracket for the infinite-component Toda system

In this subsection for the purpose of illustration we will explicitly describe the R-matrix Lie–Poisson bracket for the case of the Lie algebra $\mathfrak{g} = gl((\infty))$, for its double and for the R-operator corresponding to the natural Adler-Kostant-Symes decomposition used in the previous subsection. We will start from the Lie brackets first and then use the fact that the Lie–Poisson brackets of the coordinate functions can be easily recovered from the Lie brackets of the basic elements.

Let us, for the purpose of convenience introduce the following basis in the algebra $gl((\infty))$:

$$X^{(i)}(m) \equiv X_{m,i+m}, \quad i,\, m \in \mathbb{Z}.$$

The commutation relations in this basis acquires the following form:

$$[X^{(i)}(m), X^{(j)}(n)] = \delta_{i+n-m,0}X^{(i+j)}(m) - \delta_{m-n+j,0}X^{(i+j)}(n).$$

The R-operator in the case under consideration has the following form: $R = P_+ - P_-$, where $P_\pm$ are the projection operators onto the Lie subalgebras generated by $X^{(i)}(m)$, $i \geq 0$ and $X^{(j)}(n)$, $j < 0$ respectively.

The R-bracket on $gl((\infty))$ can be written as follows:

$$[X^{(i)}(m), X^{(j)}(n)]_R = 2(1-\sigma(i)-\sigma(j))\big(\delta_{i+n-m,0}X^{(i+j)}(m) - \delta_{m-n+j,0}X^{(i+j)}(n)\big),$$

where $\sigma(i) = 1$ if $i < 0$, $\sigma(i) = 0$ if $i \geq 0$.

For the double of $gl((\infty))$, namely for the direct sum $gl((\infty)) \oplus gl((\infty))$ we obtain the following $\mathcal{R}$-bracket written for the basic elements $X_s^{(i)}(m)$, $s \in 1, 2$:

$$[X_1^{(i)}(m), X_1^{(j)}(n)]_R = 2(1-\sigma(i)-\sigma(j))\big(\delta_{i+n-m,0}X_1^{(i+j)}(m) - \delta_{m-n+j,0}X_1^{(i+j)}(n)\big), \quad (25a)$$

$$[X_2^{(i)}(m), X_2^{(j)}(n)]_R = 2(\sigma(i)+\sigma(j)-1)\big(\delta_{i+n-m,0}X_2^{(i+j)}(m) - \delta_{m-n+j,0}X_2^{(i+j)}(n)\big), \quad (25b)$$

$$[X_1^{(i)}(m), X_2^{(j)}(n)]_R = 2(\sigma(i)-1)\big(\delta_{i+n-m,0}X_1^{(i+j)}(m) - \delta_{m-n+j,0}X_1^{(i+j)}(n)\big) + \\ + 2\sigma(j)\big(\delta_{i+n-m,0}X_2^{(i+j)}(m) - \delta_{m-n+j,0}X_2^{(i+j)}(n)\big). \quad (25c)$$

The Lie–Poisson brackets for the coordinate functions $l_s^{(i)}(m)$ readily follow from the commutation relations (25)

$$\{l_1^{(i)}(m), l_1^{(j)}(n)\}_R = 2(1-\sigma(i)-\sigma(j))\big(\delta_{i+n-m,0}l_1^{(i+j)}(m) - \delta_{m-n+j,0}l_1^{(i+j)}(n)\big), \quad (26a)$$

$$\{l_2^{(i)}(m), l_2^{(j)}(n)\}_R = 2(\sigma(i)+\sigma(j)-1)\big(\delta_{i+n-m,0}l_2^{(i+j)}(m) - \delta_{m-n+j,0}l_2^{(i+j)}(n)\big), \quad (26b)$$

$$\{l_1^{(i)}(m), l_2^{(j)}(n)\}_R = 2(\sigma(i)-1)\big(\delta_{i+n-m,0}l_1^{(i+j)}(m) - \delta_{m-n+j,0}l_1^{(i+j)}(n)\big) + \\ + 2\sigma(j)\big(\delta_{i+n-m,0}l_2^{(i+j)}(m) - \delta_{m-n+j,0}l_2^{(i+j)}(n)\big). \quad (26c)$$

The Lie–Poisson brackets of the Toda system are obtained by putting in these relations $l_1^{(-1)}(i) = c_i$, $l_1^{(k)}(i) = 0$, $k < -1$, $l_2^{(j)}(i) = 0$, $j > 1$. These brackets coincide with the first Poisson structure of the 2D Toda hierarchy found in [10].

