Linearly degenerate Hamiltonian PDEs and a new class of solutions to the WDVV associativity equations

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Abstract

We define a new class of solutions to the WDVV associativity equations. This class is selected by the property that one of the commuting PDEs associated with such a WDVV solution is linearly degenerate. We reduce the problem of classification of such linearly degenerate solutions of WDVV to a particular case of the so-called algebraic Riccati equation and, in this way we arrive at a complete classification of irreducible solutions.

1 Introduction

E.Witten–R.Dijkgraaf–E.Verlinde–H.Verlinde (WDVV) associativity equations is the following overdetermined system of partial differential equations for a function $F = F(\mathbf{v})$, $\mathbf{v} = (v^1, \dots, v^n)$

$$\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial v^\mu \partial v^\gamma \partial v^\delta} = \frac{\partial^3 F}{\partial v^\delta \partial v^\beta \partial v^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial v^\mu \partial v^\gamma \partial v^\alpha}, \quad \alpha,\, \beta,\, \gamma,\, \delta = 1, \dots, n \tag{1.1}$$

$$\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^1} = \eta_{\alpha\beta}.$$

Here

$$\left(\eta_{\alpha\beta}\right)_{1\le\alpha,\,\beta\le n} \quad \text{and} \quad \left(\eta^{\alpha\beta}\right)_{1\le\alpha,\,\beta\le n}$$

are constant symmetric nondegenerate matrices that are mutually inverse,

$$\eta_{\alpha\lambda}\eta^{\lambda\beta} = \delta_\alpha^\beta.$$

Summation over repeated Greek indices will be assumed within this section.

Recall [5] that solutions to the WDVV associativity equations are in one-to-one correspondence with n-parametric families of n-dimensional commutative associative algebras

$$\mathcal{A}_{\mathbf{v}} = \mathrm{span}(e_1, \ldots, e_n)$$

with a unity $e = e_1$ equipped with a symmetric nondegenerate invariant bilinear form $(\ ,\ )$ such that the structure constants are expressed via the third derivatives of a function F called *potential*

$$\begin{aligned}
&e_\alpha \cdot e_\beta = c_{\alpha\beta}^\gamma(\mathbf{v}) e_\gamma, \quad \alpha,\, \beta = 1, \ldots, n \\
&e_1 \cdot e_\alpha = e_\alpha \quad \text{for any} \quad \alpha \\
&(e_\alpha, e_\beta) = \eta_{\alpha\beta} \\
&(e_\alpha \cdot e_\beta, e_\gamma) = (e_\alpha, e_\beta \cdot e_\gamma) = \eta_{\gamma\lambda} c_{\alpha\beta}^\lambda(\mathbf{v}) = \frac{\partial^3 F(\mathbf{v})}{\partial v^\alpha \partial v^\beta \partial v^\gamma}.
\end{aligned}$$

If in addition the function F satisfies certain quasi-homogeneity condition then one arrives at the local description of *Frobenius manifolds* (see details in [5]). A natural metric

$$ds^2 = \eta_{\alpha\beta} dv^\alpha dv^\beta \tag{1.2}$$

(not necessarily positive definite) is defined on these manifolds. The variables $v^1, \ldots, v^n$ are *flat coordinates* for this metric. The algebra $\mathcal{A}_{\mathbf{v}}$ is identified with the tangent space to the manifold at the point $\mathbf{v}$,

$$e_\alpha \leftrightarrow \frac{\partial}{\partial v^\alpha}.$$

See [5] for more details about the coordinate-free geometric description of Frobenius manifolds.

A solution to the associativity equations (1.1) is called *semisimple* if the algebra $\mathcal{A}_{\mathbf{v}}$ has no nilpotent elements for a generic point $\mathbf{v}$. In the semisimple case it was proved [4] existence of local *canonical coordinates* $u_i = u_i(\mathbf{v})$, $i = 1, \ldots, n$ such that the multiplication table takes the following standard form

$$\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}.$$

The metric (1.2) becomes diagonal in the canonical coordinates

$$ds^2 = \sum_{i=1}^n h_i^2(\mathbf{u}) du_i^2. \tag{1.3}$$

Moreover, this is a *Egorov* metric, i.e., the *rotation coefficients*

$$\gamma_{ij}(\mathbf{u}) = \frac{1}{h_j}\frac{\partial h_i}{\partial u_j} \tag{1.4}$$

are symmetric in i, j

$$\gamma_{ji} = \gamma_{ij}. \tag{1.5}$$

They satisfy the following system of *Darboux–Egorov equations*

$$\frac{\partial \gamma_{ij}}{\partial u_k} = \gamma_{ik}\gamma_{kj}, \quad i,\ j,\ k \quad \text{distinct} \tag{1.6}$$

$$\sum_{k=1}^{n} \frac{\partial \gamma_{ij}}{\partial u_k} = 0, \quad i \neq j. \tag{1.7}$$

Any solution to the Darboux–Egorov equations comes from a semisimple solution to the WDVV associativity equations. The reconstruction procedure of the latter involves solutions to the following system of linear differential equations for a vector-function $\psi = (\psi_1(\mathbf{u}), \dots, \psi_n(\mathbf{u}))$

$$\frac{\partial \psi_i}{\partial u_j} = \gamma_{ij}\psi_j, \quad i \neq j \tag{1.8}$$

$$\sum_{k=1}^{n} \frac{\partial \psi}{\partial u_k} = 0. \tag{1.9}$$

Denote $\psi_{i\alpha} = \psi_{i\alpha}(\mathbf{u})$, $\alpha = 1, \dots, n$ a system of n linearly independent solutions to (1.8), (1.9). The reconstruction depends on a choice of one of these solutions to be identified with the Lamé coefficients of the invariant metric (1.2); let it correspond to $\alpha = 1$

$$h_i = \psi_{i1}.$$

Then

$$\eta_{\alpha\beta} = \sum_{i=1}^{n} \psi_{i\alpha}\psi_{i\beta}$$

$$dv_\alpha = \sum_{i=1}^{n} \psi_{i\alpha}\psi_{i1} du_i$$

$$\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\gamma} = \sum_{i=1}^{n} \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}}.$$

