

Hamiltonian systems of hydrodynamic type in $2 + 1$ dimensions

E.V. Ferapontov¹, A. Moro¹ and V.V. Sokolov²

¹ Department of Mathematical Sciences
Loughborough University
Loughborough, Leicestershire LE11 3TU
United Kingdom

² Landau Institute for Theoretical Physics
Kosygina 2, 119334 Moscow, Russia

e-mails:

E.V.Ferapontov@lboro.ac.uk
A.Moro@lboro.ac.uk
vsokolov@landau.ac.ru

Abstract

We investigate multi-dimensional Hamiltonian systems associated with constant Poisson brackets of hydrodynamic type. A complete list of two- and three-component integrable Hamiltonians is obtained. All our examples possess dispersionless Lax pairs and an infinity of hydrodynamic reductions.

MSC: 35L40, 35L65, 37K10.

Keywords: Multi-dimensional Systems of Hydrodynamic Type, Hamiltonian Structures, Dispersionless Lax Pairs, Hydrodynamic Reductions.

1 Introduction

Over the past three decades there has been a significant progress in the theory of $(1 + 1)$ -dimensional quasilinear systems,

$$u_t^i + v_j^i(\mathbf{u})u_x^j = 0, \quad (1)$$

which are representable in the Hamiltonian form $u_t^i + P^{ij}h_j = 0$. Here $h(\mathbf{u})$ is a Hamiltonian density, $h_j = \partial_{u^j}h$, and P^{ij} is a Hamiltonian operator of differential-geometric type,

$$P^{ij} = g^{ij}(\mathbf{u})\frac{d}{dx} + b_k^{ij}(\mathbf{u})u_x^k,$$

generated by a metric g^{ij} (assumed non-degenerate) and its Levi-Civita connection Γ_{jk}^i via $b_k^{ij} = -g^{is}\Gamma_{sk}^j$. It was demonstrated in [6] that the metric g^{ij} must necessarily be flat, and in the flat coordinates of g^{ij} the operator P^{ij} takes a constant coefficient form $P^{ij} = \epsilon^i\delta^{ij}\frac{d}{dx}$. In the same coordinates, Hamiltonian systems take a Hessian form $u_t^i + \epsilon^i h_{ij}u_x^j = 0$. It was observed that many particularly important examples arising in applications are diagonalizable, that is, reducible to the Riemann invariant form $R_t^i + v^i(\mathbf{R})R_x^i = 0$. We recall that there exists a simple tensor criterion of the diagonalizability for an arbitrary hyperbolic system (1). Let us first calculate the Nijenhuis tensor of the matrix v_j^i ,

$$\mathcal{N}_{jk}^i = v_j^p\partial_{u^p}v_k^i - v_k^p\partial_{u^p}v_j^i - v_p^i(\partial_{u^j}v_k^p - \partial_{u^k}v_j^p), \quad (2)$$

and introduce the Haantjes tensor

$$\mathcal{H}_{jk}^i = \mathcal{N}_{pr}^i v_j^p v_k^r - \mathcal{N}_{jr}^p v_p^i v_k^r - \mathcal{N}_{rk}^p v_p^i v_j^r + \mathcal{N}_{jk}^p v_r^i v_p^r. \quad (3)$$

It was observed in [16] that a $(1, 1)$ -tensor v_j^i with mutually distinct eigenvalues is diagonalizable if and only if the corresponding Haantjes tensor \mathcal{H} is identically zero. As demonstrated by Tsarev, a combination of the diagonalizability with the Hamiltonian property implies the integrability: all diagonalizable Hamiltonian systems possess an infinity of conservation laws and commuting flows, and can be solved by the generalized hodograph transform. We refer to [27, 6] for further discussion and references.

The aim of our paper is to generalize this approach to $(2 + 1)$ -dimensional Hamiltonian systems

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y = 0, \quad (4)$$

which are representable in the form $\mathbf{u}_t + Ph_{\mathbf{u}} = 0$ where $h(\mathbf{u})$ is a Hamiltonian density, and P is a two-dimensional Hamiltonian operator of differential-geometric type,

$$P^{ij} = g^{ij}(\mathbf{u})\frac{d}{dx} + b_k^{ij}(\mathbf{u})u_x^k + \tilde{g}^{ij}(\mathbf{u})\frac{d}{dy} + \tilde{b}_k^{ij}(\mathbf{u})u_y^k;$$

such operators are generated by a pair of metrics g^{ij} , \tilde{g}^{ij} and the corresponding Levi-Civita connections Γ_{jk}^i , $\tilde{\Gamma}_{jk}^i$ via $b_k^{ij} = -g^{is}\Gamma_{sk}^j$, $\tilde{b}_k^{ij} = -\tilde{g}^{is}\tilde{\Gamma}_{sk}^j$. The theory of multi-dimensional Poisson brackets was constructed in [6, 19, 20]. The main difference from the one-dimensional situation is that, although both metrics g^{ij} and \tilde{g}^{ij} must necessarily be flat, they can no longer be reduced to a constant coefficient form simultaneously: there exist obstruction tensors. The obstruction tensors are known to vanish if either one of the metrics is positive definite, or a pair of metrics is non-singular in the sense of [20], that is, the mutual eigenvalues of g^{ij} and \tilde{g}^{ij} are distinct.

In both cases, the operator P^{ij} can be transformed to a constant coefficient form. In the two-component situation any non-singular Hamiltonian operator can be cast into a canonical form

$$P = \begin{pmatrix} d/dx & 0 \\ 0 & d/dy \end{pmatrix}$$

by an appropriate linear change of the independent variables x, y . The corresponding Hamiltonian systems take the form

$$u_t^1 + (h_1)_x = 0, \quad u_t^2 + (h_2)_y = 0. \quad (5)$$

The ‘simplest’ non-trivial integrable Hamiltonian density is $h(u^1, u^2) = u^1 u^2 - \frac{1}{6}(u^1)^3$ (we point out that, up to certain natural equivalence, there exist no other integrable densities which are polynomial in u^1, u^2). The corresponding equations (5) take the form

$$u_t^1 - u^1 u_x^1 + u_x^2 = 0, \quad u_t^2 + u_y^1 = 0,$$

see Sect. 4.1. This system appears in the context of the genus zero universal Whitham hierarchy, [17, 18]. Setting $u^1 = -\varphi_{xt}$, $u^2 = \varphi_{xy}$ one obtains a second order PDE

$$\varphi_{tt} - \varphi_{xy} + \frac{1}{2}\varphi_{xt}^2 = 0,$$

which is one of the Hirota equations of the dispersionless Toda hierarchy [9]. The same equation appeared in [22] in the classification of integrable Egorov’s hydrodynamic chains. Other examples of integrable Hamiltonian densities expressible in elementary functions include

$$h(u^1, u^2) = \frac{1}{2}(u^1 - u^2)^2 + e^{u^2}, \quad h(u^1, u^2) = u^2 \sqrt{u^1} + \alpha(u^1)^{5/2}, \quad h(u^1, u^2) = (u^1 u^2)^{2/3},$$

etc. The problem of classification of integrable two-component Hamiltonian systems (5) was first addressed in [10] based on the method of hydrodynamic reductions. We recall that a multi-dimensional quasilinear system (4) is said to be integrable if it possesses an infinity of n -component hydrodynamic reductions parametrized by n arbitrary functions of a single variable (see Sect. 2 for more details). It was demonstrated in [10] that this requirement imposes strong restrictions on the corresponding Hamiltonian density $h(u^1, u^2)$. In Sect. 4 we provide a complete list of integrable Hamiltonian densities (Theorem 1), as well as the associated dispersionless Lax pairs (Sect. 4.1). The ‘generic’ density is expressed in terms of the Weierstrass elliptic functions.

In the three-component situation we consider Hamiltonian operators of the form

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} \lambda^1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix} \frac{d}{dy}, \quad (6)$$

here λ^i are constant and pairwise distinct; the corresponding Hamiltonian systems are

$$u_t^i + (h_i)_x + \lambda^i (h_i)_y = 0. \quad (7)$$

There is a new phenomenon arising in the multi-component case: it was observed in [12] that the necessary condition for integrability of an n -component quasilinear system (4) is the vanishing of the Haantjes tensor for an arbitrary matrix of the form

$$(\alpha A + \beta B + \gamma I_n)^{-1} (\tilde{\alpha} A + \tilde{\beta} B + \tilde{\gamma} I_n).$$

In fact, it is sufficient to require the vanishing of the Haantjes tensor for a two-parameter family $(kA + I_n)^{-1}(lB + I_n)$. We point out that in the two-component case the Haantjes tensor vanishes automatically. On the contrary, in the multi-component situation the vanishing of the Haantjes tensor is a very strong restriction. Systems with this property will be called ‘diagonalizable’ (we would like to stress that matrices A and B do not commute in general, and cannot be diagonalized simultaneously). In Sect. 5 we obtain a complete list of diagonalizable three-component Hamiltonian systems (7) (Theorem 3). It turns out that in this case the diagonalizability conditions are very restrictive, and imply the integrability. For technical reasons, the classification results take much simpler form when expressed in terms of the Legendre transform H of the Hamiltonian density h , rather than the Hamiltonian density h itself (recall that $H = \sum u^i h_i - h$, $H_i = u^i$, $u_i = h_i$; we use variables u_i with lower indices for the arguments of H). We demonstrate that the Legendre transform H of the ‘generic’ integrable Hamiltonian density h is given by the formula

$$H = \sum_{j \neq i} \frac{\lambda^i - \lambda^j}{a_i^2 a_j^2} V(a_i u_i, a_j u_j)$$

where

$$V(x, y) = Z(x - y) + \epsilon Z(x - \epsilon y) + \epsilon^2 Z(x - \epsilon^2 y);$$

here a_i are arbitrary constants, $\epsilon = e^{2\pi i/3}$, and $Z'' = \zeta$ where ζ is the Weierstrass zeta-function: $\zeta' = -\wp$, $(\wp')^2 = 4\wp^3 - g_3$. Notice that we are dealing with an incomplete elliptic curve, $g_2 = 0$, and that the expression for V is real. The above formula for H has a natural multi-component extension, which is also integrable. This formula possesses a number of remarkable degenerations which are listed in Theorems 1 and 3. In particular, one has

$$H = \sum_{j \neq i} \frac{\lambda^i - \lambda^j}{a_i^2 a_j^2} (a_i u_i - a_j u_j) \ln(a_i u_i - a_j u_j).$$

We prove that all examples appearing in the classification possess dispersionless Lax pairs and an infinity of hydrodynamic reductions (Theorems 4 and 5 in Sect. 5.1 and 5.2). It is important to stress that, in $1+1$ dimensions, integrable Hamiltonians are parametrized by $\frac{n(n-1)}{2}$ arbitrary functions of two variables. On the contrary, in $2+1$ dimensions, the moduli spaces of integrable Hamiltonians are finite-dimensional. Furthermore, the results Sect. 6 (Theorems 6 and 7) make it tempting to conjecture that there exists no non-trivial integrable Hamiltonian systems of hydrodynamic type in $3+1$ dimensions.

The analysis of the integrability conditions is considerably simplified after a transformation of a given Hamiltonian system into the so-called Godunov, or symmetric, form. This construction is briefly reviewed in Sect. 3.

The necessary information on hydrodynamic reductions and dispersionless Lax pairs is summarized in Sect. 2.

2 Hydrodynamic reductions and dispersionless Lax pairs

Applied to a $(2+1)$ -dimensional system (4), the method of hydrodynamic reductions consists of seeking multi-phase solutions in the form

$$\mathbf{u}(x, y, t) = \mathbf{u}(R^1(x, y, t), \dots, R^n(x, y, t))$$

where the ‘phases’ $R^i(x, y, t)$ are required to satisfy a pair of $(1 + 1)$ -dimensional systems of hydrodynamic type,

$$R_t^i = \nu^i(\mathbf{R}) R_y^i, \quad R_x^i = \mu^i(\mathbf{R}) R_y^i.$$

Solutions of this form, known as ‘non-linear interactions of n planar simple waves’ [25, 4, 24], have been extensively discussed in gas dynamics; later, they reappeared in the context of the dispersionless KP hierarchy, see [14, 15] and references therein. Technically, one ‘decouples’ a $(2 + 1)$ -dimensional system (4) into a pair of commuting n -component $(1 + 1)$ -dimensional systems. Substituting the ansatz $\mathbf{u}(R^1, \dots, R^n)$ into (4) one obtains

$$(\nu^i I_n + \mu^i A + B) \partial_i \mathbf{u} = 0, \quad i = 1, \dots, n, \quad (8)$$

$\partial_i = \partial/\partial R^i$, implying that both characteristic speeds ν^i and μ^i satisfy the dispersion relation

$$\det(\nu I_n + \mu A + B) = 0, \quad (9)$$

which defines an algebraic curve of degree n on the (ν, μ) -plane. Moreover, ν^i and μ^i have to satisfy the commutativity conditions

$$\frac{\partial_j \nu^i}{\nu^j - \nu^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad (10)$$

$i \neq j$, see [27]. It was observed in [10] that the requirement of the existence of ‘sufficiently many’ hydrodynamic reductions imposes strong restrictions on the system (4), and provides an efficient classification criterion. To be precise, we will call a system (4) integrable if, for any n , it possesses infinitely many n -component hydrodynamic reductions parametrized by n arbitrary functions of a single variable. Thus, integrable systems are required to possess an infinity of n -phase solutions which can be viewed as natural dispersionless analogs of algebro-geometric solutions of soliton equations.