5 Quadratic and cubic Poisson structures on Double

In this section we will discuss the prolongation of the second and third degree Poisson brackets from $\mathfrak{g}$ to its classical double $\mathcal{G}$ and the consistency of the corresponding brackets.

5.1 Quadratic Poisson structure

It is known that for some classical R-operators on $\mathfrak{g}$ it is possible to define, besides the linear R-bracket, also a second degree Poisson bracket important in the theory of classical integrable systems.

Hereafter we assume existence of an identification between $\mathfrak{g}$ and $\mathfrak{g}^*$. Moreover we assume that the Lie algebra $\mathfrak{g}$ possesses also the structure of an associative algebra. The following Theorem holds true [15], [16]:

Theorem 5.1 *Let the classical R-operator and its skew-symmetric part $\frac{1}{2}(R - R^*)$ satisfy modified classical Yang-Baxter equation on $\mathfrak{g}$. Then*

(i) The formula

$$\{F_1, F_2\}_2 = \langle L, [R(L\nabla F_1 + \nabla F_1 L), \nabla F_2]\rangle - \langle L, [R(L\nabla F_2 + \nabla F_2 L), \nabla F_1]\rangle \tag{27}$$

defines a Poisson bracket on $\mathfrak{g}$.

(ii) The Casimir functions of $\mathfrak{g}$ mutually commute with respect to the brackets (27).

(iii) The Hamiltonian equations with respect to the Casimir functions I_k of $\mathfrak{g}$ are written in the Lax form:

$$\frac{dL}{dt_k} = [R(L\nabla I_k + \nabla I_k L), L]$$

(iv) The Poisson brackets (27) and (3) are compatible.

Recall that two Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ on the same space are called *compatible* if an arbitrary linear combination

$$a_1\{\ ,\ \}_1 + a_2\{\ ,\ \}_2$$

is again a Poisson bracket.

It occurred that the Theorem 5.1 can be extended to the double of $\mathfrak{g}$:

Theorem 5.2 *Let the classical R-operator and its skew-symmetric part $\frac{1}{2}(R - R^*)$ satisfy modified classical Yang-Baxter equation on $\mathfrak{g}$. Then*

(i) The formula

$$\{F_1(\mathcal{L}), F_2(\mathcal{L})\}_2 = \langle \mathcal{L}, [\mathcal{R}(\mathcal{L}\widetilde{\nabla} F_1 + \widetilde{\nabla} F_1 \mathcal{L}), \nabla F_2]\rangle - \langle \mathcal{L}, [\mathcal{R}(\mathcal{L}\widetilde{\nabla} F_2 + \widetilde{\nabla} F_2 \mathcal{L}), \nabla F_1]\rangle \tag{28}$$

defines a Poisson bracket on $\mathcal{G}$.

(ii) The Casimir functions of $\mathcal{G}$ mutually commute with respect to the brackets (28).

(iii) The Hamiltonian equations with respect to the Casimir functions $I_{k,\epsilon}$ of $\mathcal{G}$ are written in the Lax form:

$$\frac{d\mathcal{L}}{dt_k^{\epsilon}} = [R(\mathcal{L}\widetilde{\nabla} I_{k,\epsilon} + \widetilde{\nabla} I_{k,\epsilon}\mathcal{L}), \mathcal{L}], \quad \epsilon = 1,\, 2.$$

(iv) The Poisson brackets (28) and (4) are compatible.