Observe also the following formula for the differentials of the second derivatives

$$\Omega_{\alpha\beta} = \frac{\partial^2 F}{\partial v^\alpha \partial v^\beta} \tag{1.10}$$

of the potential F

$$d\Omega_{\alpha\beta} = \sum_{i=1}^{n} \psi_{i\alpha}\psi_{i\beta} du_i. \qquad (1.11)$$

As it was found in [3] the Darboux–Egorov system (1.6)–(1.7) can be identified with a particular reduction of the well-known in the theory of integrable PDEs n-wave system written in the form suggested in [2]. It can also be embedded into the framework of the nKP system (see, e.g., [10]). All known particular solutions to the associativity equations correspond to further reduction from the n-wave system to a system of ODEs. For example the semisimple Frobenius manifolds come out from the *scaling* reduction

$$\sum_{k=1}^{n} u_k \frac{\partial \gamma_{ij}}{\partial u_k} = -\gamma_{ij}, \quad i \neq j.$$

It corresponds to the quasi-homogeneity axiom of the theory of Frobenius manifolds [4], [5]. Other particular classes of solutions (solitons, algebro-geometric solutions, degenerate Frobenius manifolds) also naturally arise in the framework of the n-wave system.

In this paper we introduce another class of solutions to WDVV. In order to describe this class let us first recall the connection between the associativity equations and integrable hierarchies. Let $\theta = \theta(\mathbf{v})$ be a solution to the following system of linear differential equations

$$\frac{\partial^2 \theta}{\partial v^\alpha \partial v^\beta} = c_{\alpha\beta}^{\gamma} \frac{\partial^2 \theta}{\partial v^1 \partial v^\gamma}, \quad \alpha, \beta = 1, \ldots, n. \qquad (1.12)$$

Consider a system of linear PDEs for the vector-function $\mathbf{v} = \mathbf{v}(x,t)$

$$\mathbf{v}_t = [\nabla\theta(\mathbf{v})]_x. \qquad (1.13)$$

It is a Hamiltonian PDE with the Hamiltonian

$$H = \int \theta(\mathbf{v})\, dx$$

and the Poisson bracket

$$\{v^\alpha(x), v^\beta(y)\} = \eta^{\alpha\beta}\delta'(x-y).$$

All these Hamiltonian systems of the form (1.12), (1.13) commute pairwise. Moreover the Hamiltonians (1.12) satisfy certain *completeness* conditions. So any of the systems (1.13) can be considered as a completely integrable Hamiltonian PDE.

In the semisimple case all such PDEs diagonalize in the canonical coordinates

$$\mathbf{u}_t = \Lambda(\mathbf{u})\mathbf{u}_x, \quad \Lambda(\mathbf{u}) = \mathrm{diag}\left(\lambda_1(\mathbf{u}), \ldots, \lambda_n(\mathbf{u})\right). \qquad (1.14)$$

Thus the canonical coordinates are *Riemann invariants* for the quasilinear systems (1.13). For a generic solution to (1.12) the characteristic velocities are pairwise distinct

$$\lambda_i(\mathbf{u}) \neq \lambda_j(\mathbf{u}), \quad i \neq j \tag{1.15}$$

for a generic point $\mathbf{u}$.

Definition 1.1 *The semisimple solution $F(\mathbf{v})$ to the WDVV associativity equations is called* linearly degenerate *if among the commuting PDEs* (1.12)–(1.14) *there exists at least one satisfying* (1.15) *along with the condition*

$$\frac{\partial \lambda_i(\mathbf{u})}{\partial u_i} = 0, \quad i = 1, \ldots, n. \tag{1.16}$$

The motivation for our terminology is that one of the quasilinear systems of the commuting family (1.12)–(1.14) is linearly degenerate, i.e., the i-th characteristic velocity λ_i does not depend on the i-th Riemann invariant u_i for every i from $i = 1$ to $i = n$.

The main goal of the present paper is to classify linearly degenerate solutions to the WDVV associativity equations. Such a solution is called *reducible* if, for some i one has

$$\gamma_{ij}(\mathbf{u}) \equiv 0 \quad \forall\, j \neq i.$$

In the opposite case it will be called *irreducible.* It suffices to classify irreducible linearly degenerate solutions.

Theorem 1.2 *The rotation coefficients of an irreducible linearly degenerate solution to the WDVV associativity equation has the form*

$$\begin{aligned} \gamma_{ij}(u) &= \frac{\left[G(1 - \frac{1}{\rho}\tanh\rho U \cdot G)^{-1}\right]_{ij}}{\cosh\rho u_i \cosh\rho u_j}, \quad i,\, j = 1, \ldots, n, \quad i \neq j \\ U &= \mathrm{diag}(u_1, \ldots, u_n) \end{aligned} \tag{1.17}$$

where G is a symmetric matrix satisfying

$$G^2 = \rho^2 \cdot 1, \tag{1.18}$$

ρ is an arbitrary complex parameter.