We recall that a system (4) is said to possess a dispersionless Lax pair

$$\psi_t = f(\mathbf{u}, \psi_y), \quad \psi_x = g(\mathbf{u}, \psi_y), \quad (11)$$

if it can be recovered from the consistency condition $\psi_{xt} = \psi_{tx}$ (we point out that the dependence of f and g on ψ_y is generally non-linear). Lax pairs of this type first appeared in the construction of the universal Whitham hierarchy, see [17] and references therein. It was observed in [28] that such non-linear Lax pairs arise from the usual ‘solitonic’ Lax pairs in the dispersionless limit, and the cases of polynomial/rational dependence of f and g on ψ_y were investigated. In particular, a Hamiltonian formulation of such systems was uncovered, requiring a non-local Hamiltonian density. It was demonstrated in [10, 13] that, for a number of particularly interesting classes of systems, the existence of a dispersionless Lax pair is *equivalent* to the existence of hydrodynamic reductions and, thus, to the integrability.

Setting $\psi_y = p$ and calculating the consistency condition $\psi_{xt} = \psi_{tx}$ by virtue of (4), one arrives at the following relations for $f(\mathbf{u}, p)$ and $g(\mathbf{u}, p)$:

$$\text{grad} f + \text{grad} g A = 0, \quad \text{grad} g [f_p I_n + g_p A + B] = 0; \quad (12)$$

here grad is the gradient with respect to \mathbf{u} . In particular, this shows that f_p and g_p satisfy the dispersion relation (9), and the vector $\text{grad} g$ belongs to the left characteristic cone of the system (4). Thus, as p varies, the equations $\nu = f_p$, $\mu = g_p$ parametrize the dispersion curve (9), while $\text{grad} g$ parametrizes the left characteristic cone.

Throughout this paper we assume that the dispersion relation (9) defines an *irreducible algebraic curve*. This condition is satisfied for most examples discussed in the literature so far.

3 Transformation of a Hamiltonian system into Godunov's form

Recall that a system of hydrodynamic type (4) is said to be symmetrizable, or reducible to Godunov's form [8], if it possesses a conservative representation of the form

$$(\partial_{u^i} p)_t + (\partial_{u^i} q)_x + (\partial_{u^i} r)_y = 0;$$

here the potentials p, q and r are certain functions of \mathbf{u} . Any such system possesses an extra conservation law $L(p)_t + L(q)_x + L(r)_y = 0$ where L denotes Legendre's transform. Equations in Godunov's form play important role in the general theory of multi-dimensional hyperbolic conservation laws [5].

Given a Hamiltonian system (7) we perform the Legendre transform, $H = L(h) = u^i h_i - h$, $H_i = u^i$, $u_i = h_i$, to obtain a system in Godunov's form,

$$(H_i)_t + (u_i)_x + \lambda^i (u_i)_y = 0,$$

which corresponds to the choice $p = H$, $q = \sum u_i^2/2$, $r = \sum \lambda^i u_i^2/2$. We assume that the Legendre transform is well-defined, that is, all partial derivatives h_i are functionally independent. This condition is equivalent to the requirement that the Hessian matrix of h is non-degenerate, which is automatically satisfied under the assumption of the irreducibility of the dispersion relation. It turns out that the integrability conditions take much simpler form when represented in terms of the Legendre transform $H = L(h)$, rather than the Hamiltonian density h itself. Thus, in what follows we will work with systems represented in Godunov's form (to make the equations look formally 'evolutionary' we will relabel the independent variables as $x, y, t \rightarrow T, X, Y$). This results in

$$(u_i)_T + \lambda^i (u_i)_X + (H_i)_Y = 0; \tag{13}$$

Systems of this type can be viewed as describing n linear waves (traveling with constant speeds λ^i in the X, T -plane) which are non-linearly coupled in the Y -direction.

4 Integrable Hamiltonians in 2 + 1 dimensions: two-component case

In this section we classify two-component Hamiltonian systems (5). The corresponding Legendre transform is

$$v_T + (H_v)_Y = 0, \quad w_X + (H_w)_Y = 0; \tag{14}$$

here $v = u_1, w = u_2$. We point out that this case was addressed previously in [10], although the classification was only sketched. Here we provide a complete list of integrable potentials $H(v, w)$, and calculate the corresponding dispersionless Lax pairs. For systems (14) the integrability conditions constitute an over-determined system of fourth order PDEs for the potential $H(v, w)$:

$$\begin{aligned} H_{vw} H_{vvv} &= 2H_{vv} H_{vvw}, \\ H_{vw} H_{vvv} &= 2H_{vv} H_{vw}, \\ H_{vw} H_{vvw} &= H_{vv} H_{vw} + H_{vv} H_{ww}, \\ H_{vw} H_{vww} &= 2H_{vv} H_{ww}, \\ H_{vw} H_{www} &= 2H_{vw} H_{ww}. \end{aligned} \tag{15}$$

The system (15) is in involution, and its solution space is 10-dimensional [10]. We point out that the transformations

$$v \rightarrow av + b, \quad w \rightarrow cw + d, \quad H \rightarrow \alpha H + \beta v^2 + \gamma w^2 + \mu v + \nu w + \delta$$

generate a 10-dimensional group of Lie-point symmetries of the system (15). These transformations correspond to obvious linear changes of the independent variables X, Y, T in the equations (14). One can show that the action of the symmetry group on the moduli space of solutions of the system (15) possesses an open orbit. The classification of integrable potentials $H(v, w)$ will be performed up to this equivalence. Moreover, we will not be interested in the potentials which are either quadratic in v, w and generate linear systems (14), or separable potentials of the form $f(v) + g(w)$ giving rise to reducible systems. Our main result is the following complete list of integrable potentials:

Theorem 1 *The ‘generic’ solution of the system (15) is given by the formula*

$$H(v, w) = Z(v + w) + \epsilon Z(v + \epsilon w) + \epsilon^2 Z(v + \epsilon^2 w); \quad (16)$$

here $\epsilon = e^{2\pi i/3}$ and $Z''(s) = \zeta(s)$ where ζ is the Weierstrass zeta-function: $\zeta' = -\wp$, $(\wp')^2 = 4\wp^3 - g_3$. Degenerations of this solution correspond to

$$H(v, w) = \frac{1}{2}v^2\zeta(w), \quad (17)$$

$$H(v, w) = (v + w) \ln(v + w), \quad (18)$$

as well as the following polynomial potentials:

$$H(v, w) = v^2w^2, \quad (19)$$

$$H(v, w) = vw^2 + \frac{\alpha}{5}w^5, \quad \alpha = \text{const}, \quad (20)$$

and

$$H(v, w) = vw + \frac{1}{6}w^3. \quad (21)$$

Remark. The ‘elliptic’ examples (16) and (17) possess a specialization $g_3 = 0$: $\wp(w) \rightarrow 1/w^2$, $\zeta(w) \rightarrow 1/w$, $\sigma(w) \rightarrow w$, etc. This results in the potentials

$$H(v, w) = (v + w) \log(v + w) + \epsilon(v + \epsilon w) \log(v + \epsilon w) + \epsilon^2(v + \epsilon^2 w) \log(v + \epsilon^2 w) \quad (22)$$

and

$$H(v, w) = \frac{v^2}{2w},$$

respectively. Dispersionless Lax pairs for the equations (14) corresponding to the potentials (16)-(21) are calculated in Sect. 4.1.

Proof of Theorem 1:

The system (15) can be solved as follows. The first two equations imply that $H_{vvv}/H_{vw}^2 = \text{const}$. Similarly, the last two equations imply $H_{www}/H_{vw}^2 = \text{const}$. Setting $H_{vw} = e$ one can parametrise the third order derivatives of H in the form

$$H_{vvv} = \frac{1}{2}me^2, \quad H_{vvw} = e_v, \quad H_{vww} = e_w, \quad H_{www} = \frac{1}{2}ne^2, \quad (23)$$

here m, n are arbitrary constants. The compatibility conditions for these equations, plus the equation (15)₃, result in the following overdetermined system for e :

$$(\ln e)_{vw} = \frac{mn}{4}e^2, \quad e_{vv} = mee_w, \quad e_{ww} = nee_v. \quad (24)$$

The general solution of the first (Liouville) equation has the form

$$e^2 = \frac{4}{mn} \frac{p'(v)q'(w)}{(p(v) + q(w))^2}, \quad (25)$$

one has to consider separately the case $e = \text{const}$ (up to the equivalence transformations, this results in the potential (21)), as well as the case when e depends on one variable only, say, on w (this leads to the potential (20)). Let us assume that both constants m and n are nonzero (the cases when either of them vanishes will be discussed later). By scaling v and w one can assume $m = n = 1$. Setting

$$(p')^3 = P^2(p), \quad (q')^3 = Q^2(q), \quad (26)$$

(here $P(p)$ and $Q(q)$ are functions to be determined), one obtains from the last two equations (24) the following functional-differential equations for P and Q :

$$P''(p+q)^2 - 4P'(p+q) + 6P = 2Q'(p+q) - 6Q, \quad Q''(p+q)^2 - 4Q'(p+q) + 6Q = 2P'(p+q) - 6P;$$

these equations imply that both P and Q are cubic polynomials in p and q ,

$$P = ap^3 + bp^2 + cp + d, \quad Q = aq^3 - bq^2 + cq - d,$$

where a, b, c, d are arbitrary constants. Notice that the right hand side of (25) possesses the following $SL(2, R)$ -invariance,

$$p \rightarrow \frac{\alpha p + \beta}{\gamma p + \delta}, \quad q \rightarrow -\frac{\alpha p - \beta}{\gamma p - \delta},$$

which can be used to bring the polynomials $P(p)$ and $Q(q)$ to canonical forms. There are three cases to consider.

Three distinct roots: in this case one can reduce both $P(p)$ and $Q(q)$ to quadratics, so that the ODEs (26) assume the form

$$(p')^3 = \frac{27}{2}(p^2 + g_3)^2 \quad \text{and} \quad (q')^3 = \frac{27}{2}(q^2 + g_3)^2,$$

respectively. Thus, $p = \wp'(v)$, $q = \wp'(w)$ where \wp is the Weierstrass \wp -function: $(\wp')^2 = 4\wp^3 - g_3$ (we point out that the value of g_3 is not really essential, and can be normalized to ± 1). Setting

$$H_{vw} = e = -\frac{12\wp(v)\wp(w)}{\wp'(v) + \wp'(w)}$$

and integrating (23) with respect to v and w we obtain

$$H_{vv} = -6\zeta(w) - \frac{12\wp^2(w)}{\wp'(v) + \wp'(w)}, \quad H_{vw} = -\frac{12\wp(v)\wp(w)}{\wp'(v) + \wp'(w)}, \quad H_{ww} = -6\zeta(v) - \frac{12\wp^2(v)}{\wp'(v) + \wp'(w)}, \quad (27)$$

here the zeta-function is defined as $\zeta' = -\wp$. Since the \wp -function on the elliptic curve $y^2 = 4x^3 - g_3$ satisfies the automorphic property $\wp(\epsilon z) = \epsilon\wp(z)$, $\epsilon^3 = 1$, one can rewrite (27) in the following equivalent form:

$$\begin{aligned} H_{vv} &= -2\left(\zeta(v+w) + \epsilon\zeta(v+\epsilon w) + \epsilon^2\zeta(v+\epsilon^2 w)\right), \\ H_{vw} &= -2\left(\zeta(v+w) + \epsilon^2\zeta(v+\epsilon w) + \epsilon\zeta(v+\epsilon^2 w)\right), \\ H_{ww} &= -2\left(\zeta(v+w) + \zeta(v+\epsilon w) + \zeta(v+\epsilon^2 w)\right). \end{aligned}$$

Up to a constant multiple, these formulae give rise to (16).

Double root: in this case both $P(p)$ and $Q(q)$ can be reduced to p and q , so that the ODEs (26) take the form $(p')^3 = 27p^2$ and $(q')^3 = 27q^2$, respectively. This leads to $p = v^3$, $q = w^3$, and a straightforward integration of (23) gives

$$H_{vv} = -\frac{6w^2}{v^3 + w^3}, \quad H_{vw} = \frac{6vw}{v^3 + w^3}, \quad H_{ww} = -\frac{6v^2}{v^3 + w^3};$$

notice that these formulae can be obtained as a degeneration of (27) corresponding to $g_3 = 0$. Up to a constant multiple, this leads to the potential (22).

Triple root: in this case both $P(p)$ and $Q(q)$ can be reduced to constants, so that the ODEs (26) take the form $(p')^3 = 1$ and $(q')^3 = 1$, respectively. This leads to $e = 2/(v+w)$, which, up to a constant multiple, results in the potential (18).

If $m = 0$, $n \neq 0$ (without any loss of generality we will again set $n = 1$), equations (24) can be solved in the form $e = 6v\wp(w)$ where \wp is the Weierstrass \wp -function: $(\wp')^2 = 4\wp^3 - g_3$. The corresponding potential H is given by $H = -3v^2\zeta(w)$. Up to a multiple, this is the case (17).

In the simplest case $m = n = 0$ equations (24) imply

$$e = (\alpha v + \beta)(\gamma w + \delta),$$

and the elementary integration of equations (23) results in

$$H(v, w) = \left(\frac{1}{2}\alpha v^2 + \beta v\right)\left(\frac{1}{2}\gamma w^2 + \delta w\right);$$

here $\alpha, \beta, \gamma, \delta$ are arbitrary constants. Using the equivalence transformations one can reduce H to either $H = v^2w^2$ (both α and γ are nonzero) or $H = vw^2$ ($\alpha = 0$). These are the polynomial cases (19) and a subcase of (20), respectively. This finishes the proof of Theorem 1.

4.1 Dispersionless Lax pairs

In this section we calculate dispersionless Lax pairs for systems (14) corresponding to the potentials (16)-(21) of Theorem 1. We point out that, in spite of the deceptive simplicity of some of these potentials, the corresponding Lax pairs are quite non-trivial.