Remark 8. The Lie–Poisson bracket (28) can be written in terms of the operators R, $R_\pm$ more explicitly as follows:

$$\begin{aligned}\{F_1(L_1,L_2),F_2(L_1,L_2)\}_2 = &\langle L_1,[R(L_1\nabla_1F_1+\nabla_1F_1L_1)-R_-(L_2\nabla_2F_1+\nabla_2F_1L_2),\nabla_1F_2]\rangle+\\ &+\langle L_2,[R_+(L_1\nabla_1F_1+\nabla_1F_1L_1)-R(L_2\nabla_2F_1+\nabla_2F_1L_2),\nabla_2F_2]\rangle-\\ &-\langle L_1,[R(L_1\nabla_1F_2+\nabla_1F_2L_1)-R_-(L_2\nabla_2F_2+\nabla_2F_2L_2),\nabla_1F_1]\rangle-\\ &-\langle L_2,[R_+(L_1\nabla_1F_2+\nabla_1F_2L_1)-R(L_2\nabla_2F_2+\nabla_2F_2L_2),\nabla_2F_1]\rangle.\end{aligned}$$

Proof. Let us at first note that if $\mathfrak{g}$ is an associative algebra then $\mathcal{G}$ is also an associative algebra with respect to the natural structure of direct sum of associative algebras. In order to prove the theorem it will suffice to apply the Theorem 5.1. Namely, to derive from the assumptions about the classical R-operator on $\mathfrak{g}$ that the classical R-operator $\mathcal{R}$ and its skew-symmetric part $\frac{1}{2}(\mathcal{R}-\mathcal{R}^*)$ satisfy modified classical Yang-Baxter equation on the double $\mathcal{G}$. The first part of the statement follows automatically from the results of [5]. It remains to show that $\frac{1}{2}(\mathcal{R}-\mathcal{R}^*)$ satisfies modified classical Yang-Baxter equation on $\mathcal{G}$. We will do this by direct calculation. We have that

$$\mathcal{A}\equiv\frac{1}{2}(\mathcal{R}-\mathcal{R}^*)=\frac{1}{2}\begin{pmatrix}A & -S\\ S & -A\end{pmatrix},\quad A=R-R^*,\quad S=R+R^*.$$

Substituting this expression into the modified classical Yang-Baxter equation one obtains that it is equivalent to the following three conditions

$$A([A(X),Y]+[X,A(Y)])-[A(X),A(Y)]=4[X,Y],\quad \forall X,Y\in\mathfrak{g}, \tag{29a}$$

$$-S([S(X),Y])-A([X,S(Y)])+[A(X),S(Y)]=0,\quad \forall X,Y\in\mathfrak{g}, \tag{29b}$$

$$S([A(X),Y]+[X,A(Y)])-[S(X),S(Y)]=0,\quad \forall X,Y\in\mathfrak{g}. \tag{29c}$$

The equation (29a) follows from the very condition of the theorem. Besides, using the definition of the operators A and S it is straightforward to show that the conditions (29) are equivalent to the following three equations

$$R([R(X),Y]+[X,R(Y)])-[R(X),R(Y)]=[X,Y],\quad \forall X,Y\in\mathfrak{g}, \tag{30a}$$

$$R^*([R^*(X),Y])-R^*([X,R(Y)])+[R^*(X),R(Y)]=[X,Y],\quad \forall X,Y\in\mathfrak{g}, \tag{30b}$$

$$R([R^*(X),Y]+[X,R^*(Y)])-[R^*(X),R^*(Y)]=-[X,Y],\quad \forall X,Y\in\mathfrak{g}. \tag{30c}$$

The equation (30a) is a modified classical Yang-Baxter equation for R. The equation (30b) is derived using classical modified Yang-Baxter equation (30a) and the existence of the non-degenerate invariant form on $\mathfrak{g}$. Finally the identity (30c) is derived using the modified Yang-Baxter equation for A.

Theorem is proven.