For $\rho = 0$ the above formulae are considered in the sense of a limit

$$\frac{1}{\rho}\tanh\rho U \to U, \quad \cosh\rho u_i \to 1.$$

The paper is organized as follows. In Section 2 we recall the necessary constructions of the theory of the WDVV associativity equations. We derive the main system of differential equations (2.8) of the theory of linearly degenerate solutions to WDVV. In Section 3 we solve the main system and describe a symmetry group of it acting by fractional linear transformations. In Section 4 we select those solutions to the main system that give rise to WDVV and derive the algebraic Riccati equation. Using the symmetries of this equation we classify all irreducible linearly degenerate solutions to the WDVV associativity equations.

Acknowledgments.

Authors thank Eugeni Ferapontov and Sergey Tsarev for their stimulating and clarifying discussions.

This work is partially supported by the European Research Council Advanced Grant FroM-PDE, by the Russian Federation Government Grant No. 2010-220-01-077 and by PRIN 2008 Grant "Geometric methods in the theory of nonlinear waves and their applications" of Italian Ministry of Universities and Researches.

MVP is partially supported by ag. #11.G34.31.0005, by the grant of Presidium of RAS "Fundamental Problems of Nonlinear Dynamics" and by the RFBR grant 11-01-00197. MVP and SAZ are grateful to the SISSA in Trieste (Italy) where some part of this work has been done.

2 Linearly degenerate solutions to the WDVV associativity equations

Let

$$\Gamma = (\gamma_{ij}(\mathbf{u}))_{1\le i,\, j\le n}$$

be the symmetric matrix of rotation coefficients[1] (1.4) of a linearly degenerate irreducible solution to the associativity equations.

Lemma 2.1 *The matrix valued function $\Gamma = \Gamma(\mathbf{u})$ satisfies the following differential equations*

$$\frac{\partial \Gamma}{\partial u_k} = \Gamma E_k \Gamma + \sigma_k(u_k) E_k, \quad k = 1, \dots, n \tag{2.1}$$

with some functions $\sigma_1(u_1)$, ..., $\sigma_n(u_n)$. Here E_k is the matrix with only one nonzero entry

$$(E_k)_{ij} = \delta_{ik}\delta_{jk}. \tag{2.2}$$

[1]Actually, in the differential geometry of curvilinear orthogonal coordinate systems only the off-diagonal entries of the matrix Γ are called rotation coefficients. However, in our case it will be convenient to also add the diagonal entries $\gamma_{ii} = \partial \log h_i / \partial u_i$.

Proof By construction the equations

$$\frac{\partial \gamma_{ij}}{\partial u_k} = \gamma_{ik}\gamma_{kj} \tag{2.3}$$

hold true for distinct values of the indices i, j, k. Let us first establish validity of (2.3) also for $k = i$ or $k = j$ with $i \neq j$ or for $i = j$ but $k \neq i$.

According to [4] the characteristic velocities $\lambda_k(\mathbf{u})$ of the commuting PDEs (1.12)–(1.14) can be represented in the form

$$\lambda_k(\mathbf{u}) = \frac{\phi_k(\mathbf{u})}{h_k(\mathbf{u})}, \quad k = 1, \dots, n \tag{2.4}$$

where the vector-function $\phi = (\phi_1(\mathbf{u}), \dots \phi_n(\mathbf{u}))$ satisfies the system of linear differential equations

$$\frac{\partial \phi_i}{\partial u_j} = \gamma_{ij}\phi_j, \quad i \neq j. \tag{2.5}$$

In particular $\phi_k = h_k$ is one of solutions to (2.5). Let ϕ be the solution to (2.5) corresponding to the linearly degenerate member of the commuting family (1.12)–(1.14). Differentiating the equation

$$\frac{\partial}{\partial u_k}\left(\frac{\phi_k}{h_k}\right) = 0$$

in u_i with $i \neq k$ one derives the following equation

$$\frac{h_i}{h_k}(\lambda_i - \lambda_k)\gamma_{ik}\frac{\partial}{\partial u_k}\left[\log \gamma_{ik} - \log h_k\right] = 0.$$

Due to the assumptions of irreducibility and (1.15) we arrive at the equation

$$\frac{\partial \log \gamma_{ik}}{\partial u_k} = \frac{\partial \log h_k}{\partial u_k} = \gamma_{kk}.$$

This proves (2.3) for $k = j$, $i \neq j$. Next, assuming $k \neq i$ one has

$$\frac{\partial \gamma_{ii}}{\partial u_k} = \frac{\partial}{\partial u_i}\frac{\partial \log h_i}{\partial u_k} = \frac{\partial}{\partial u_i}\left(\gamma_{ik}\frac{h_k}{h_i}\right) = \gamma_{ik}^2.$$

Thus the equation (2.3) with $i = j$ and $k \neq i$ is also fixed. The last step is to verify that the difference $\sigma_i := \partial\gamma_{ii}/\partial u_i - \gamma_{ii}^2$ depends only on u_i. Indeed, for $k \neq i$

$$\frac{\partial}{\partial u_k}\left(\frac{\partial \gamma_{ii}}{\partial u_i} - \gamma_{ii}^2\right) = \frac{\partial}{\partial u_i}\frac{\partial \gamma_{ii}}{\partial u_k} - 2\gamma_{ii}\gamma_{ik}^2 = \frac{\partial \gamma_{ik}^2}{\partial u_i} - 2\gamma_{ii}\gamma_{ik}^2 = 0.$$

$\square$

Let us now describe a class of transformations

$$u_k \mapsto \tilde{u}_k, \quad \gamma_{ij} \mapsto \tilde{\gamma}_{ij}$$

leaving invariant the system (2.1).