Potential (21): The corresponding system (14) takes the form

$$v_T + w_Y = 0, \quad w_X + w w_Y + v_Y = 0; \tag{28}$$

it arises in the genus zero case of the universal Whitham hierarchy [17, 18]. This system possesses the Lax pair

$$\psi_T = \frac{1}{2} \ln(\psi_Y + w/2), \quad \psi_X = \psi_Y^2 + v/2.$$

A simple calculation shows that the Legendre transform of the potential $H(v, w) = vw + \frac{1}{6}w^3$, defined by the formulae

$$u^1 = H_v, \quad u^2 = H_w, \quad h(u^1, u^2) = vH_v + wH_w - h,$$

is also polynomial:

$$h(u^1, u^2) = u^1 u^2 - \frac{1}{6}(u^1)^3.$$

We point out that all other examples of integrable potentials $H(v, w)$ produce non-polynomial Hamiltonian densities $h(u^1, u^2)$.

Potential (20): The corresponding system (14) takes the form

$$v_T + (w^2)_Y = 0, \quad w_X + 2(vw)_Y + \alpha(w^4)_Y = 0. \quad (29)$$

For $\alpha = 0$ it possesses the Lax pair

$$\psi_T = -\frac{w^2}{2\psi_Y^2}, \quad \psi_X = \psi_Y^4 - 2v\psi_Y.$$

Setting $v = u_Y$, $w^2 = -u_T$ one can rewrite (29) (when $\alpha = 0$) as a single second order PDE

$$u_{XT} + 2u_Y u_{TY} + 4u_T u_{YY} = 0.$$

Up to a rescaling $X \rightarrow -2X$ this equation is a particular case of the generalized dispersionless Harry Dym equation [1, 23]. For $\alpha \neq 0$ the Lax pair modifies to

$$\psi_T = f\left(\frac{w}{\psi_Y}\right), \quad \psi_X = \psi_Y^4 - 2v\psi_Y,$$

where the function $f(s)$ satisfies the equation $f'(s) = -s/(\alpha s^3 + 1)$ (for $\alpha = 0$ one recovers the previous formula). The first equation of this Lax pair appeared in [23] as a generating function of conservation laws for the Kupershmidt hydrodynamic chain. Without any loss of generality one can set $\alpha = -1$, which gives

$$f(s) = \frac{1}{3}(\ln(s-1) + \epsilon^2 \ln(s-\epsilon) + \epsilon \ln(s-\epsilon^2)), \quad \epsilon^3 = 1.$$

Potential (19): The corresponding system (14) takes the form

$$v_T + 2(vw^2)_Y = 0, \quad w_X + 2(v^2w)_Y = 0. \quad (30)$$

It possesses the Lax pair

$$\psi_T = w^2 a(\psi_Y), \quad \psi_X = -v^2 b(\psi_Y)$$

where the dependence of a and b on $\psi_Y \equiv \xi$ is governed by the ODEs

$$a' = -4\frac{a}{b} - 2, \quad b' = 4\frac{b}{a} + 2.$$

To solve these equations we proceed as follows. Expressing b from the first equation, $b = -4a/(a' + 2)$, and substituting into the second one arrives at a second order ODE $2aa'' - 3(a')^2 + 12 = 0$. It can be integrated once, $(a')^2 = 4ca^3 + 4$, where c is a constant of integration.

Without any loss of generality we will set $c = 1$. Thus, a is the Weierstrass \wp -function: $a = \wp(\xi, 0, -4) = \wp(\xi)$. The corresponding b is given by $b = -4\wp/(\wp' + 2)$. Notice that this expression for b equals $\wp(\xi + c)$ where c is the zero of \wp -function such that $\wp(c) = 0$, $\wp'(c) = 2$ (use the addition theorem to calculate $\wp(\xi + c)$). Ultimately, we obtain the Lax pair

$$\psi_T = w^2\wp(\psi_Y), \quad \psi_X = -v^2\wp(\psi_Y + c).$$

Setting $V = v^2$, $W = w^2$ one can rewrite (30) in the form where the non-linearity is quadratic:

$$V_t + 2WV_Y + 4VW_Y = 0, \quad W_x + 2VW_Y + 4WV_Y = 0.$$

Potential (18): The corresponding system (14) takes the form

$$v_T + \frac{v_Y + w_Y}{v + w} = 0, \quad w_X + \frac{v_Y + w_Y}{v + w} = 0.$$

It possesses the Lax pair

$$\psi_T = -\ln(w + \psi_Y), \quad \psi_X = \ln(v - \psi_Y).$$

This system also arises in the genus zero case of the universal Whitham hierarchy [17, 18]; its dispersionful analogue was constructed in [26].

Potential (17): The corresponding system (14) takes the form

$$v_T + \zeta(w)v_Y - v\wp(w)w_Y = 0, \quad w_X - \wp(w)vv_Y - \frac{1}{2}v^2\wp'(w)w_Y = 0.$$

One can show that it possesses the Lax pair

$$\psi_T = -f(w, \psi_Y), \quad \psi_X = -\frac{1}{2}v^2b(\psi_Y)$$

where, setting $\psi_Y \equiv \xi$, the function $f(w, \xi)$ has to satisfy the equations

$$f_w = \frac{2b(\xi)\wp(w)}{b'(\xi) + \wp'(w)}, \quad f_\xi = \zeta(w) + \frac{2\wp^2(w)}{b'(\xi) + \wp'(w)}.$$

We point out that the consistency condition $f_{\xi w} = f_{w\xi}$ implies a second order ODE $2bb'' - 3(b')^2 - 3g_3 = 0$ which, upon integration, gives

$$(b'(\xi))^2 = 4b^3(\xi) - g_3,$$

(the constant of integration is not essential). Thus, one can set $b = \wp(\xi)$ so that the equations for f take the form

$$f_w = \frac{2\wp(\xi)\wp(w)}{\wp'(\xi) + \wp'(w)}, \quad f_\xi = \zeta(w) + \frac{2\wp^2(w)}{\wp'(\xi) + \wp'(w)},$$

compare with (27)! Thus,

$$f(w, \xi) = \frac{1}{3} \ln \sigma(\xi + w) + \frac{\epsilon}{3} \ln \sigma(\xi + \epsilon w) + \frac{\epsilon^2}{3} \ln \sigma(\xi + \epsilon^2 w),$$

where σ is the Weierstrass sigma-function: $\sigma'/\sigma = \zeta$. Ultimately, the Lax pair takes the form

$$\psi_T = \frac{1}{3} \ln \sigma(\psi_Y + w) + \frac{\epsilon}{3} \ln \sigma(\psi_Y + \epsilon w) + \frac{\epsilon^2}{3} \ln \sigma(\psi_Y + \epsilon^2 w), \quad \psi_X = -\frac{1}{2} v^2 \wp(\psi_Y).$$

Potential (16): the equations corresponding to $H/3$ take the form

$$\begin{aligned} v_T + \left(\zeta(w) + \frac{2\wp^2(w)}{\wp'(v) + \wp'(w)} \right) v_Y + \frac{2\wp(v)\wp(w)}{\wp'(v) + \wp'(w)} w_Y &= 0, \\ w_X + \frac{2\wp(v)\wp(w)}{\wp'(v) + \wp'(w)} v_Y + \left(\zeta(v) + \frac{2\wp^2(v)}{\wp'(v) + \wp'(w)} \right) w_Y &= 0. \end{aligned}$$

One can show that the corresponding Lax pair is given by the equations

$$\psi_T = f(w, \psi_Y), \quad \psi_X = g(v, \psi_Y)$$

where, setting $\psi_Y = \xi$, the first order partial derivatives of f and g are given by

$$f_w = -\frac{2\wp(\xi)\wp(w)}{\wp'(\xi) + \wp'(w)}, \quad f_\xi = -\zeta(w) - \frac{2\wp^2(w)}{\wp'(\xi) + \wp'(w)}$$

and

$$g_v = -\frac{2\wp(\xi)\wp(v)}{\wp'(\xi) - \wp'(v)}, \quad g_\xi = -\zeta(v) + \frac{2\wp^2(v)}{\wp'(\xi) - \wp'(v)},$$

respectively. Explicitly, one has

$$\begin{aligned} f(w, \xi) &= -\frac{1}{3} \ln \sigma(\xi + w) - \frac{\epsilon}{3} \ln \sigma(\xi + \epsilon w) - \frac{\epsilon^2}{3} \ln \sigma(\xi + \epsilon^2 w), \\ g(v, \xi) &= \frac{1}{3} \ln \sigma(\xi - v) + \frac{\epsilon}{3} \ln \sigma(\xi - \epsilon v) + \frac{\epsilon^2}{3} \ln \sigma(\xi - \epsilon^2 v). \end{aligned}$$

Notice that the expression for $f(w, \xi)$ coincides with the one from the previous case. This means that the corresponding Hamiltonian systems commute with each other — the fact which is, in a sense, unexpected.

Potential (22): this is the $g_3 = 0$ degeneration of the potential (16). The system corresponding to $H/3$ takes the form

$$v_T + \frac{w^2}{v^3 + w^3} v_Y - \frac{vw}{v^3 + w^3} w_Y = 0, \quad w_X - \frac{vw}{v^3 + w^3} v_Y + \frac{v^2}{v^3 + w^3} w_Y = 0;$$

it possesses the Lax pair

$$\psi_T = f(w/\psi_Y), \quad \psi_X = g(v/\psi_Y)$$

where the dependence of f and g on their arguments is specified by $f'(s) = s/(s^3 - 1)$, $g'(s) = s/(s^3 + 1)$. Explicitly, one has

$$\begin{aligned} f(s) &= \frac{1}{3} (\ln(s - 1) + \epsilon^2 \ln(s - \epsilon) + \epsilon \ln(s - \epsilon^2)), \\ g(s) &= -\frac{1}{3} (\ln(s + 1) + \epsilon^2 \ln(s + \epsilon) + \epsilon \ln(s + \epsilon^2)). \end{aligned}$$

5 Integrable Hamiltonians in 2+1 dimensions: three-component case

In this section we classify three-component integrable equations of the form (13),

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_T + \begin{pmatrix} \lambda^1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_X + \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_Y = 0, \quad (31)$$

assuming that the constants λ^i are pairwise distinct. As mentioned in the introduction, the integrability of the system (31) implies the vanishing of the Haantjes tensor for any matrix of the two-parameter family $(kA + I_3)^{-1}(lB + I_3)$. Here $A = \text{diag}(\lambda^i)$ and $B = (H_{ij})$. To formulate the integrability conditions in a compact form we introduce the following notation:

$$R_1 = \frac{H_{12}H_{13}}{H_{23}}(\lambda^2 - \lambda^3), \quad R_2 = \frac{H_{12}H_{23}}{H_{13}}(\lambda^3 - \lambda^1), \quad R_3 = \frac{H_{13}H_{23}}{H_{12}}(\lambda^1 - \lambda^2);$$

we will see below that all mixed partial derivatives H_{ij} must be non-zero, otherwise the system is either linear, or reducible. Moreover, we will need the quantities

$$I = \Delta^2 - 4(\lambda^2 - \lambda^3)(\lambda^3 - \lambda^1)H_{12}^2 - 4(\lambda^3 - \lambda^1)(\lambda^1 - \lambda^2)H_{23}^2 - 4(\lambda^1 - \lambda^2)(\lambda^2 - \lambda^3)H_{13}^2$$

and

$$J = (\lambda^2 - \lambda^3)H_{12}^2H_{13}^2 + (\lambda^3 - \lambda^1)H_{23}^2H_{12}^2 + (\lambda^1 - \lambda^2)H_{13}^2H_{23}^2 - H_{12}H_{23}H_{13}\Delta$$

where

$$\Delta = (\lambda^2 - \lambda^3)H_{11} + (\lambda^3 - \lambda^1)H_{22} + (\lambda^1 - \lambda^2)H_{33}.$$

Our first result is the following

Theorem 2 *The system (31) with an irreducible dispersion curve is diagonalizable if and only if the potential H satisfies the relations*

$$\begin{aligned} J &= 0, \quad H_{123} = 0, \\ \frac{\partial}{\partial u_1} ((\lambda^3 - \lambda^2)H_{11} + R_2 + R_3) &= 0, \\ \frac{\partial}{\partial u_2} ((\lambda^1 - \lambda^3)H_{22} + R_1 + R_3) &= 0, \\ \frac{\partial}{\partial u_3} ((\lambda^2 - \lambda^1)H_{33} + R_1 + R_2) &= 0. \end{aligned} \quad (32)$$

Notice that, in contrast to the two-component situation (15), these relations are third order in the derivatives of H . We will demonstrate below that the necessary conditions (32) are, in fact, sufficient for the integrability, and imply the existence of dispersionless Lax pairs and an infinity of hydrodynamic reductions.