Remark 9. Note that the Theorem 5.2 means that the quadratic Poisson structure, whenever exists on $\mathfrak{g}$ can be always extended to its double $\mathcal{G}$. In particular such an extension exists for skew-symmetric R-operators on $\mathfrak{g}$ as the corresponding operator $\mathcal{R}$ on $\mathcal{G}$ is also skew-symmetric.

5.2 Cubic Poisson structure

In this subsection we will describe the cubic Poisson structure on $\mathfrak{g}$ and its prolongation to $\mathcal{G}$. We will use the following Theorem [15], [16] valid under the same assumptions about the Lie algebra $\mathfrak{g}$.

Theorem 5.3 *Let R be the classical R-operator. Then*

(i) The following formula:

$$\{F_1(L), F_2(L)\}_3 = \langle L, [R(L\nabla F_1 L + L\nabla F_1 L), \nabla F_2]\rangle - \langle L, [R(L\nabla F_2 L + L\nabla F_2 L), \nabla F_1]\rangle \quad (31)$$

defines a Poisson bracket on $\mathfrak{g}$.

(ii) The Casimir functions of $\mathfrak{g}$ mutually commute with respect to the brackets (31).

(iii) The Hamiltonian equations with respect to the Casimir functions I_k of $\mathfrak{g}$ are written in the Lax form:

$$\frac{dL}{dt_k} = [R(L\nabla I_k L), L]$$

(iv) If the skew-symmetric part $\frac{1}{2}(R - R^*)$ of the operator R satisfies modified classical Yang-Baxter equation on $\mathfrak{g}$. Then the Poisson brackets (31), (27), and (3) are compatible.

A similar statement holds true also for the classical double of $\mathfrak{g}$.

Corollary 5.1 *Let R be the classical R-operator. Then*

(i) the following formula:

$$\{F_1(\mathcal{L}), F_2(\mathcal{L})\}_3 = \langle \mathcal{L}, [\mathcal{R}(\mathcal{L}\widetilde{\nabla} F_1 \mathcal{L} + \mathcal{L}\widetilde{\nabla} F_1 \mathcal{L}), \widetilde{\nabla} F_2]\rangle - \langle \mathcal{L}, [\mathcal{R}(\mathcal{L}\widetilde{\nabla} F_2 \mathcal{L} + \mathcal{L}\widetilde{\nabla} F_2 \mathcal{L}), \widetilde{\nabla} F_1]\rangle \quad (32)$$

defines Poisson bracket on $\mathcal{G}$.

(ii) The Casimir functions of $\mathcal{G}$ mutually commute with respect to the brackets (28).

(iii) The Hamiltonian equations with respect to the Casimir functions $I_{k,\epsilon}$ of $\mathcal{G}$ are written in the Lax form:

$$\frac{d\mathcal{L}}{dt_k^{\epsilon}} = [R(\mathcal{L}\widetilde{\nabla} I_{k,\epsilon}\mathcal{L}), \mathcal{L}].$$

(iv) if the skew-symmetric part $\frac{1}{2}(R - R^*)$ of the operator R satisfies modified classical Yang-Baxter equation on $\mathfrak{g}$ then the Poisson brackets (32), (28), and (5) are compatible.

Remark 10. The Lie–Poisson bracket (32) can be written in terms of the operators R, $R_{\pm}$ more explicitly as follows:

$$\begin{aligned}
&\{F_1(L_1, L_2), F_2(L_1, L_2)\}_3 = \\
&= \langle L_1, [R(L_1\nabla_1 F_1 L_1) - R_-(L_2\nabla_2 F_1 L_2), \nabla_1 F_2]\rangle + \langle L_2, [R_+(L_1\nabla_1 F_1 L_1) - R(L_2\nabla_2 F_1 L_2), \nabla_2 F_2]\rangle - \\
&- \langle L_1, [R(L_1\nabla_1 F_2 L_1) - R_-(L_2\nabla_2 F_2 L_2), \nabla_1 F_1]\rangle - \langle L_2, [R_+(L_1\nabla_1 F_2 L_1) - R(L_2\nabla_2 F_2 L_2), \nabla_2 F_1]\rangle.
\end{aligned}$$