Lemma 2.2 *The substitution*

$$\tilde{u}_k = f_k(u_k), \quad k = 1, \ldots, n$$
$$\tilde{\gamma}_{ij} = \frac{\gamma_{ij}}{\sqrt{f_i'(u_i) f_j'(u_j)}} - \frac{f_i''(u_i)}{2[f_i'(u_i)]^2}\,\delta_{ij}, \quad i,\, j = 1, \ldots, n \tag{2.6}$$

with arbitrary nonconstant smooth functions $f_1(u_1),\ \ldots,\ f_n(u_n)$ *leaves invariant the form of equations* (2.1)

$$\frac{\partial \tilde{\Gamma}}{\partial \tilde{u}_k} = \tilde{\Gamma} E_k \tilde{\Gamma} + \tilde{\sigma}_k(\tilde{u}_k) E_k, \quad k = 1, \ldots, n.$$

with

$$f_k'^{\,2}\, \tilde{\sigma}_k = \sigma_k - \frac{1}{2} S_{u_k}\left(f_k\right). \tag{2.7}$$

Here $S_u(f)$ *is the Schwarzian derivative of a function* $f = f(u)$

$$S_u(f) = \frac{f'''}{f'} - \frac{3}{2}\frac{f''^2}{f'^2}.$$

Proof is given by a straightforward computation. □

Corollary 2.3 *The system* (2.1) *by a suitable transformation of the form* (2.6) *can be reduced to the form*

$$\frac{\partial \tilde{\Gamma}}{\partial \tilde{u}_k} = \tilde{\Gamma} E_k \tilde{\Gamma}, \quad k = 1, \ldots, n. \tag{2.8}$$

Proof The needed transformation $\tilde{u}_k = f_k(u_k)$ is determined from the Schwarzian equations

$$S_{u_k}(f_k) = 2\sigma_k(u_k), \quad k = 1, \ldots, n.$$

□

Recall that the solution to the general Schwarzian equation $S_u(f(u)) = 2\sigma(u)$ is represented as the ratio of two solutions to the linear second order equation

$$y'' + \sigma(u) y = 0.$$

Remark 2.4 *The system* (2.8) *appeared [8] in the investigation of the so-called cold gas reductions of the nonlocal kinetic equation derived as the thermodynamical limit of the averaged multi-phase solutions of the KdV equation by the Whitham approach.*

In the next section we will solve the system (2.8).

3 Main system

In this section we will describe solutions to the main system

$$\frac{\partial \Gamma}{\partial u_k} = \Gamma\, E_k \Gamma, \quad k = 1, \dots, n. \tag{3.1}$$

Here

$$\Gamma = (\gamma_{ij}(\mathbf{u}))_{1 \le i,j \le n} \tag{3.2}$$

is a symmetric matrix (the tildes of the previous section have been omitted now). The compatibility conditions

$$\frac{\partial}{\partial u_l}\frac{\partial \Gamma}{\partial u_k} = \frac{\partial}{\partial u_k}\frac{\partial \Gamma}{\partial u_l} \tag{3.3}$$

for any k, l can be readily verified. So, locally, any solution to (3.1) is uniquely determined by the initial data

$$\Gamma^0 = \Gamma(\mathbf{u}^0). \tag{3.4}$$

Here $\mathbf{u}^0$ is any point in the space of independent variables. Therefore the space of solutions to the system (3.1) has dimension $n(n+1)/2$.

Without loss of generality one can assume $\mathbf{u}^0 = 0$. The solution to the system (3.1) with a given initial data at the point $\mathbf{u} = 0$ can be written explicitly.

Proposition 3.1 *The solution* $\Gamma = \Gamma(\mathbf{u})$ *to the main system* (3.1) *with the initial data*

$$\Gamma(0) = G \tag{3.5}$$

with a given symmetric matrix $G = (g_{ij})$ *is given by the following formula*

$$\Gamma = G\,(1 - UG)^{-1} \tag{3.6}$$

where 1 *is the* $n \times n$ *identity matrix,*

$$U = \operatorname{diag}\,(u_1, \dots, u_n)\,.$$

Proof Symmetry of the matrix (3.6) is tantamount to the equality

$$G\,(1 - UG)^{-1} = (1 - G\,U)^{-1}\,G. \tag{3.7}$$

To prove the latter we multiply it by $(1 - G\,U)$ on the left and by $(1 - UG)$ on the right to arrive at an obvious identity

$$(1 - G\,U)\,G = G\,(1 - UG) = G - G\,UG.$$

Clearly $\Gamma(0) = G$. Applying the well known rule of differentiating of the inverse matrix

$$\begin{aligned}\frac{\partial \Gamma}{\partial u_k} &= -G\,(1 - UG)^{-1}\,\frac{\partial\,(1 - UG)}{\partial u_k}\,(1 - UG)^{-1} \\ &= G\,(1 - UG)^{-1}\,E_k G\,(1 - UG)^{-1} = \Gamma\, E_k \Gamma\end{aligned}$$

one completes the proof of the Proposition. □

Example 3.2 *For the matrix of rank 1, $g_{ij} = \omega_i \omega_j$ one obtains the following solution to the main system*

$$\gamma_{ij} = \frac{\omega_i \omega_j}{1 - \sum_{k=1}^{n} \omega_k^2 u_k}. \tag{3.8}$$

Let us now describe a subclass of the transformations (2.6) leaving invariant the main system (3.1).