Remark. The condition $J = 0$, which is equivalent to $R_1 + R_2 + R_3 = \Delta$, has a simple geometric interpretation as the condition of reducibility of the left characteristic cone of the system (31) (see Sect. 2 for definitions). Indeed, the left characteristic cone consists of all vectors $\mathbf{g} = (g_1, g_2, g_3)$ which satisfy the relation

$$\mathbf{g}(\nu I_3 + \mu A + B) = 0. \quad (33)$$

Excluding ν and μ , one obtains a single algebraic relation among g_1, g_2, g_3 ,

$$\begin{aligned} & (H_{13}(g_1)^2 g_2 + H_{23}g_1(g_2)^2 + H_{33}g_1g_2g_3) (\lambda^1 - \lambda^2) + \\ & (H_{21}(g_2)^2 g_3 + H_{13}g_2(g_3)^2 + H_{11}g_1g_2g_3) (\lambda^2 - \lambda^3) + \\ & (H_{23}(g_3)^2 g_1 + H_{12}g_3(g_1)^2 + H_{22}g_1g_2g_3) (\lambda^3 - \lambda^1) = 0, \end{aligned} \quad (34)$$

which is the equation of the left characteristic cone. The condition $J = 0$ is equivalent to its degeneration into a line and a conic:

$$\begin{aligned} & [H_{12}H_{13}g_1 + H_{12}H_{23}g_2 + H_{13}H_{23}g_3] \\ & [H_{13}H_{23}(\lambda^1 - \lambda^2)g_1g_2 + H_{12}H_{23}(\lambda^3 - \lambda^1)g_1g_3 + H_{12}H_{13}(\lambda^2 - \lambda^3)g_2g_3] = 0. \end{aligned} \quad (35)$$

We point out that, by virtue of (33), the left characteristic cone and the dispersion curve are birationally equivalent. This implies that the dispersion curve is necessarily rational, although not reducible (the linear factor of the left characteristic cone corresponds to a singular point on the dispersion curve — see Sect. 5.2 for explicit formulae).

Proof of Theorem 2:

To simplify the calculation of the Haantjes tensor we multiply the matrix $(kA + I_3)^{-1}(lB + I_3)$ by $(k\lambda^1 + 1)(k\lambda^2 + 1)(k\lambda^3 + 1)$. This results in the matrix $\tilde{A}(lB + I_3)$ where $\tilde{A} = \text{diag}[(k\lambda^2 + 1)(k\lambda^3 + 1), (k\lambda^1 + 1)(k\lambda^3 + 1), (k\lambda^1 + 1)(k\lambda^2 + 1)]$. Since the multiplication by a scalar does not effect the vanishing of the Haantjes tensor, we will work with the matrix $\tilde{A}(lB + I_3)$ which has an advantage of being polynomial in k and l . Using computer algebra we calculate components of the Haantjes tensor \mathcal{H} (which are certain polynomials in k and l) and set them equal to zero. First of all, one can verify that all components of the form \mathcal{H}_{ij}^i vanish identically, so that the only nonzero components are \mathcal{H}_{jk}^i , $i \neq j \neq k$. In the following we will focus on the analysis of the component \mathcal{H}_{12}^3 : it turns out the vanishing of \mathcal{H}_{12}^3 alone implies the vanishing of the full Haantjes tensor. Let us compute coefficients at different powers of the parameter l and set them equal to zero. At the order l^0 , all terms in \mathcal{H}_{12}^3 vanish identically since \tilde{A} is a constant diagonal matrix. The coefficient at l^1 is a polynomial in k , however, setting its coefficients equal to zero we obtain only one independent relation:

$$H_{123} = 0.$$

Similarly, two extra relations come from the analysis of l^2 -terms, three relations from l^3 -terms, and four relations from l^4 -terms. Ultimately, we end up with a set of 9 linear homogeneous equations for the 9 third order derivatives H_{iii} , H_{ijj} .

From these 9 relations it readily follows that if one of the mixed derivatives equals zero, say, $H_{12} = 0$, then either $H_{13}H_{23} = 0$ or $H_{ijk} = 0$ for all i, j, k . In the first case the system (31) decouples into a pair of independent 1×1 and 2×2 subsystems. The second case corresponds to linear systems with constant coefficients. Therefore, from now on we assume $H_{ij} \neq 0$ for any $i \neq j$.

The set of 9 relations so obtained is rather complicated, and the calculation of the corresponding 9×9 determinant is computationally intense. A simpler equivalent set of relations can be derived as follows: first, divide \mathcal{H}_{12}^3 by $(\lambda^1 k + 1)(\lambda^2 k + 1)^2(\lambda^3 k + 1)^2$ (which is a common multiple), then equate to zero the coefficient of l^2 at $k = -1/\lambda^1, -1/\lambda^2$ (the coefficient at $k = -1/\lambda^3$ appears to be a linear combination of the previous two), the coefficient of l^3 at $k = -1/\lambda^1, -1/\lambda^2, -1/\lambda^3$ and the coefficient of l^4 at $k = 0, -1/\lambda^1, -1/\lambda^2, -1/\lambda^3$. As a result,

we arrive at a simpler set of 9 linearly independent relations that are nothing but linear combinations of the previous ones. If the determinant of this system is non-zero, then all remaining derivatives H_{iii} and H_{ij} vanish identically. This is the case of linear systems. Thus, to obtain non-linear examples, one has to require the vanishing of the determinant. It is straightforward to verify that this determinant factorizes as follows:

$$J^4 \left(I^2 - 64(\lambda^1 - \lambda^2)(\lambda^2 - \lambda^3)(\lambda^3 - \lambda^1)J \right) = 0.$$

Thus, there are two cases to consider. If

$$I^2 - 64(\lambda^1 - \lambda^2)(\lambda^2 - \lambda^3)(\lambda^3 - \lambda^1)J = 0, \quad (36)$$

then the dispersion relation of the system (31) is reducible. To show this we introduce the quantities

$$\begin{aligned} \Omega_1 &= \Delta H_{12} - 2H_{13}H_{23}(\lambda^1 - \lambda^2), \\ \Omega_2 &= \Delta H_{23} - 2H_{12}H_{13}(\lambda^2 - \lambda^3), \\ \Omega_3 &= \Delta^2 - 4H_{13}^2(\lambda^1 - \lambda^2)(\lambda^2 - \lambda^3), \\ \Omega_4 &= H_{12}^2(\lambda^3 - \lambda^2) + H_{23}^2(\lambda^1 - \lambda^2), \end{aligned}$$

which can be verified to satisfy the quadratic identity

$$(\lambda^2 - \lambda^3)\Omega_1^2 + (\lambda^2 - \lambda^1)\Omega_2^2 + \Omega_3\Omega_4 = 0. \quad (37)$$

In terms of these quantities, the equation (36) can be rewritten as follows:

$$(\Omega_3 - 4(\lambda^1 - \lambda^3)\Omega_4)^2 + 16(\lambda^1 - \lambda^2)(\lambda^1 - \lambda^3)\Omega_2^2 = 0, \quad (38)$$

or, equivalently,

$$(\Omega_3 + 4(\lambda^1 - \lambda^3)\Omega_4)^2 + 16(\lambda^1 - \lambda^3)(\lambda^2 - \lambda^3)\Omega_1^2 = 0; \quad (39)$$

one has to use the identity (37) to verify the equivalence of (38) and (39). Let us assume that $\lambda^1 < \lambda^2 < \lambda^3$. Since we are interested in real-valued solutions, the equation (38) implies

$$\Omega_2 = 0, \quad \Omega_3 = 4(\lambda^1 - \lambda^3)\Omega_4; \quad (40)$$

(one should use (39) if $\lambda^2 < \lambda^1 < \lambda^3$). In this case the identity (37) takes the form

$$(\lambda^2 - \lambda^3)\Omega_1^2 + 4(\lambda^1 - \lambda^3)\Omega_4^2 = 0,$$

so that

$$\Omega_1 = 0, \quad \Omega_4 = 0.$$

These conditions lead to potentials of the form

$$H = u_2(\gamma u_1 + \delta u_3) + f(\gamma u_1 + \delta u_3);$$

here the constants γ and δ satisfy the relation $(\lambda^2 - \lambda^1)\delta^2 + (\lambda^2 - \lambda^3)\gamma^2 = 0$, and f is an arbitrary function of the indicated argument. This ansatz, however, implies the reducibility of the dispersion relation as discussed in [12]. Thus, we are left with the second branch $J = 0$,

in which case the rank of the system drops to 5, and we end up with the equations (32). This finishes the proof of Theorem 2.

The main result of this Section is a complete list of integrable potentials $H(u_1, u_2, u_3)$ which come from a detailed analysis of the equations (32). The classification will be performed up to the following equivalence transformations, which constitute a group of point symmetries of the relations (32).

Equivalence transformations:

transformations of the variables u_i : $u_i \rightarrow au_i + b_i$;

transformations of the potential H :

$$H \rightarrow \alpha H + \beta \sum u_i^2/2 + \gamma \sum \lambda^i u_i^2/2 + \mu_i u_i + \delta,$$

the latter corresponding to $Y \rightarrow \alpha Y + \beta T + \gamma X$ in the equations (31). Moreover, relations (32) are invariant under arbitrary permutations of indices. Finally, we will not be interested in the potentials which are either quadratic in u_i and generate linear systems (31), or separable potentials, e.g., $H = f(u_1) + g(u_2, u_3)$, giving rise to reducible systems.

Theorem 3 *The ‘generic’ solution of the equations (32) is given by the formula*

$$H = - \sum_{j \neq i} \frac{\lambda^i - \lambda^j}{6a_i^2 a_j^2} V(a_i u_i, a_j u_j) \quad (41)$$

where

$$V(x, y) = Z(x - y) + \epsilon Z(x - \epsilon y) + \epsilon^2 Z(x - \epsilon^2 y); \quad (42)$$

here $\epsilon = e^{2\pi i/3}$ and $Z'' = \zeta$ where ζ is the Weierstrass zeta-function: $\zeta' = -\wp$, $(\wp')^2 = 4\wp^3 - g_3$. Degenerations of this solution correspond to

$$H = - \sum_{j \neq i} \frac{\lambda^i - \lambda^j}{3a_i^2 a_j^2} \tilde{V}(a_i u_i, a_j u_j) \quad (43)$$

where

$$\tilde{V}(x, y) = (x - y) \ln(x - y) + \epsilon(x - \epsilon y) \ln(x - \epsilon y) + \epsilon^2(x - \epsilon^2 y) \ln(x - \epsilon^2 y),$$

and

$$H = - \sum_{j \neq i} \frac{\lambda^i - \lambda^j}{a_i^2 a_j^2} (a_i u_i - a_j u_j) \ln(a_i u_i - a_j u_j), \quad (44)$$

respectively. Further examples include

$$H = \frac{\lambda^1 - \lambda^2}{a_2^2} u_1^2 \zeta(a_2 u_2) + \frac{\lambda^1 - \lambda^3}{a_3^2} u_1^2 \zeta(a_3 u_3) - \frac{2\lambda^2 - \lambda^3}{3} \frac{\lambda^2 - \lambda^3}{a_2^2 a_3^2} V(a_2 u_2, a_3 u_3) \quad (45)$$

where V is the same as in (42). This potential possesses a degeneration

$$H = (\lambda^1 - \lambda^2) u_1^2 u_2^2 + (\lambda^2 - \lambda^3) \zeta(u_3 + c) u_2^2 - (\lambda^3 - \lambda^1) \zeta(u_3) u_1^2, \quad (46)$$

here $\zeta' = -\wp$, $(\wp')^2 = 4\wp^3 + 4$, and c is the zero of \wp such that $\wp(c) = 0$, $\wp'(c) = 2$. It possesses a further quartic degeneration,

$$H = (\lambda^1 - \lambda^2) u_1^2 u_2^2 + (\lambda^2 - \lambda^3) u_2^2 u_3^2 + (\lambda^3 - \lambda^1) u_3^2 u_1^2. \quad (47)$$

We have also found the following (non-symmetric) examples:

$$H = (pu_1 + qu_3) \ln(pu_1 + qu_3) - \frac{1}{6}p(\lambda^1 - \lambda^2)(\lambda^1 - \lambda^3)u_1^3 - \frac{1}{6}q(\lambda^3 - \lambda^1)(\lambda^3 - \lambda^2)u_3^3 + p(\lambda^3 - \lambda^2)u_2u_3 + q(\lambda^2 - \lambda^1)u_1u_2, \quad (48)$$

$$H = (\lambda^2 - \lambda^1)u_2u_1^2 + (\lambda^2 - \lambda^3)u_2u_3^2 + \frac{1}{10}(\lambda^2 - \lambda^3)(\lambda^3 - \lambda^1)u_3^5 + \frac{u_1^2}{u_3}, \quad (49)$$

and

$$H = (\lambda^2 - \lambda^1)u_2u_1^2 + (\lambda^2 - \lambda^3)u_2u_3^2 + \frac{p}{15q^2}(\lambda^2 - \lambda^1)(\lambda^1 - \lambda^3)u_1^5 + \frac{q}{15p^2}(\lambda^2 - \lambda^3)(\lambda^3 - \lambda^1)u_3^5 + u_3G\left(\frac{u_1}{u_3}\right), \quad (50)$$

where

$$G(x) = (px + q) \log(px + q) + \epsilon(px + \epsilon q) \log(px + \epsilon q) + \epsilon^2(px + \epsilon^2 q) \log(px + \epsilon^2 q).$$

Up to the equivalence transformations, the above examples exhaust the list of integrable potentials. We claim that all examples appearing in the classification possess dispersionless Lax pairs and an infinity of hydrodynamic reductions (this will be demonstrated in Sect. 5.1–5.2).

Proof of Theorem 3:

We can assume that all mixed partial derivatives H_{ij} are non-zero. It follows from (32) that

$$\frac{\partial^3}{\partial u_1 \partial u_2 \partial u_3} \left(\frac{H_{12}H_{13}}{H_{23}} \right) = \frac{\partial^3}{\partial u_1 \partial u_2 \partial u_3} \left(\frac{H_{12}H_{23}}{H_{13}} \right) = \frac{\partial^3}{\partial u_1 \partial u_2 \partial u_3} \left(\frac{H_{13}H_{23}}{H_{12}} \right) = 0. \quad (51)$$

The further analysis depends on the value of the expression

$$\frac{\partial H_{12}}{\partial u_2} \frac{\partial H_{23}}{\partial u_3} \frac{\partial H_{13}}{\partial u_1} + \frac{\partial H_{12}}{\partial u_1} \frac{\partial H_{23}}{\partial u_2} \frac{\partial H_{13}}{\partial u_3}, \quad (52)$$

which appears as a denominator when solving the equations (51).