Proof. In order to prove the Corollary, let us first observe that, using the identification between $\mathfrak{g}$ and $\mathfrak{g}^*$ as linear spaces and $\mathfrak{g}$-modules one can also identify the spaces $\mathcal{G}$ and $\mathcal{G}^*$ as linear spaces and $\mathcal{G}$-modules. As it was explained above the double $\mathcal{G}$ of an associative algebra also possesses a structure of an associative algebra. Now in order to prove the Corollary it suffices to apply the Theorem 5.3 to the classical double $\mathcal{G}$ and take into account that, as it was proven in the Theorem 5.2, the antisymmetric part $\frac{1}{2}(\mathcal{R} - \mathcal{R}^*)$ satisfies the modified classical Yang-Baxter equation on $\mathcal{G}$ if $\frac{1}{2}(R - R^*)$ satisfies modified classical Yang-Baxter equation on $\mathfrak{g}$. This proves the Corollary.

Remark 11. Note, that the second and third degree Poisson structures exist and are compatible with the linear Poisson structure for all associative algebras (provided that R and $\frac{1}{2}(R-R^*)$ satisfy the modified Yang-Baxter equation). They produce commuting hamiltonian flows written in the Lax form. Nevertheless their usage in the soliton theory is not as straightforward and universal as the usage of the linear Poisson R-bracket. Indeed, in order to obtain the phase space of the soliton equation under consideration one has, as it was explained above, to restrict oneself to certain linear subspaces coinciding with the quotient algebras of the corresponding linear R-bracket, i.e., to Poisson subspaces with respect to the linear bracket. On the other hand, these linear subspaces in general are not Poisson subspaces of the quadratic and cubic brackets. Moreover, in order to restrict these brackets to the corresponding subspaces, one has to apply additionally Dirac's reduction. For the case of the infinite-component Toda system this procedure was considered in [10].

At the end of this section let us finally explain why second and third order Poisson structures exist and are compatible with the linear Poisson structure for the case of infinite-component Toda system (i.e. before the restriction onto the Poisson subspaces of linear bracket). As far as infinite-component Toda model is connected with the double of $gl((\infty))$ we need to prove the following:

Proposition 5.1 *On the double of $gl((\infty))$ equipped with the Adler-Kostant-Symmes R-operator $R = P^+ - P^-$, where P^+ and P^- are projection operators onto the algebra of upper triangular and strictly lower triangular matrices, exist quadratic and cubic Poisson structures compatible with linear Poisson brackets and given by the formulas (28) and (32) correspondingly.*

Proof. To prove the proposition we note that the double of $gl((\infty))$ is evidently an associative algebra. Hence in order to apply the Theorem 5.2 and Corollary 5.1 for the case at hand we have to show that $\frac{1}{2}(R-R^*)$ satisfies classical Yang-Baxter equation on $gl((\infty))$. In our case we have that: $R = P^+ - P^-$, where P^+ and P^- are projection operators onto the algebra of upper triangular and strictly lower triangular matrices respectively. Let us also note that this R-operator can be written as $R = P_+ + P_0 - P_-$ where P_+ is the projection operator onto the Lie algebra of strictly upper triangular matrices, P_0 is the projection onto the Lie subalgebra of diagonal matrices and $P_- \equiv P^-$. Using the explicit form of the invariant pairing (bilinear form on $gl((\infty))$) it is easy to see that $P^*_\pm = P_\mp$, $P^*_0 = P_0$, and, hence:

$$\frac{1}{2}(R - R^*) = P_+ - P_-.$$

As it follows from the results of [17] (see also [9]) if $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_0 + \mathfrak{g}_-$ is a triangular decomposition of a Lie algebra $\mathfrak{g}$, i.e. $\mathfrak{g}_\pm$, $\mathfrak{g}_0$ are closed Lie subalgebras and $\mathfrak{g}_0$-modules, then any operator of the form: $P_+ + R_0 - P_-$, where $P_\pm$ are projection operators onto $\mathfrak{g}_\pm$, is a solution of modified classical Yang-Baxter equation on $\mathfrak{g}$ if R_0 is a solution of the modified classical Yang-Baxter equation on $\mathfrak{g}_0$. Moreover, if the subalgebra $\mathfrak{g}_0$ is abelian then any operator R_0 (including trivial one) is a solution of the modified Yang-Baxter equation on $\mathfrak{g}_0$. That is why operator $P_+ - P_-$ is in this case a particular solution of the modified Yang-Baxter equation on $\mathfrak{g}$.