Proposition 3.3 *The main system* (3.1) *is invariant with respect to the transformations* (2.6) iff $f_k(u_k)$ *for every $k = 1, \ldots, n$ is a fractional linear transformation*

$$f_k(u_k) = \frac{a_k u_k + b_k}{c_k u_k + d_k}, \quad a_k d_k - b_k c_k = 1. \tag{3.9}$$

Proof It is well known that the general solution to the homogeneous Schwarzian equation

$$\frac{f'''}{f'} - \frac{3}{2}\frac{f''^2}{f'^2} = 0.$$

is given by fractional linear functions. □

Corollary 3.4 *The main system* (3.1) *is invariant with respect to the transformations*

$$\tilde{u}_k = \frac{a_k u_k + b_k}{c_k u_k + d_k}, \quad \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \in SL_2(\mathbb{R}), \quad k = 1, \ldots, n$$
$$\tilde{\gamma}_{ij} = (c_i u_i + d_i)(c_j u_j + d_j)\gamma_{ij} + c_i(c_i u_i + d_i)\delta_{ij}, \quad i, j = 1, \ldots, n. \tag{3.10}$$

Observe the matrix version of the transformation (3.10)

$$\tilde{U} = (A\,U + B)\,(C\,U + D)^{-1}, \quad \tilde{\Gamma} = (C\,U + D)\,\Gamma(C\,U + D) + C(C\,U + D)$$
$$A = \mathrm{diag}(a_1, \ldots, a_n), \quad B = \mathrm{diag}(b_1, \ldots, b_n), \quad C = \mathrm{diag}(c_1, \ldots, c_n), \quad D = \mathrm{diag}(d_1, \ldots, d_n) \tag{3.11}$$
$$AD - BC = 1.$$

Example 3.5 *The substitution*

$$\tilde{u}_k = \omega_k^2 u_k, \quad \tilde{\gamma}_{ij} = \frac{\gamma_{ij}}{\omega_i \omega_j}$$

reduces the solution (3.8) *to the standard form*

$$\tilde{\gamma}_{ij} = \frac{1}{1 - \sum_{k=1}^{n} \tilde{u}_k}, \quad i, j = 1, \ldots, n. \tag{3.12}$$

The action of $[SL_2(\mathbb{R})]^n$ transformations (3.10) on the solutions (3.6) is given by the following analogue of Siegel modular transformations.

Proposition 3.6 *Let the symmetric matrix G satisfy the condition*

$$\det(A + B\,G) \neq 0.$$

Then the transformation (3.10) *transforms the solution* $\Gamma(\mathbf{u})$ *with the initial data* $\Gamma(0) = G$ *to*

$$\tilde{\Gamma} = \tilde{G}\left(1 - \tilde{U}\tilde{G}\right)^{-1}$$

with

$$\tilde{G} = (C + D\,G)(A + B\,G)^{-1}. \tag{3.13}$$

Proof An easy calculation with the help of (3.11) yields

$$\tilde{\Gamma} = \left(-C\,\tilde{U} + A\right)^{-1} G\left[A + B\,G - \tilde{U}\,(C + D\,G)\right]^{-1} + C\,\left(-C\,\tilde{U} + A\right)^{-1}.$$

Computing the initial data of this solution at $\tilde{\mathbf{u}} = 0$ we arrive at $\tilde{\Gamma}(0) = \tilde{G}$ with the matrix $\tilde{G}$ given by the formula (3.13). □

Definition 3.7 *Two solutions Γ and $\tilde{\Gamma}$ to the main system are called* equivalent *if they are related by a symmetry transformation* (3.11). *Two symmetric matrices G and $\tilde{G}$ related by the transformation* (3.13) *will also be called equivalent.*

Observe a useful identity

$$(C + D\,G)(A + B\,G)^{-1} = (A + G\,B)^{-1}(C + G\,D) \tag{3.14}$$

equivalent to the symmetry of the matrix G.

4 From the main system back to linearly degenerate solutions to the associativity equations

In this section we will address the problem of selection of those solutions to the main system (3.1) that come from a linearly degenerate solution to the associativity equations.

For a given symmetric matrix valued function $\Gamma(\mathbf{u})$ satisfying (3.1) we are looking for a substitution of the form (2.6) such that the transformed matrix $\tilde{\Gamma}$ satisfies also the last equation (1.7) of the Darboux–Egorov system, that is

$$\sum_{k=1}^{n} \frac{\partial \tilde{\Gamma}}{\partial \tilde{u}_k} = \text{a diagonal matrix.} \tag{4.1}$$

Recall that validity of the first part

$$\frac{\partial\tilde{\gamma}_{ij}}{\partial\tilde{u}_k} = \tilde{\gamma}_{ik}\tilde{\gamma}_{kj} \quad \text{for} \quad i,\, j,\, k \quad \text{distinct}$$

follows from the main system, due to the Lemma 2.2.

Applying the Lemma 2.2 we arrive at the following simple statement.

Proposition 4.1 *Let* $\Gamma(\mathbf{u})$ *be a solution to the main system* (3.1)*. Suppose that the functions* $f_1(u_1),\ \ldots,\ f_n(u_n)$ *are chosen in such a way that the transformed matrix* (2.6) *satisfies* (4.1)*. Then the off-diagonal entries of the transformed matrix* $\tilde{\Gamma}$ *are rotation coefficients of some Egorov metric.*

Introduce the diagonal matrices

$$\begin{aligned} S = &\quad \text{diag}\,(s_1, \ldots, s_n)\,, \quad s_i = \frac{1}{f_i'} \\ S' = &\ \text{diag}\,(s_1', \ldots, s_n')\,, \quad s_i' = \frac{ds_i}{du_i} = -\frac{f_i''}{[f_i']^2}. \end{aligned} \tag{4.2}$$

Here and in sequel we will use short notations

$$f_i' = f_i'(u_i), \quad f_i'' = f_i''(u_i) \quad \text{etc.}$$

In these notations the transformation law (2.6) reads

$$\tilde{\Gamma} = S^{1/2}\Gamma S^{1/2} + \frac{1}{2}S'. \tag{4.3}$$

Thus the condition (4.1) can be represented in the form

$$\Gamma S\Gamma + \frac{1}{2}S'\Gamma + \frac{1}{2}\Gamma S' + P = 0 \tag{4.4}$$

for some diagonal matrix P.