Case I. The expression (52) is nonzero. In this case equations (51) are equivalent to

$$\begin{aligned} F_{u_1, u_2} &= \frac{Ku_1}{K} F_{u_2} + \frac{Gu_2}{G} F_{u_1} - \frac{Ku_1}{K} \frac{Gu_2}{G} F, \\ G_{u_2, u_3} &= \frac{Fu_2}{F} G_{u_3} + \frac{Ku_3}{K} G_{u_2} - \frac{Fu_2}{F} \frac{Ku_3}{K} G, \\ K_{u_3, u_1} &= \frac{Gu_3}{G} K_{u_1} + \frac{Fu_1}{F} K_{u_3} - \frac{Gu_3}{G} \frac{Fu_1}{F} K, \end{aligned}$$

where $F = 1/H_{12}$, $G = 1/H_{23}$, $K = 1/H_{13}$. Keeping in mind that $F_3 = G_1 = K_2 = 0$, we can rewrite these equations in the form

$$\left(\frac{F}{GK} \right)_{12} = 0, \quad \left(\frac{G}{FK} \right)_{23} = 0, \quad \left(\frac{K}{FG} \right)_{13} = 0, \quad F_3 = G_1 = K_2 = 0. \quad (53)$$

The system (53) possesses obvious symmetries

$$\begin{aligned} F &\rightarrow f_1(u_1)f_2(u_2)F, & G &\rightarrow f_2(u_2)f_3(u_3)G, & K &\rightarrow f_1(u_1)f_3(u_3)K, \\ u_1 &\rightarrow g_1(u_1), & u_2 &\rightarrow g_2(u_2), & u_3 &\rightarrow g_3(u_3); \end{aligned} \quad (54)$$

here f_i and g_i are six arbitrary functions of the indicated arguments. As a first step, we introduce the new variables

$$p = \frac{K_1}{K} - \frac{F_1}{F}, \quad q = \frac{F_2}{F} - \frac{G_2}{G}, \quad r = \frac{G_3}{G} - \frac{K_3}{K},$$

which are nothing but the invariants of the first ‘half’ of the symmetry group (54). In terms of p, q, r , the equations (53) take the form

$$q_1 = -p_2 = pq, \quad r_2 = -q_3 = qr, \quad p_3 = -r_1 = pr. \quad (55)$$

This system is straightforward to solve: assuming $p \neq 0$ (the case when $p = q = r = 0$ will be a particular case of the general formula), one has $q = -p_2/p$, $r = p_3/p$, along with the three commuting Monge-Ampère equations for p ,

$$p_{23} = 0, \quad (\ln p)_{12} = p_2, \quad (\ln p)_{13} = -p_3. \quad (56)$$

The integration of the last two equations implies $p_1/p = p + 2\varphi(u_1, u_3)$ and $p_1/p = -p + 2\psi(u_1, u_2)$, respectively. Thus, $p = \psi(u_1, u_2) - \varphi(u_1, u_3)$, and the substitution back into the above equations gives $\psi_1(u_1, u_2) - \psi^2(u_1, u_2) = \varphi_1(u_1, u_3) - \varphi^2(u_1, u_3)$. The separation of variables provides a pair of Riccati equations, $\psi_1 = \psi^2 + V(u_1)$ and $\varphi_1 = \varphi^2 + V(u_1)$. Thus, $\psi = -[\ln v]_1$, $\varphi = -[\ln \tilde{v}]_1$, where v and \tilde{v} are two arbitrary solutions of the linear ODE $v_{11} + V(u_1)v = 0$. Therefore, we can represent ψ and φ in the form

$$\psi = -[\ln(q_2(u_2)p_1(u_1) - p_2(u_2)q_1(u_1))]_1, \quad \varphi = -[\ln(q_3(u_3)p_1(u_1) - q_1(u_1)p_2(u_2))]_1,$$

where $p_1(u_1)$ and $q_1(u_1)$ form a basis of solutions of the linear ODE. Introducing $w_i(u_i) = q_i(u_i)/p_i(u_i)$, one obtains the final formula

$$p = \psi - \varphi = \frac{w'_1(w_3 - w_2)}{(w_2 - w_1)(w_3 - w_1)},$$

leading to

$$q = \frac{w'_2(w_1 - w_3)}{(w_2 - w_1)(w_2 - w_3)}, \quad r = \frac{w'_3(w_2 - w_1)}{(w_3 - w_1)(w_3 - w_2)}.$$

Here $w_i(u_i)$ can be viewed as three arbitrary functions of one argument. The corresponding F, G, H are given by

$$F = s_1 s_2 (w_1 - w_2), \quad G = s_2 s_3 (w_2 - w_3), \quad K = s_1 s_3 (w_3 - w_1),$$

where $s_i(u_i)$ are three extra arbitrary functions. This implies the ansatz

$$H_{12} = \frac{P(u_1)Q(u_2)}{f(u_1) - g(u_2)}, \quad H_{23} = \frac{Q(u_2)R(u_3)}{g(u_2) - h(u_3)}, \quad H_{13} = \frac{P(u_1)R(u_3)}{h(u_3) - f(u_1)}, \quad (57)$$

(with the obvious identification $w_1(u_1) \rightarrow f(u_1)$, $s_1(u_1) \rightarrow 1/P(u_1)$, etc). We have to consider different cases depending on how many functions among f, g, h are constant.

Subcase 1: $f' = g' = h' = 0$. Without any loss of generality one can assume

$$H_{12} = P(u_1)Q(u_2), \quad H_{23} = Q(u_2)R(u_3), \quad H_{13} = P(u_1)R(u_3).$$

Substituting this ansatz into (32) one can show that the functions P, Q, R must necessarily be linear. Up to the equivalence transformations, this leads to a unique quartic potential (47):

$$H = (\lambda^1 - \lambda^2)u_1^2 u_2^2 + (\lambda^2 - \lambda^3)u_2^2 u_3^2 + (\lambda^3 - \lambda^1)u_3^2 u_1^2.$$

Subcase 2: $f' = g' = 0$. Without any loss of generality one can assume the following ansatz:

$$H_{12} = P(u_1)Q(u_2), \quad H_{23} = Q(u_2)h_1(u_3), \quad H_{13} = P(u_1)h_2(u_3). \quad (58)$$

The substitution into (32) implies that P and Q must necessarily be linear. Up to the equivalence transformations, this results in the potential

$$H = (\lambda^1 - \lambda^2)u_1^2u_2^2 + (\lambda^2 - \lambda^3)b(u_3)u_2^2 + (\lambda^3 - \lambda^1)a(u_3)u_1^2,$$

where the functions a and b satisfy the ODEs

$$a'' = 4\frac{a'}{b} - 2, \quad b'' = 4\frac{b'}{a} - 2, \quad a'b' = 2(a + b).$$

The special case $a = b = u_3^2$ brings us back to the quartic potential from the previous subcase. The generic solution of these ODEs takes the form $a(u_3) = -\zeta(u_3)$, $b(u_3) = \zeta(u_3 + c)$ where ζ is the Weierstrass ζ -function, $\zeta' = -\wp$, $(\wp')^2 = 4\wp^3 + 4$, and c is the zero of \wp such that $\wp(c) = 0$, $\wp'(c) = 2$. This is the case (46).

Subcase 3: $f' = 0$. The analysis of this case leads to the ansatz

$$H = (\lambda^1 - \lambda^2)u_1^2a(u_2) + (\lambda^3 - \lambda^1)u_1^2b(u_3) + h(u_2, u_3)$$

where

$$a(u_2) = \frac{1}{a_2^2}\zeta(a_2u_2), \quad b(u_3) = -\frac{1}{a_3^2}\zeta(a_3u_3),$$

(here a_2, a_3 are arbitrary constants), and the second order derivatives of $h(u_2, u_3)$ are given by

$$\begin{aligned} H_{23} &= 4\frac{\lambda^2 - \lambda^3}{a_2a_3} \frac{\wp(a_2u_2)\wp(a_3u_3)}{\wp'(a_2u_2) - \wp'(a_3u_3)}, \\ H_{22} &= 4\frac{\lambda^2 - \lambda^3}{a_3^2} \left(\frac{1}{2}\zeta(a_3u_3) - \frac{\wp^2(a_3u_3)}{\wp'(a_2u_2) - \wp'(a_3u_3)} \right), \\ H_{33} &= 4\frac{\lambda^3 - \lambda^2}{a_2^2} \left(\frac{1}{2}\zeta(a_2u_2) - \frac{\wp^2(a_2u_2)}{\wp'(a_3u_3) - \wp'(a_2u_2)} \right). \end{aligned}$$

This is the case (45).

Generic subcase: $f'(x)g'(x)h'(x) \neq 0$. From (57) and (32) we find all third order derivatives of H . The compatibility conditions $\partial_i H_{jji} = \partial_j H_{iij}$ give rise to six functional-differential equations for the functions f, g, h, P, Q, R . It follows from (32) that

$$\begin{aligned} \partial_{u_1} \left(R_1 + (\lambda^1 - \lambda^3)H_{22} + (\lambda^2 - \lambda^1)H_{33} \right) &= 0, \\ \partial_{u_2} \left(R_2 + (\lambda^2 - \lambda^1)H_{33} + (\lambda^3 - \lambda^2)H_{11} \right) &= 0, \\ \partial_{u_3} \left(R_3 + (\lambda^3 - \lambda^2)H_{11} + (\lambda^1 - \lambda^3)H_{22} \right) &= 0. \end{aligned} \quad (59)$$

These give us three more equations for f, g, h, P, Q, R , so that we have nine equations altogether. Substituting the values of the third order derivatives of H into the first equation (59), taking the numerator and dividing by the common factor $P(u_1)^2Q(u_2)^2R(u_3)$, we get a fourth degree

polynomial in f, g, h, P, Q, R , and first order derivatives thereof. Applying to this polynomial the differential operator

$$\frac{1}{f'(u_1)g'(u_2)}\partial_1\partial_2\frac{1}{f'(u_1)g'(u_2)h'(u_3)}\partial_1\partial_2\partial_3,$$

we arrive at a separation of variables,

$$\frac{(\lambda^2 - \lambda^3)(P'''(u_1)f'(u_1) - P''(u_1)f''(u_1))}{f'(u_1)^3} = \frac{(\lambda^3 - \lambda^1)(Q'''(u_2)g'(u_2) - Q''(u_2)g''(u_2))}{g'(u_2)^3} = 2c.$$

Integrating twice, we obtain

$$(\lambda^2 - \lambda^3)P' = cf^2 + a_1f + b_1, \quad (\lambda^3 - \lambda^1)Q' = cg^2 + a_2g + b_2.$$

Analogously,

$$(\lambda^1 - \lambda^2)R' = ch^2 + a_3g + b_3.$$

Using these relations we eliminate all derivatives of P, Q and R from our nine equations. As a result, we obtain a linear system of nine equations for the three unknowns P, Q, R . This system is consistent (that is, the rank of the extended matrix is ≤ 3) if and only if $a_1 = a_2 = a_3 = a$, $b_1 = b_2 = b_3 = b$, and

$$\begin{aligned} & 4(ch^2 + bh + a)h'^2f'' - 4(cf^2 + bf + a)f'^2h'' + \\ & (f - h)\left(2c(f^2 + fh + h^2) + 3b(f + h) + 6a\right)f''h'' = 0, \\ & 4(cf^2 + bf + a)f'^2g'' - 4(cg^2 + bg + a)g'^2f'' + \\ & (g - f)\left(2c(g^2 + gf + f^2) + 3b(g + f) + 6a\right)g''f'' = 0, \\ & 4(cg^2 + bg + a)g'^2h'' - 4(ch^2 + bh + a)h'^2g'' + \\ & (h - g)\left(2c(h^2 + hg + g^2) + 3b(h + g) + 6a\right)h''g'' = 0. \end{aligned} \tag{60}$$

Hence, we have either $c = b = a = 0$ or $f'' = g'' = h'' = 0$, otherwise $f''g''h'' \neq 0$. If $c = b = a = 0$ then the linear system for P, Q, R becomes homogeneous. Its rank equals two if and only if $f'' = g'' = h'' = 0$. In this case

$$Pf'(\lambda^3 - \lambda^2) = Qg'(\lambda^1 - \lambda^3) = Rh'(\lambda^2 - \lambda^1) = \text{const}. \tag{61}$$

If $f'' = g'' = h'' = 0$ then the rank of the system also equals two. The requirement that the rank of the extended matrix equals two as well leads to $c = b = a = 0$. Thus, this case reduces to the previous one.

Suppose now that $f''g''h'' \neq 0$. Solving the linear system for P, Q, R we get

$$P = \frac{2(cf^2 + bf + a)f'}{(\lambda^2 - \lambda^3)f''}, \quad Q = \frac{2(cg^2 + bg + a)g'}{(\lambda^3 - \lambda^1)g''}, \quad R = \frac{2(ch^2 + bh + a)h'}{(\lambda^1 - \lambda^2)h''}.$$

Separating the variables in (60) we ultimately obtain

$$f'^3 = c_1S^2(f), \quad g'^3 = c_2S^2(g), \quad h'^3 = c_3S^2(h), \tag{62}$$

and

$$P = \frac{(\lambda^1 - \lambda^2)(\lambda^1 - \lambda^3)S(f)}{2f'}, \quad Q = \frac{(\lambda^2 - \lambda^1)(\lambda^2 - \lambda^3)S(g)}{2g'}, \quad R = \frac{(\lambda^3 - \lambda^1)(\lambda^3 - \lambda^2)S(h)}{2h'},$$

where $S(x)$ is a polynomial of degree ≤ 3 , and c_i are arbitrary constants (the polynomial $S(z)$ can be recovered from $(\lambda^1 - \lambda^2)(\lambda^1 - \lambda^3)(\lambda^2 - \lambda^3)S' = 6(cz^2 + bz + a)$). Notice that the case (61) is a particular case of the above with $S = \text{const}$.