On the other hand it is clear that the decomposition of the algebra $\mathfrak{g} = gl((\infty))$ into the strictly upper-triangular and strictly lower-triangular and diagonal part is a triangular decomposition with abelian (diagonal) part $\mathfrak{g}_0$. Hence for the considered R-operator

$$\frac{1}{2}(R - R^*) = P_+ - P_-$$

is indeed a solution of mYBE on $gl((\infty))$ and by the virtue of the Theorem 5.2 and Corollary 5.1 there indeed exist quadratic and cubic Poisson structures on $gl((\infty))$ and its double.

Proposition is proven.

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References

[1] Flaschka H., Newell A., Ratiu T., Kac–Moody Lie algebras and soliton equations. II-III. *Phys. D* **9**, no. 3, 300–323; 324–332 (1983).

[2] Newell A. "Solitons in mathematics and physics." University of Arizona, Society for industrial and Applied mathematics 1985

[3] Semenov Tian-Shansky M. What is clasical r-matrix, *Funct An and Apl*, **v 17**, 259 (1983).

[4] Reyman A.G., Semenov Tian-Shansky M. A. "Group theoretical methods in the theory of finite-dimensional integrable systems," VINITI: Cont. probl. math. Fundamental trends **6**, 145-147 (1989)

[5] Reyman A.G., Semenov Tian-Shansky M. A. "Integrable systems: Group-theoretical approach", Moscow-Izhevsk 2003.

[6] Kac V., Infinite-dimensional Lie algebras, Cambridge University Press, Cambridge, 1990.

[7] Skrypnyk T., Quasigraded Lie algebras, Kostant-Adler scheme and integrable hierarchies, *Theor Math Phys* **142** (2005), 275.

[8] Skrypnyk T., Dual R-matrix integrability, *Theoretical and Mathematical Physics*, **v. 155**, p.633-645 (2008).

[9] Skrypnyk T., Classical R-operators and integrable Thirring-type equations, *Symmetry, Integrability and Geometry*, v. 4, Paper 011, (2008).

[10] Carlet G. The Hamiltonian structures of the two-dimensional Toda lattice and R-matrices. *Lett. Math. Phys.* **71**, no. 3, 209–226 (2005).

[11] Ueno K., Takasaki K., Toda lattice hierarchy. I-II. *Proc. Japan Acad.* Ser. A Math. Sci. **59** , no. 5, 167–170; no. 6, 215–218 (1983).

[12] Mikhailov A. Ob integriruemosti dvumernogo obobshcheniya tsepochki Toda. *Pisma v ZhETF* **30**, 443, (1979).

[13] Mikhailov A., Olshanetsky M., Perelomov A. Two-dimensional generalized Toda lattice. *Comm. Math. Phys.* **79** no. 4, 473–488, (1981).

[14] Nirov Kh., Razumov A. Toda equations associated with loop groups of complex classical Lie groups. ıNuclear Phys. B **782** no. 3, 241–275 (2007).

[15] Oevel W., Ragnisco O., R-matrices and higher Poisson brackets for integrable systems., *Phys. A* **161** no. 1, 181-220 (1989).

[16] Li L-C., Parmentier S., Nonlinear Poisson structures and r-matrices,*Commun. Math.Phys.* **125**, p. 545-563 (1989).

[17] Guil F., Mañas M., The homogeneous Heisenberg subalgebra and equations of AKS type, *Lett Math Phys* **19** (1990), 89-95.