Definition 4.2 *The solution* Γ *is called* reducible *if, for some* i *one has*

$$\gamma_{ij} \equiv 0 \quad \text{for any} \quad j \neq i.$$

Otherwise it is called irreducible*.*

A reducible solution essentially depends on a smaller number of variables.

Theorem 4.3 *For an irreducible solution*

$$\Gamma = G(1-UG)^{-1} = (1-GU)^{-1}G$$

a transformation (2.6) *satisfying* (4.1) *exists* iff *the matrix G satisfies the quadratic equation*

$$GRG + QG + GQ + P = 0 \tag{4.5}$$

for some constant diagonal matrices

$$P = \mathrm{diag}(p_1, \ldots, p_n), \quad Q = \mathrm{diag}(q_1, \ldots, q_n), \quad R = \mathrm{diag}(r_1, \ldots, r_n). \tag{4.6}$$

The transformation in question is determined by

$$\frac{d\tilde{u}_i}{du_i} = \frac{1}{p_i u_i^2 + 2q_i u_i + r_i}, \quad i = 1, \ldots, n. \tag{4.7}$$

Proof Differentiating (4.4) in u_i with the help of (3.1) and using the obvious formulae

$$\frac{\partial S}{\partial u_i} = s_i' E_i, \quad \frac{\partial S'}{\partial u_i} = s_i'' E_i$$

etc. one obtains

$$\left(\frac{1}{2}s_i'' - p_i\right)(\Gamma E_i + E_i \Gamma) + \frac{\partial P}{\partial u_i} = 0. \tag{4.8}$$

All the entries of the matrix $\Gamma E_i + E_i \Gamma$ vanish except for the i-th row and i-th column coinciding with $(\gamma_{1i}, \ldots, \gamma_{ni})$. Due to the irreducibility assumption from (4.8) it follows that

$$p_i = \frac{1}{2}s_i''. \tag{4.9}$$

Substituting again into (4.8) yields

$$\frac{\partial P}{\partial u_i} = 0.$$

Repeating this procedure for every $i = 1, \ldots, n$ one proves that the matrix P is constant. Using (4.9) we conclude that $s_i = s_i(u_i)$ is a quadratic polynomial, $s_i = p_i u_i^2 + 2q_i u_i + r_i$. Finally, multiplying eq. (4.4) on the left by $1 - GU$ and on the right by $1 - UG$ we arrive at the quadratic equation (4.5). □

Definition 4.4 *A symmetric matrix G is called* admissible *if it satisfies the matrix quadratic equation* (4.5)*. The solution $\Gamma = G(1-UG)^{-1}$ is called admissible if the parameter matrix G is admissible.*

The matrix quadratic equation (4.5) for the symmetric matrix G is a particular case of the so-called *algebraic Riccati equation* (see, e.g., [9]). The class of such equations is invariant with respect to fractional linear transformations, as it follows from

Lemma 4.5 *If the symmetric matrix G satisfies a matrix quadratic equation*

$$GRG + QG + GQ + P = 0$$

with some diagonal matrices P, Q, R then so does the equivalent matrix $\tilde{G} = (C + D\,G)(A + B\,G)^{-1}$

$$\tilde{G}\tilde{R}\tilde{G} + \tilde{Q}\tilde{G} + \tilde{G}\tilde{Q} + \tilde{P} = 0$$

with

$$\begin{aligned} \tilde{P} &= D^2 P - 2CDQ + C^2 R \\ \tilde{Q} &= -BDP + (AD + BC)Q - ACR \\ \tilde{R} &= B^2 P - 2ABQ + A^2 R. \end{aligned} \tag{4.10}$$

Proof is straightforward with the help of the identity (3.14). □

Corollary 4.6 *The class of admissible solutions to the main system* (3.1) *is invariant with respect to the* $[SL_2]^n$ *action* (3.11).

The entries $\Delta_1, \ldots, \Delta_n$ of the diagonal matrix

$$\Delta = Q^2 - PR \tag{4.11}$$

are invariants of the $[SL_2]^n$ action (4.10).

The next step is to parameterize linearly degenerate solutions to the associativity equations by solutions to the algebraic Riccati equation (4.5) with prescribed coefficients satisfying

$$|p_i|^2 + |q_i|^2 + |r_i|^2 \neq 0, \quad i = 1, \ldots, n. \tag{4.12}$$

Let us first simplify the matrix quadratic equation with the help of the transformations (4.10).

Lemma 4.7 *1) For an irreducible admissible matrix G the matrix quadratic equation* (4.5) *is equivalent, up to transformations* (4.10) *to*

$$G^2 = \Delta \tag{4.13}$$

where Δ is given by (4.11).