We point out that the right hand sides of (57) possess the following $SL(2, R)$ -invariance,

$$f \rightarrow \frac{\alpha f + \beta}{\gamma f + \delta}, \quad g \rightarrow \frac{\alpha g + \beta}{\gamma g + \delta}, \quad h \rightarrow \frac{\alpha h + \beta}{\gamma h + \delta}, \quad P \rightarrow \frac{P}{\gamma f + \delta}, \quad Q \rightarrow \frac{Q}{\gamma g + \delta}, \quad R \rightarrow \frac{R}{\gamma h + \delta},$$

which can be used to bring the polynomial S to a canonical form. There are three cases to consider.

Three distinct roots: in this case one can reduce S to a quadratic, $S(x) = x^2 + g_3$, so that the ODEs (62) imply $f = \wp'(a_1 u_1)$, $g = \wp'(a_2 u_2)$, $h = \wp'(a_3 u_3)$ where $27a_i^3 = 2c_i$ and \wp is the Weierstrass \wp -function: $(\wp')^2 = 4\wp^3 - g_3$. Up to a constant multiple, this leads to

$$H_{ij} = \frac{\lambda^i - \lambda^j}{a_i a_j} \frac{\wp(a_i u_i) \wp(a_j u_j)}{\wp'(a_i u_i) - \wp'(a_j u_j)}, \quad H_{ii} = \sum_{j \neq i} \frac{\lambda^i - \lambda^j}{a_j^2} \left(\frac{1}{2} \zeta(a_j u_j) - \frac{\wp^2(a_j u_j)}{\wp'(a_i u_i) - \wp'(a_j u_j)} \right).$$

The corresponding potential $H(\mathbf{u})$ is given by (41).

Double root: in this case one can assume $S(x) = x$, so that the ODEs (62) imply $f = (a_1 u_1)^3$, $g = (a_2 u_2)^3$, $h = (a_3 u_3)^3$, here $27a_i^3 = c_i$. Up to a constant multiple, this leads to

$$H_{ij} = \frac{(\lambda^i - \lambda^j) u_i u_j}{(a_i u_i)^3 - (a_j u_j)^3}, \quad H_{ii} = - \sum_{j \neq i} \frac{(\lambda^i - \lambda^j) u_j^2}{(a_i u_i)^3 - (a_j u_j)^3}.$$

The corresponding potential $H(\mathbf{u})$ is given by (43).

Triple root: in this case S can be reduced to $S = 1$, so that the ODEs (62) imply $f = a_1 u_1$, $g = a_2 u_2$, $h = a_3 u_3$, here $a_i^3 = c_i$. Up to a constant multiple, this leads to

$$H_{ij} = \frac{\lambda^i - \lambda^j}{a_i a_j (a_i u_i - a_j u_j)}, \quad H_{ii} = - \sum_{j \neq i} \frac{\lambda^i - \lambda^j}{a_j^2 (a_i u_i - a_j u_j)}, \quad (63)$$

and the corresponding potential $H(\mathbf{u})$ is given by (44). Notice, however, that for this potential the expression (52) equals zero. Formally, it should be considered as an example from the Case II below.

Case II. This is the case when the expression (52) equals zero, although both terms in (52) are nonzero:

$$H_{122} H_{233} H_{113} = -H_{112} H_{223} H_{133} \neq 0; \quad (64)$$

an integrable example from this class is provided by

$$H_{ij} = \frac{\lambda^i - \lambda^j}{a_i a_j (a_i u_i - a_j u_j)}; \quad (65)$$

it appears in the 'triple root' case above. A detailed analysis below shows that this case possesses no other non-trivial solutions. Rewriting (64) in the form

$$\frac{H_{122}}{H_{112}} \frac{H_{233}}{H_{223}} \frac{H_{113}}{H_{133}} = -1$$

one can set

$$\frac{H_{122}}{H_{112}} = -\frac{l(u^1)}{m(u^2)}, \quad \frac{H_{233}}{H_{223}} = -\frac{m(u^2)}{n(u^3)}, \quad \frac{H_{113}}{H_{133}} = -\frac{n(u^3)}{l(u^1)}.$$

Thus,

$$H_{12} = \frac{1}{P(x)}, \quad H_{23} = \frac{1}{Q(y)}, \quad H_{13} = \frac{1}{R(z)}, \quad (66)$$

where $x = \alpha(u_1) - \beta(u_2)$, $y = \beta(u_2) - \gamma(u_3)$ and $z = -x - y$ for some functions α, β, γ such that $\alpha' = 1/l$, $\beta' = 1/m$, $\gamma' = 1/n$. Substituting (66) into (32) and integrating once, one gets

$$\begin{aligned} H_{11} &= \frac{(\lambda^1 - \lambda^2)P^2 + (\lambda^3 - \lambda^1)R^2}{(\lambda^2 - \lambda^3)PQR} + \mu(u_2, u_3), \\ H_{22} &= \frac{(\lambda^1 - \lambda^2)P^2 + (\lambda^2 - \lambda^3)Q^2}{(\lambda^3 - \lambda^1)PQR} + \nu(u_1, u_3), \\ H_{33} &= \frac{(\lambda^3 - \lambda^2)Q^2 + (\lambda^3 - \lambda^1)R^2}{(\lambda^1 - \lambda^2)PQR} + \eta(u_1, u_2). \end{aligned} \quad (67)$$

Expressing six partial derivatives of the functions $\mu(u_2, u_3)$, $\nu(u_1, u_3)$, $\eta(u_1, u_2)$ from the six compatibility conditions $\partial_j H_{ii} = \partial_i H_{ij}$, and substituting them into the equations $\partial_1 J = 0$ and $\partial_2 J = 0$, we obtain

$$\begin{aligned} w_1 \alpha'(u_1) + w_2 \gamma'(u_3) &= 0, & w_3 \beta'(u_2) + w_4 \gamma'(u_3) &= 0, \\ \partial_2(w_1) \alpha'(u_1) + \partial_2(w_2) \gamma'(u_3) &= 0, & \partial_1(w_3) \beta'(u_2) + \partial_1(w_4) \gamma'(u_3) &= 0, \end{aligned} \quad (68)$$

where

$$\begin{aligned} w_1 &= (\lambda^2 - \lambda^3)Q(RP'Q' + QP'R' - PQ'R'), & w_2 &= (\lambda^2 - \lambda^1)P(RP'Q' - QP'R' + PQ'R'), \\ w_3 &= (\lambda^1 - \lambda^3)R(RP'Q' + QP'R' - PQ'R'), & w_4 &= (\lambda^2 - \lambda^1)P(RP'Q' - QP'R' - PQ'R'). \end{aligned}$$

The equations (68)₂ are obtained from (68)₁ upon differentiation by u_2 and u_1 , respectively. Since α', β' and γ' are nonzero, the system (68) is consistent iff P, Q and R satisfy the following conditions:

$$w_1 \partial_2(w_2) - w_2 \partial_2(w_1) = 0, \quad w_3 \partial_1(w_4) - w_4 \partial_1(w_3) = 0. \quad (69)$$

Let us observe that the equations (51) (which also hold in this case) can be rewritten as follows:

$$\frac{p'}{p} = \frac{pq}{r} - \frac{rq}{p} + \frac{r'}{r}, \quad \frac{q'}{q} = \frac{pq}{r} - \frac{rp}{q} + \frac{r'}{r}, \quad \frac{p'}{p} = \frac{rp}{q} - \frac{rq}{p} + \frac{q'}{q}; \quad (70)$$

here we use the notation $p = P'/P$, $q = Q'/Q$, $r = R'/R$, and prime denotes the derivative of functions with respect to their arguments. Note that only two of the above equations are independent. Differentiating, for instance, the first two equations in (70) by x and y and eliminating r and r' , one ends up at the following relations involving p and q :

$$\begin{aligned} \left[\left(\frac{p'}{p} \right)' + \left(\frac{q'}{q} \right)' \right] (p^2 - q^2) &= \left[\left(\frac{p'}{p} \right)^2 - \left(\frac{q'}{q} \right)^2 \right] (p^2 + q^2), \\ \left[\left(\frac{p'}{p} \right)^2 - \left(\frac{q'}{q} \right)^2 \right] &= 2(p^2 - q^2) - \left[\left(\frac{p'}{p} \right)' - \left(\frac{q'}{q} \right)' \right]. \end{aligned} \quad (71)$$

Similarly, one can eliminate p and p' obtaining the analog of (71) for q and r . All these relations imply

$$p^2 \left(\frac{p'}{p} \right)' - p^4 = q^2 \left(\frac{q'}{q} \right)' - q^4 = r^2 \left(\frac{r'}{r} \right)' - r^4. \quad (72)$$

Thus, p , q and r must satisfy an ODE of the form

$$f f'' - (f')^2 - f^4 + k = 0$$

where $f = f(\zeta)$ and k is an arbitrary constant. If $k = 0$ then

$$f = \frac{\nu}{\sin \nu \zeta} \quad \text{or} \quad f = \frac{1}{\zeta}$$

where ν is an arbitrary constant. Note that the second solution is a limit of the first as $\nu \rightarrow 0$. If $k \neq 0$ we have

$$f = \frac{k^{\frac{1}{4}}}{\tanh k^{\frac{1}{4}} \zeta} \quad \text{or} \quad f = \frac{1}{\nu} \operatorname{sn} \left(\nu \sqrt{k} \zeta; \frac{1}{\nu^4 k} \right)$$

where sn is the Jacobi elliptic sine function: $(\operatorname{sn}')^2 = (1 - \operatorname{sn}^2)(1 - 1/(\nu^4 k) \operatorname{sn}^2)$. Using $p = P'/P$ and integrating once, we obtain

$$P = c_3 x, \quad P = c_3 \frac{\tan \nu x}{\nu}, \quad P = c_3 \frac{\sinh k^{\frac{1}{4}} x}{k^{\frac{1}{4}}}, \quad \text{or} \quad P = c_3 \operatorname{dn} - c_3 \frac{cn}{\nu^2 \sqrt{k}}. \quad (73)$$

Analogously, Q and R can be obtained from these formulae by cycling the indices $c_3 \rightarrow c_1 \rightarrow c_2$ and the variables $x \rightarrow y \rightarrow z$. Here cn and dn are the Jacobi elliptic functions $cn(\nu \sqrt{k} x; 1/(\nu^4 k))$ and $dn(\nu \sqrt{k} x; 1/(\nu^4 k))$, and c_1 , c_2 and c_3 are arbitrary constants. It turns out that only linear and trigonometric solutions in (73) satisfy the condition (69). Thus, hyperbolic and elliptic solutions can be dropped. The substitution of the linear solution into one of the equations (68)₁ implies that the functions α , β and γ must be linear. One recovers the solution (63) by setting $c_1 = a_2 a_3 / (\lambda^2 - \lambda^3)$, $c_2 = a_1 a_3 / (\lambda^3 - \lambda^1)$, $c_3 = a_1 a_2 / (\lambda^1 - \lambda^2)$. Finally, the substitution of the trigonometric solution (73) also implies that α , β and γ must be linear, however, the compatibility conditions for the systems (66) and (67) are not satisfied.

Case III. This is the case when both terms in (52) equal zero separately:

$$\frac{\partial H_{12}}{\partial u_2} \frac{\partial H_{23}}{\partial u_3} \frac{\partial H_{13}}{\partial u_1} = \frac{\partial H_{12}}{\partial u_1} \frac{\partial H_{23}}{\partial u_2} \frac{\partial H_{13}}{\partial u_3} = 0. \quad (74)$$

Up to permutations of indices, we have to consider the following three subcases.

Subcase 1: $H_{12} = \operatorname{const} \neq 0$. It follows from (32) that

$$(\lambda^1 - \lambda^2) H_{13} H_{233} = (\lambda^3 - \lambda^1) H_{12} H_{223}, \quad (\lambda^2 - \lambda^1) H_{23} H_{133} = (\lambda^3 - \lambda^2) H_{12} H_{113}.$$

Differentiating the first equation with respect to u_1 we obtain $H_{113} H_{233} = 0$. If $H_{233} = 0$ then the first equation implies $H_{23} = \operatorname{const}$. Otherwise, it follows from the second equation that $H_{13} = \operatorname{const}$. Without any loss of generality we assume that $H_{23} = \operatorname{const} \neq 0$. Setting $H_{12} = q(\lambda^2 - \lambda^1)$, $H_{23} = p(\lambda^3 - \lambda^2)$, $p, q = \operatorname{const}$, and substituting into (32) one arrives, up to the equivalence transformations, at the following potential H :

$$H(u_1, u_2, u_3) = (p u_1 + q u_3) \ln(p u_1 + q u_3) - \frac{1}{6} p (\lambda^1 - \lambda^2) (\lambda^1 - \lambda^3) u_1^3 -$$

$$-\frac{1}{6}q(\lambda^3 - \lambda^1)(\lambda^3 - \lambda^2)u_3^3 + p(\lambda^3 - \lambda^2)u_2u_3 + q(\lambda^2 - \lambda^1)u_1u_2.$$

Subcase 2: $H_{12} = f(u_1)$, $H_{23} = g(u_3)$. One can prove that in this case

$$H(u_1, u_2, u_3) = \alpha u_1^2 u_2 + \beta u_2 u_3^2 + \gamma u_1^5 + \delta u_3^5 + u_3 G\left(\frac{u_1}{u_3}\right)$$

for some constants $\alpha, \beta, \gamma, \delta$. The function G has to satisfy an equation of the form

$$G''(x) = \frac{z_1}{z_2 + z_3 x^3},$$

where z_i are some constants. If $z_3 = 0$ we have $G(x) = x^2$. In this case

$$\alpha = (\lambda^2 - \lambda^1), \quad \beta = (\lambda^2 - \lambda^3), \quad \gamma = 0, \quad \delta = \frac{1}{10}(\lambda^2 - \lambda^3)(\lambda^3 - \lambda^1),$$

which gives (49). The case $z_2 = 0$ is equivalent to the above. Otherwise,

$$G(x) = (px + q) \log(px + q) + \epsilon(px + \epsilon q) \log(px + \epsilon q) + \epsilon^2(px + \epsilon^2 q) \log(px + \epsilon^2 q).$$

In this case

$$\alpha = (\lambda^2 - \lambda^1), \quad \beta = (\lambda^2 - \lambda^3), \quad \gamma = \frac{p}{15q^2}(\lambda^2 - \lambda^1)(\lambda^1 - \lambda^3), \quad \delta = \frac{q}{15p^2}(\lambda^2 - \lambda^3)(\lambda^3 - \lambda^1),$$

which gives (50).

Subcase 3: $H_{12} = f(u_1)$, $H_{13} = g(u_1)$. A direct calculation shows that this case gives no non-trivial examples.

This finishes the proof of Theorem 3.

5.1 Dispersionless Lax pairs

In this section we prove that the diagonalizability conditions (32) imply the existence of the dispersionless Lax pairs (Theorem 4), and explicitly calculate Lax pairs for some of the most ‘symmetric’ examples appearing in the classification list of Theorem 3.

Example 1. Let us consider the quartic potential (47),

$$H = (\lambda^1 - \lambda^2)u_1^2 u_2^2 + (\lambda^2 - \lambda^3)u_2^2 u_3^2 + (\lambda^3 - \lambda^1)u_3^2 u_1^2,$$

which is a three-component generalization of the potential (19) from Theorem 1 (we have verified that this example possesses no natural four-component extensions). The corresponding system (31) has a Lax pair

$$\psi_T = \lambda_1 a_1(\xi) u_1^2 + \lambda_2 a_2(\xi) u_2^2 + \lambda_3 a_3(\xi) u_3^2, \quad \psi_X = -a_1(\xi) u_1^2 - a_2(\xi) u_2^2 - a_3(\xi) u_3^2;$$

here $\xi = \psi_Y$ and the functions $a_i(\xi)$ satisfy the ODEs

$$a_1' = \frac{4a_1}{a_3} + 2, \quad a_2' = \frac{4a_2}{a_1} + 2, \quad a_3' = \frac{4a_3}{a_2} + 2, \quad a_1 a_2 + a_2 a_3 + a_3 a_1 = 0.$$

Equivalently,

$$a_3 = \frac{4a_1}{a_1' - 2}, \quad a_2 = -\frac{4a_1}{a_1' + 2}, \quad 2a_1 a_1'' = 3a_1'^2 - 12.$$

Without any loss of generality one can set

$$a_1 = \wp(\xi), \quad a_2 = \wp(\xi + c), \quad a_3 = \wp(\xi - c)$$

where \wp is the Weierstrass \wp -function, $(\wp')^2 = 4\wp^3 + 4$, and c is the zero of \wp such that $\wp(c) = 0$, $\wp'(c) = 2$.

Example 2. Let us consider the potential (44),

$$H = - \sum_{j \neq i} \frac{\lambda^i - \lambda^j}{a_i^2 a_j^2} (a_i u_i - a_j u_j) \ln(a_i u_i - a_j u_j),$$

which is a three-component generalization of the potential (18) from Theorem 1. The corresponding system (31) possesses the Lax pair

$$\psi_T = - \sum \frac{\lambda^i}{a_i^2} \ln(a_i u_i - \psi_Y), \quad \psi_X = \sum \frac{1}{a_i^2} \ln(a_i u_i - \psi_Y).$$

This Lax pair appeared previously in [21].

Example 3. Let us consider the potential (41),

$$H = - \sum_{j \neq i} \frac{\lambda^i - \lambda^j}{6a_i^2 a_j^2} V(a_i u_i, a_j u_j).$$

One can show that the corresponding system (31) has the Lax pair

$$\psi_T = - \sum \frac{\lambda^i}{a_i^2} f(a_i u_i, \psi_Y), \quad \psi_X = \sum \frac{1}{a_i^2} f(a_i u_i, \psi_Y)$$

where the dependence of $f(u, \xi)$ on its arguments (here $\xi = \psi_Y$) is governed by

$$f_u = \frac{\wp(u)\wp(\xi)}{\wp'(u) - \wp'(\xi)}, \quad f_\xi = \frac{\wp^2(u)}{\wp'(\xi) - \wp'(u)} - \frac{1}{2}\zeta(u).$$

Explicitly, one has

$$f(u, \xi) = \frac{1}{6} \ln \sigma(u - \xi) + \frac{\epsilon}{6} \ln \sigma(\epsilon u - \xi) + \frac{\epsilon^2}{6} \ln \sigma(\epsilon^2 u - \xi),$$

here σ is the Weierstrass sigma-function: $\sigma'/\sigma = \zeta$. In a different parametrization, this Lax pair appeared in [21] in the classification of dispersionless Lax pairs with movable singularities. We point out that both examples 2 and 3 generalize to n -component case in a straightforward way (allowing the summation to go from 1 to n).

In fact, the following general result holds:

Theorem 4 *Any system (31) satisfying the diagonalizability conditions (32) possesses a dispersionless Lax pair.*

Proof:

We look for a Lax pair in the form (11). The compatibility condition $\psi_{tx} = \psi_{xt}$ results in the following set of relations:

$$f_1 = \lambda^1 g_1, \quad f_2 = \lambda^2 g_2, \quad f_3 = \lambda^3 g_3, \quad (75)$$

and

$$\begin{aligned} f_p g_1 &= H_{11} g_1 + H_{21} g_2 + H_{31} g_3 + g_p f_1, \\ f_p g_2 &= H_{12} g_1 + H_{22} g_2 + H_{32} g_3 + g_p f_2, \\ f_p g_3 &= H_{13} g_1 + H_{23} g_2 + H_{33} g_3 + g_p f_3, \end{aligned} \quad (76)$$

where we have set $p = \psi_y$, $f_i = \partial_i f$, and $g_i = \partial_i g$. The relations (75) and (76) are equivalent to (12). Eliminating f_p and g_p from (76), one obtains a single algebraic constraint among the components g_1, g_2, g_3 , which coincides with the left characteristic cone (34). The expressions for f_p and g_p obtained from the first two equations (76) take the form

$$\begin{aligned} f_p &= \frac{(H_{11} g_1 + H_{12} g_2 + H_{13} g_3) \lambda^2 g_2 - (H_{12} g_1 + H_{22} g_2 + H_{23} g_3) \lambda^1 g_1}{g_1 g_2 (\lambda^2 - \lambda^1)}, \\ g_p &= \frac{(H_{11} g_1 + H_{12} g_2 + H_{13} g_3) g_2 - (H_{12} g_1 + H_{22} g_2 + H_{23} g_3) g_1}{g_1 g_2 (\lambda^2 - \lambda^1)}. \end{aligned} \quad (77)$$

Using the compatibility conditions $f_{ij} = f_{ji}$ and $f_{ip} = f_{pi}$, we can express all second order derivatives of g in the form

$$\begin{aligned} g_{12} &= g_{13} = g_{23} = 0, \\ g_{11} &= \frac{g_1 (H_{111} g_1 + H_{112} g_2 + H_{113} g_3)}{H_{12} g_2 + H_{13} g_3}, \\ g_{22} &= \frac{g_2 (H_{221} g_1 + H_{222} g_2 + H_{223} g_3)}{H_{12} g_1 + H_{23} g_3}, \\ g_{33} &= \frac{g_1 (H_{123} g_1 + H_{223} g_2 + H_{332} g_3) (\lambda^3 - \lambda^1)}{H_{13} g_2 (\lambda^3 - \lambda^2) + H_{23} g_1 (\lambda^1 - \lambda^3)} \\ &\quad + \frac{g_2 (H_{113} g_1 + H_{123} g_2 + H_{331} g_3) (\lambda^2 - \lambda^3)}{H_{13} g_2 (\lambda^3 - \lambda^2) + H_{23} g_1 (\lambda^1 - \lambda^3)}. \end{aligned} \quad (78)$$

It was already mentioned that the condition $J = 0$ implies the decomposition of the left characteristic cone (34) into a linear and quadratic factors, see (35). We will assume that g_1, g_2, g_3 lie on the quadratic branch,

$$\Gamma = H_{13} H_{23} (\lambda^1 - \lambda^2) g_1 g_2 + H_{12} H_{23} (\lambda^3 - \lambda^1) g_1 g_3 + H_{12} H_{13} (\lambda^2 - \lambda^3) g_2 g_3 = 0. \quad (79)$$

One can verify that the differential consequences

$$\frac{\partial \Gamma}{\partial u_1} = 0, \quad \frac{\partial \Gamma}{\partial u_2} = 0, \quad \frac{\partial \Gamma}{\partial u_3} = 0, \quad \frac{\partial \Gamma}{\partial p} = 0 \quad (80)$$

hold identically modulo (78), (79) and (32). Finally, using computer algebra, it is straightforward to verify that the consistency conditions for the system (78) are satisfied identically modulo (79) and (32). This completes the proof of theorem 4.

5.2 Hydrodynamic reductions

The aim of this section is to prove that all examples listed in Theorem 3 possess infinitely many n -component hydrodynamic reductions parametrized by n arbitrary functions of a single variable. To do so one has to demonstrate the consistency of the relations (8), (10) where the characteristic speeds ν^i and μ^i satisfy the dispersion relation $\det(\nu I_3 + \mu A + B) = 0$, and $\partial_i \mathbf{u}$ is the right eigenvector of the matrix $\nu^i I_3 + \mu^i A + B$ — see Sect. 2.

Theorem 5 *The diagonalizability conditions (32) are necessary and sufficient for the existence of an infinity of n -component hydrodynamic reductions parametrized by n arbitrary functions of a single variable.*

Proof:

The necessity follows from the general result of [12] which states that, for a quasilinear system (4), the diagonalizability is a necessary condition for the existence of an infinity of hydrodynamic reductions.

The first step to demonstrate the sufficiency is to explicitly parametrize the dispersion curve (9), which we know to be a *rational* curve of degree three (see the Remark after Theorem 2). This can be done as follows. Let us first calculate the singular point ν_0, μ_0 on the dispersion curve. It corresponds to the situation when the rank of the matrix $\nu I_3 + \mu A + B$ drops to one. The associated left eigenvectors constitute a two-dimensional plane given by the first factor in the equation of the left characteristic cone (35). A simple calculation shows that ν_0 and μ_0 can be obtained from the linear system

$$\begin{aligned}\nu_0 + \lambda^1 \mu_0 &= \frac{H_{12}H_{13}}{H_{23}} - H_{11}, \\ \nu_0 + \lambda^2 \mu_0 &= \frac{H_{12}H_{23}}{H_{13}} - H_{22}, \\ \nu_0 + \lambda^3 \mu_0 &= \frac{H_{13}H_{23}}{H_{12}} - H_{33};\end{aligned}$$

notice that these three relations are linearly dependent, indeed, multiplying the first by $\lambda^2 - \lambda^3$, the second by $\lambda^3 - \lambda^1$, the third by $\lambda^1 - \lambda^2$ and adding them together, one obtains $J = 0$, see (32). Next, we parametrize the quadratic branch of the left characteristic cone (35) in the form

$$g_1 = \frac{1}{(\lambda^1 + s)H_{23}}, \quad g_2 = \frac{1}{(\lambda^2 + s)H_{13}}, \quad g_3 = \frac{1}{(\lambda^3 + s)H_{12}},$$

here s is a parameter. The corresponding relation (33) is equivalent to

$$\begin{aligned}\nu + \mu \lambda^1 + H_{11} + \frac{\lambda^1 + s}{\lambda^2 + s} \frac{H_{12}H_{23}}{H_{13}} + \frac{\lambda^1 + s}{\lambda^3 + s} \frac{H_{13}H_{23}}{H_{12}} &= 0, \\ \nu + \mu \lambda^2 + H_{22} + \frac{\lambda^2 + s}{\lambda^1 + s} \frac{H_{12}H_{13}}{H_{23}} + \frac{\lambda^2 + s}{\lambda^3 + s} \frac{H_{13}H_{23}}{H_{12}} &= 0, \\ \nu + \mu \lambda^3 + H_{33} + \frac{\lambda^3 + s}{\lambda^1 + s} \frac{H_{12}H_{13}}{H_{23}} + \frac{\lambda^3 + s}{\lambda^2 + s} \frac{H_{12}H_{23}}{H_{13}} &= 0;\end{aligned}$$

we point out that these three relations are also linearly dependent. Solving them for $\nu(s)$ and $\mu(s)$ one obtains a rational parametrization of the dispersion curve:

$$\begin{aligned}\nu(s) &= \nu_0 - \frac{s}{\lambda^1 + s} \frac{H_{12}H_{13}}{H_{23}} - \frac{s}{\lambda^2 + s} \frac{H_{12}H_{23}}{H_{13}} - \frac{s}{\lambda^3 + s} \frac{H_{13}H_{23}}{H_{12}}, \\ \mu(s) &= \mu_0 - \frac{1}{\lambda^1 + s} \frac{H_{12}H_{13}}{H_{23}} - \frac{1}{\lambda^2 + s} \frac{H_{12}H_{23}}{H_{13}} - \frac{1}{\lambda^3 + s} \frac{H_{13}H_{23}}{H_{12}};\end{aligned}$$

here ν_0 and μ_0 are coordinates of the singular point. Thus, the characteristic speeds $\nu^i(\mathbf{R})$ and $\mu^i(\mathbf{R})$ can be represented in the form

$$\nu^i(\mathbf{R}) = \nu(s^i), \quad \mu^i(\mathbf{R}) = \mu(s^i) \quad (81)$$

where s^i , which are the parameter values of n points on the dispersion curve, are certain functions of the Riemann invariants: $s^i = s^i(\mathbf{R})$. Since in our case the matrix $\nu I_3 + \mu A + B$ is symmetric, the left characteristic cone coincides with the right characteristic cone. Thus, the right eigenvector corresponding to the point ν^i, μ^i on the dispersion curve is

$$\left(\frac{1}{(\lambda^1 + s^i)H_{23}}, \frac{1}{(\lambda^2 + s^i)H_{13}}, \frac{1}{(\lambda^3 + s^i)H_{12}} \right)^t,$$

and the relations (8) take the form

$$\partial_i u_2 = \frac{\lambda^1 + s^i}{\lambda^2 + s^i} \frac{H_{23}}{H_{13}} \partial_i u_1, \quad \partial_i u_3 = \frac{\lambda^1 + s^i}{\lambda^3 + s^i} \frac{H_{23}}{H_{12}} \partial_i u_1. \quad (82)$$