2) For an admissible irreducible G the matrix Δ is proportional to the identity matrix

$$\Delta_1 = \cdots = \Delta_n =: \rho^2. \tag{4.14}$$

Proof If all entries of the matrix R are different from zero then the equation (4.5) can be reduced to the canonical form (4.13) by a transformation

$$G \mapsto AGA + B$$

with suitable diagonal matrices A and B. This is a particular class of transformation (4.10). If $r_i = 0$ for some i then one can assume that $p_i \neq 0$. Let us apply the transformation of the form (3.13) with

$$A = 1 - E_i, \quad B = -E_i, \quad C = E_i, \quad D = 1 - E_i$$

$$G \mapsto \tilde{G} = [1 + E_i(1 - G)]\, [1 - E_i(1 + G)]^{-1}$$

where the matrix E_i is of the form (2.2). Such a transformation is applicable only if the matrix

$$1 - E_i(1 + G)$$

does not degenerate. It is easy to see that the determinant of this matrix is equal to $\pm g_{ii} = \gamma_{ii}(0)$. If $g_{ii} = 0$ but the solution is irreducible then one can perform a shift $\mathbf{u} \mapsto \mathbf{u} + \mathbf{u}^0$ arriving at a matrix $G' = \Gamma(\mathbf{u}^0)$ with $g'_{ii} \neq 0$. After the transformation one obtains $\tilde{r}_i = p_i \neq 0$.

In order to prove the second part of the Lemma it suffices to observe that any eigenvector f of the matrix G with the eigenvalue λ is also an eigenvector of G^2 with the eigenvalue λ^2. So, if e_i and e_j are the i-th and j-th basic vectors and $\Delta_i \neq \Delta_j$ then these vectors belong respectively to the sum of the root subspaces $R(\sqrt{\Delta_i}) \oplus R(-\sqrt{\Delta_i})$ and $R(\sqrt{\Delta_j}) \oplus R(-\sqrt{\Delta_j})$ of the matrix G. As these root subspaces of a symmetric matrix are orthogonal, the matrix G must have a block-diagonal form in the same basis. □

The Main Theorem 1.2 readily follows from the above considerations.

Recall that the reconstruction of the solution to the associativity equations with given rotation coefficients (1.17) depends on the choice of a solution to the linear system (1.8), (1.9). We will now apply this procedure in order to produce examples of linearly degenerate solutions to WDVV. It is convenient to separately consider the cases $\rho \neq 0$ and $\rho = 0$.

Case 1. Eigenvalues of a symmetric matrix G satisfying $G^2 = \rho^2 \cdot 1$ are equal to $\pm\rho$. Denote k the number of eigenvalues equal to $-\rho$. Here we will consider in more details the case $k = 1$. It is more convenient to deal with the matrix

$$\tilde{G} = G - \rho \cdot 1$$

satisfying

$$\tilde{G}^2 + 2\rho\tilde{G} = 0.$$

For the case $k = 1$ this matrix can be represented in the form

$$\tilde{G} = (\omega_i \omega_j)\,, \quad \sum_{i=1}^{n} \omega_i^2 = -2\rho.$$

In this way we obtain the family of solutions of the form (3.8). The substitution

$$\tilde{u}_k = -\log\left[\omega_k^2(u_k - u_k^0)\right], \quad k = 1, \dots, n$$

with arbitrary constants u_k^0 satisfying

$$\sum_{k=1}^{n} u_k^0 = 0$$

yields the following rotation coefficients satisfying Darboux–Egorov equations

$$\tilde{\gamma}_{ij} = \frac{e^{-\frac{\tilde{u}_i + \tilde{u}_j}{2}}}{\sum_{k=1}^{n} e^{-\tilde{u}_k}}, \quad i \neq j. \tag{4.15}$$

In sequel we will omit all tildes. The system (1.8)–(1.9) can be easily solved:

$$\begin{aligned} \psi_{ii} &= \frac{2e^{-u_i}}{D} - 1, \quad D = \sum_{k=1}^{n} e^{-u_k} \\ \psi_{ij} &= \frac{2e^{-\frac{u_i+u_j}{2}}}{D}, \quad i \neq j. \end{aligned} \tag{4.16}$$

Computation of the quadratures (1.11) gives the following expression for the matrix Ω of the second derivatives of the potential (see (1.10))

$$\Omega_{ij} = u_i \delta_{ij} + \frac{4e^{-\frac{u_i+u_j}{2}}}{D}. \tag{4.17}$$

Flat coordinates are obtained by choosing a linear combination of the columns of this matrix. The choice of the first column yields a Egorov metric

$$ds^2 = \left(1 - 4\frac{e^{-u_1}}{D}\right) du_1^2 + 4\sum_{i=1}^{n} \frac{e^{-u_1-u_i}}{D^2} du_i^2$$

with the flat coordinates

$$v_1 = u_1 + \frac{4e^{-u_1}}{D}, \quad v_i = \frac{4e^{-\frac{u_1+u_i}{2}}}{D} \quad \text{for} \quad i \neq 1$$

Solving these equations for the canonical coordinates u_i

$$u_1 = v_1 - \sqrt{4-\sigma} - 2, \quad u_i = v_1 - \sqrt{4-\sigma} - 2 + 2\log\frac{2+\sqrt{4-\sigma}}{v_i} \quad \text{for} \quad i \neq 1$$

with

$$\sigma = \sum_{k=2}^{n} v_k^2$$

and integrating the quadratures (4.17) one arrives at the following expression for the potential of the corresponding linearly degenerate solution to the WDVV associativity equations

$$F = \frac{1}{6}v_1^3 + \frac{1}{2}v_1\sigma - \sum_{k=2}^{n} v_k^2 \log v_k - \frac{1}{3}(2+\sigma)\sqrt{4-\sigma} + \sigma\log(2+\sqrt{4-\sigma}). \qquad (4.18)$$

One can also obtain an explicit realization of the integrable hierarchy associated, in the sense of [4], with (4.18). Recall that the hierarchy consists of an infinite family of commuting flows labeled by pairs (α, p), $\alpha = 1, \ldots, n$, $p = 0,\ 1,\ 2, \ldots$. The flows read

$$\frac{\partial v^\gamma}{\partial t^{\alpha,p}} = \partial_x \left(\nabla^\gamma \theta_{\alpha,p+1}(v)\right). \qquad (4.19)$$

The generating functions

$$\theta_\alpha(v,z) = \sum_{p=0}^{\infty} \theta_{\alpha,p}(v) z^p$$

of $\theta_{\alpha,p}(v)$ (the *deformed flat coordinates*) can be found by quadratures

$$d\theta_\alpha(v,z) = \sum_{i=1}^{n} h_i \Psi_{i\,\alpha} du_i, \quad \alpha = 1, \ldots, n \qquad (4.20)$$

from a basis $\Psi_{i\,\alpha}(v,z)$, $\alpha = 1, \ldots, n$ of "wave functions" determined from the system

$$\frac{\partial \Psi_i}{\partial u_j} = \gamma_{ij}\Psi_j, \quad i \neq j$$

$$\sum_{k=1}^{n} \frac{\partial \Psi_i}{\partial u_k} = z\,\Psi_i$$

The basis $\Psi_{i\alpha}$ can be conveniently normalized by the conditions

$$\sum_{\alpha=1}^{n} \Psi_{i\alpha}(v,-z)\Psi_{j\alpha}(v,z) = \delta_{ij}. \qquad (4.21)$$

In our case the normalized wave functions read

$$\Psi_{i\alpha} = \frac{2e^{z\,u_\alpha}}{\sqrt{1-4z^2}}\left[\left(z - \frac{1}{2}\right)\delta_{i\alpha} + \frac{e^{-\frac{u_i+u_\alpha}{2}}}{D}\right]. \qquad (4.22)$$

That gives

$$\theta_\alpha = \frac{1}{\sqrt{1-4z^2}}\left\{\left[\frac{1}{z}\left(e^{z\,u_1} - 1\right) - e^{z\,u_1}(u_1+2) + 2\right]\delta_{\alpha\,1} + v_\alpha e^{z\,u_\alpha}\right\}, \quad \alpha = 1, \ldots, n. \qquad (4.23)$$

Case 2. We will now consider the second type solutions parametrized by symmetric matrices G satisfying $G^2 = 0$. In this case one obtains a solution to the WDVV equations also satisfying the quasihomogeneity condition.

All the eigenvalues of G are equal to 0. All the Jordan blocks are of order 1 or 2. Let us consider the simplest case of only one block of order 2. The matrix $G = (g_{ij})$ can be written in the form

$$G_{ij} = \omega_i \omega_j, \quad \sum_{i=1}^{n} \omega_i^2 = 0$$

The corresponding solution to WDVV can be obtained from the trivial (i.e., the cubic one)

$$F(v) = \frac{1}{6} \sum_{i,j,k} c_{ijk} v^i v^j v^k \tag{4.24}$$

by the inversion symmetry of [5] (see esp. Appendix B and Proposition 3.14). Here c_{ijk} are structure constants of a semisimple Frobenius algebra

$$\mathcal{A} = \mathrm{span}\,(e_1, \ldots, e_n), \quad \langle e_i \cdot e_j, e_k \rangle = c_{ijk}, \quad \langle e_i, e_j \rangle = \delta_{i+j,n+1} \tag{4.25}$$

with the unity e_1 and the trivial gradation $\deg e_i = 0$ for all i. Recall that the structure constants can be represented in the form

$$c_{ijk} = \sum_{s=1}^{n} \frac{a_{si} a_{sj} a_{sk}}{a_{s1}} \tag{4.26}$$

with a matrix (a_{ij}) satisfying

$$\sum_{s=1}^{n} a_{si} a_{sj} = \delta_{i+j,n+1}. \tag{4.27}$$

For our construction one has to choose the matrix in such a way that

$$a_{i1} = \omega_i, \quad i = 1, \ldots, n.$$

After the substitution

$$\begin{aligned} \hat{v}^1 &= \frac{1}{2} \frac{v_\alpha v^\alpha}{v^n} \\ \hat{v}^\alpha &= \frac{v^\alpha}{v^n}, \quad \alpha \neq 1,\, n \\ \hat{v}^n &= -\frac{1}{v^n} \end{aligned}$$

one obtains the needed solution $\hat{F}$ to WDVV in the form

$$\hat{F}\,(\hat{v}) = \frac{1}{2} \hat{v}^1 \hat{v}_\alpha \hat{v}^\alpha + (\hat{v}^n)^2\, F(v) = \frac{1}{2} (\hat{v}^1)^2 \hat{v}^n + \frac{1}{2} \sum_{\alpha=2}^{n-1} \hat{v}^1 \hat{v}^\alpha \hat{v}^{n-\alpha+1} + \frac{P\,(\hat{v}^2, \ldots, \hat{v}^{n-1})}{\hat{v}^n}. \tag{4.28}$$

Here[2] $P(\hat{v}^2, \ldots, \hat{v}^{n-1})$ is certain polynomial of degree 4. The potential $\hat{F}$ satisfies the quasihomogeneity condition

$$\hat{E}\,\hat{F} = \hat{F}, \quad \hat{E} = \hat{v}^1 \frac{\partial}{\partial \hat{v}^1} - \hat{v}^n \frac{\partial}{\partial \hat{v}^n}. \tag{4.29}$$

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[2] This example has also been considered in [10] in a different framework. Our formula (4.28) is in disagreement with [10].