Substituting (81) into the commutativity conditions (10) and using (82) one obtains the relations

$$\partial_j s^i = (\dots) \partial_j u_1 \quad (83)$$

$i \neq j$, where dots denote certain *rational* expression in s^i, s^j whose coefficients depend on the second and third order derivatives of the potential H . For example, in the case of the quartic potential (47) these relations take the form

$$\partial_j s^i = \frac{3(\lambda^1 + s^i)(\lambda^2 + s^i)(\lambda^3 + s^i)(\lambda^1 + s^j)}{(\lambda^1 - \lambda^2)(\lambda^1 - \lambda^3)(s^j - s^i) u_1} \partial_j u_1.$$

By virtue of (82) and (32), the consistency conditions $\partial_j \partial_i u_2 = \partial_i \partial_j u_2$ and $\partial_j \partial_i u_3 = \partial_i \partial_j u_3$ imply one and the same relation

$$\partial_i \partial_j u_1 = (\dots) \partial_i u_1 \partial_j u_1, \quad (84)$$

$i \neq j$, where, again, dots denote a *rational* expression in s^i, s^j whose coefficients depend on the second and third order derivatives of H . In the case (47), we have

$$\partial_i \partial_j u_1 = \frac{Y(s^i, s^j)}{u_1} \partial_i u_1 \partial_j u_1,$$

where

$$Y(\alpha, \beta) = \frac{6\alpha^2\beta^2 + k_1(\alpha^2\beta + \alpha\beta^2) + k_2(\alpha^2 + 4\alpha\beta + \beta^2) + k_3(\alpha + \beta) + k_4}{(\lambda^1 - \lambda^2)(\lambda^1 - \lambda^3)(\alpha - \beta)^2},$$

$$\begin{aligned}
k_1 &= 3(\lambda^2 + \lambda^3 + 2\lambda^1), & k_2 &= (\lambda^1)^2 + 2\lambda^1\lambda^2 + 2\lambda^1\lambda^3 + \lambda^2\lambda^3, \\
k_3 &= 3\lambda^1(\lambda^1\lambda^2 + \lambda^1\lambda^3 + 2\lambda^2\lambda^3), & k_4 &= 6(\lambda^1)^2\lambda^2\lambda^3.
\end{aligned}$$

The relations (83) and (84) constitute the so-called Gibbons-Tsarev-type equations which govern hydrodynamic reductions of the system (31). The last step is to verify their consistency, namely, $\partial_k \partial_j s^i = \partial_j \partial_k s^i$ and $\partial_i \partial_j \partial_k u_1 = \partial_i \partial_k \partial_j u_1$ (without any loss of generality one can set $i = 1$, $j = 2$, $s = 3$). If these consistency conditions are satisfied identically, the system (83), (84) will be in involution, with the general solution depending on $2n$ arbitrary functions of a single variable. Up to reparametrizations $R^i \rightarrow f^i(R^i)$ this gives an infinity of hydrodynamic reductions depending on n arbitrary functions.

We have verified the consistency for all examples appearing in Theorem 3. In fact, rather than considering them case-by-case, one can give a unified proof of the consistency using only the diagonalizability conditions (32). To do so one needs to bring the system (32) into a passive form. It turns out that all higher order partial derivatives of the potential H can be expressed in terms of the second order derivatives H_{ij} and the 4 third order derivatives, say, $H_{122}, H_{113}, H_{223}, H_{233}$. Second order derivatives are constrained by a single algebraic equation $J = 0$, while the values of H and its first order derivatives H_i are arbitrary. This calculation shows that the generic solution of the system (32) should depend on 13 arbitrary constants, which is in full accordance with the results of Section 5. The computation of the expressions (83) and (84), as well as the verification of the consistency conditions have been performed modulo this passive form. This means that all partial derivatives of H except the basic ones were eliminated, and the basic derivatives were considered as independent variables related by a single algebraic equation $J = 0$. An intense computer calculation shows that all compatibility conditions are identities in the basic derivatives.

6 Hamiltonian systems in 3 + 1 dimensions

In this section we establish a number of non-existence results for integrable Hamiltonian systems of hydrodynamic type in 3+1 dimensions. We will begin with a two-component case. According to the results of [19], there exists a unique two-component Hamiltonian operator of hydrodynamic type which is essentially three-dimensional. Up to a linear transformation of the independent variables it can be cast into a canonical form

$$P = \begin{pmatrix} d/dx & 0 \\ 0 & d/dy \end{pmatrix} + \begin{pmatrix} 0 & d/dz \\ d/dz & 0 \end{pmatrix}.$$

The corresponding Hamiltonian systems $\mathbf{u}_t + P(h_{\mathbf{u}}) = 0$ take the form

$$u_t^1 + (h_1)_x + (h_2)_z = 0, \quad u_t^2 + (h_2)_y + (h_1)_z = 0.$$

Applying the Legendre transform, $u_1 = h_1$, $u_2 = h_2$, $H = u^1 h_1 + u^2 h_2 - h$, one can rewrite these equations in the equivalent form

$$(u_1)_x + (u_2)_z + (H_1)_t = 0, \quad (u_2)_y + (u_1)_z + (H_2)_t = 0. \tag{85}$$

Notice that $H(u_1, u_2)$ is defined up to an arbitrary quadratic form (all quadratic terms in H can be eliminated by appropriate linear changes of the independent variables). Our first result is the following

Theorem 6 *Any integrable system (85) is necessarily linear (that is, the potential H is quadratic in u_1, u_2).*

Proof:

Our strategy will be to consider reductions of the system (85) to various $(2 + 1)$ -dimensional systems. In fact, it will be sufficient to look at reductions governing traveling wave solutions. If the original system (85) is integrable, all such reductions must be integrable as well. Since the integrability conditions for $(2 + 1)$ -dimensional two-component systems of hydrodynamic type are explicitly known [11], this will provide a set of *necessary* conditions for the integrability of the system (85). It turns out that these conditions are very strong indeed, leading to the non-existence of non-quadratic integrable potentials H .

Setting in the equations (85) $\partial_z = \mu\partial_t$, which is equivalent to seeking solutions in the form $\mathbf{u}(x, y, t + \mu z)$, one obtains a $(2 + 1)$ -dimensional Hamiltonian system

$$(u_1)_x + (H_1 + \mu u_2)_t = 0, \quad (u_2)_y + (H_2 + \mu u_1)_t = 0,$$

with the Hamiltonian density $H(u_1, u_2) + \mu u_1 u_2$. According to our philosophy we have to require that it is integrable for an arbitrary value of the parameter μ . The integrability conditions (15) readily imply that the corresponding H must be cubic in u_1, u_2 , and Theorem 1 tells us that the only two ‘suspicious’ cases to consider are $H = \frac{1}{6}u_2^3$ and $H = \frac{1}{2}u_1 u_2^2$ (recall that we ignore quadratic terms in H). In the first case the system (85) takes the form

$$(u_1)_x + (u_2)_z = 0, \quad (u_2)_y + (u_1)_z + u_2(u_2)_t = 0.$$

Setting here $x = y$ (this amounts to seeking traveling wave solutions in the form $\mathbf{u}(x + y, z, y)$), one obtains a $(2 + 1)$ -dimensional system

$$(u_1)_x + (u_2)_z = 0, \quad (u_2)_x + (u_1)_z + u_2(u_2)_t = 0. \tag{86}$$

We recall that the paper [11] provides a complete set of the integrability conditions for two-component hydrodynamic type systems represented in the form

$$\begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x + \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_y = 0.$$

The integrability conditions constitute a complicated over-determined system of PDEs for the coefficients a, b, p, q, r, s as functions of v, w . Representing the equations (86) in the form

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x + \begin{pmatrix} 0 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_z = 0$$

one can verify that these integrability conditions are not satisfied. Thus, the $(3 + 1)$ -dimensional system corresponding to $H = \frac{1}{6}u_2^3$ is not integrable. Similarly, for $H = \frac{1}{2}u_1 u_2^2$ the system (85) takes the form

$$(u_1)_x + (u_2)_z + u_2(u_2)_t = 0, \quad (u_2)_y + (u_1)_z + u_2(u_1)_t + u_1(u_2)_t = 0.$$

Setting, again, $x = y$, and changing to the new dependent variables $v = u_1 + u_2$, $w = u_2 - u_1$, one obtains the system

$$\begin{pmatrix} v \\ w \end{pmatrix}_x + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_z + \begin{pmatrix} \frac{3v+w}{4} & \frac{v-w}{4} \\ \frac{v-w}{4} & -\frac{v+3w}{4} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_t = 0,$$

which also does not satisfy the integrability conditions. This finishes the proof of Theorem 4.

Our next result shows that any three-component $(3+1)$ -dimensional integrable Hamiltonian system associated with a non-singular Poisson bracket of hydrodynamic type is either linear or reducible. Any such system can be brought to a canonical form

$$u_t^1 + (h_1)_x = 0, \quad u_t^2 + (h_2)_y = 0, \quad u_t^3 + (h_3)_z = 0, \quad (87)$$

with the Hamiltonian operator

$$\begin{pmatrix} d/dx & 0 & 0 \\ 0 & d/dy & 0 \\ 0 & 0 & d/dz \end{pmatrix}.$$

Performing the Legendre transform one obtains

$$(H_1)_t + (u_1)_x = 0, \quad (H_2)_t + (u_2)_y = 0, \quad (H_3)_t + (u_3)_z = 0,$$

or, in matrix form,

$$A_0 \mathbf{u}_t + A_1 \mathbf{u}_x + A_2 \mathbf{u}_y + A_3 \mathbf{u}_z = 0,$$

where the 3×3 matrices A_i are given by

$$A_0 = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem 7 *Any integrable $(3+1)$ -dimensional Hamiltonian system (87) is either linear or reducible.*

Proof:

As a necessary condition for integrability, one has to require the vanishing of the Haantjes tensor for an arbitrary matrix of the form

$$(A_0 + \lambda A_1 + \beta A_2 + \gamma A_3)^{-1} (A_0 + \tilde{\lambda} A_1 + \tilde{\beta} A_2 + \tilde{\gamma} A_3),$$

which is equivalent to the vanishing of the Haantjes tensor for any matrix $\Lambda(A_0 + \tilde{\Lambda})$ where Λ and $\tilde{\Lambda}$ are arbitrary 3×3 constant coefficient diagonal matrices. Computing the Haantjes tensor and equating to zero coefficients at different monomials in the diagonal entries of Λ and $\tilde{\Lambda}$, one obtains that either all third order derivatives H_{ijk} are identically zero (this corresponds to linear systems), or $H_{ij} = H_{ik} = 0$ for some $i \neq j \neq k$ (this corresponds to the reducible case).

We would like to conclude this section by formulating the following general

Conjecture *There exists no non-trivial integrable Hamiltonian systems of hydrodynamic type in $3+1$ dimensions corresponding to a local Poisson bracket of hydrodynamic type and a local Hamiltonian density.*

7 Concluding remarks

We have found a broad class of non-trivial potentials leading to integrable Hamiltonian systems of hydrodynamic type in $2 + 1$ dimensions. There is a number of natural problems arising in this context, in particular:

- Describe the structure of the corresponding Hamiltonian hierarchies. The main difficulty here is the non-locality of higher symmetries/conservation laws.
- Construct the associated Hamiltonian hydrodynamic chains. This requires the introduction of a canonical set of non-local variables reducing all higher flows of the hierarchy to infinite-component systems of hydrodynamic type.
- Construct dispersive deformations of the examples arising in the classification, especially those with ‘elliptic’ Lax pairs.
- Study the behavior of exact solutions coming from hydrodynamic reductions.

We hope to address some of these questions elsewhere.

Acknowledgements

We thank B Dubrovin, O Mokhov, A Odesskii, M Pavlov and A Veselov for clarifying discussions. This research was supported by the EPSRC grant EP/D036178/1. The work of EVF was also partially supported by the European Union through the FP6 Marie Curie RTN project ENIGMA (Contract number MRTN-CT-2004-5652), and the ESF programme MISGAM. The work of VVS was partially supported by the RFBI-grant 08-01-00461. VVS also thanks IHES, where the final part of this research was completed, for their hospitality.

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