Integrable equations of the dispersionless Hirota type and hypersurfaces in the Lagrangian Grassmannian

E.V. Ferapontov, L. Hadjikos and K.R. Khusnutdinova

Department of Mathematical Sciences
Loughborough University
Loughborough, Leicestershire LE11 3TU
United Kingdom

e-mails:

E.V.Ferapontov@lboro.ac.uk
L.Hadjikos@lboro.ac.uk
K.Khusnutdinova@lboro.ac.uk

Abstract

We investigate integrable second order equations of the form

$$F(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt}) = 0.$$

Familiar examples include the Boyer-Finley equation $u_{xx} + u_{yy} = e^{u_{tt}}$, the potential form of the dispersionless Kadomtsev-Petviashvili (dKP) equation $u_{xt} - \frac{1}{2}u_{xx}^2 = u_{yy}$, the dispersionless Hirota equation $(\alpha - \beta)e^{u_{xy}} + (\beta - \gamma)e^{u_{yt}} + (\gamma - \alpha)e^{u_{tx}} = 0$, etc. The integrability is understood as the existence of infinitely many hydrodynamic reductions. We demonstrate that the natural equivalence group of the problem is isomorphic to $Sp(6)$, revealing a remarkable correspondence between differential equations of the above type and hypersurfaces of the Lagrangian Grassmannian. We prove that the moduli space of integrable equations of the dispersionless Hirota type is 21-dimensional, and the action of the equivalence group $Sp(6)$ on the moduli space has an open orbit.

MSC: 35Q58, 37K25, 53A40, 53B25, 53Z05.

Keywords: Integrable equations, Hydrodynamic Reductions, Lagrangian Grassmannian, Hypersurfaces, Rational Normal Curve, $GL(2)$-structure.

1 Introduction

We investigate a general class of three-dimensional second order equations of the form

$$F(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt}) = 0 \qquad (1)$$

where $u = u(x, y, t)$ is a function of three independent variables. Equations of this type arise in a wide range of applications including non-linear physics, general relativity, differential geometry, integrable systems and complex analysis. For instance, the dKP equation,

$$u_{xt} - \frac{1}{2}u_{xx}^2 = u_{yy},$$

also known as the Khokhlov-Zabolotskaya equation, arises in non-linear acoustics [52], as well as in the theory of Einstein-Weyl structures [18]. The Boyer-Finley equation,

$$u_{xx} + u_{yy} = e^{u_{tt}},$$

which is descriptive of a class of self-dual 4-manifolds, has been extensively discussed in the context of general relativity [12]. The equations

$$Hess\ u = 1 \quad \text{and} \quad Hess\ u = \triangle u,$$

where $Hess$ is the determinant of the Hessian matrix of u, and $\triangle$ is the Laplacian, appear in differential geometry in the theory of affine spheres and special Lagrangian 3-folds, respectively [13, 32]. A subclass of equations of the form (1),

$$u_{tt} = f(u_{xx}, u_{xt}, u_{xy}),$$

was discussed recently in [41] in connection with hydrodynamic chains satisfying the so-called Egorov property. Equations of the form (1) typically arise as the Hirota-type relations for various $(2+1)$-dimensional dispersionless hierarchies. For instance, equations of the dispersionless Toda hierarchy [49, 14, 53],

$$(\lambda - \mu)e^{D_\lambda D_\mu u} = \lambda e^{-\partial_s D_\lambda u} - \mu e^{-\partial_s D_\mu u},$$

$$(\lambda - \mu)e^{\bar{D}_\lambda \bar{D}_\mu u} = \lambda e^{\partial_s \bar{D}_\mu u} - \mu e^{\partial_s \bar{D}_\lambda u},$$

$$e^{D_\lambda \bar{D}_\mu u} = 1 - \frac{\mu}{\lambda}e^{(\partial_s^2 + \partial_s D_\lambda - \partial_s \bar{D}_\mu)u},$$

take the form (1) after one replaces the 'vertex' operators D_λ, D_μ, $\bar{D}_\lambda$, $\bar{D}_\mu$ by ordinary partial derivatives. Equations of the dispersionless Toda hierarchy play an important role in complex analysis underlying the integrable structure of the Dirichlet problem for simply-connected domains [51, 39]. A generalization of this construction to multiply-connected domains [34] leads to a remarkable hierarchy of Hirota-like equations which constitute the so-called universal Whitham hierarchy [33]. Further examples of Hirota-type relations arise in the theory of the associativity (WDVV) equations [8, 10, 30], e.g.,

$$e^{u_{xx}+u_{xy}+u_{xt}-u_{yt}} - e^{u_{xy}+u_{yy}+u_{yt}-u_{xt}} + e^{u_{xt}+u_{yt}+u_{tt}-u_{xy}} = 0,$$

$$e^{u_{xx}+2u_{xt}+u_{xy}} - e^{u_{tt}+2u_{xt}+u_{yt}} = 1,$$

$$e^{u_{tt}-u_{xx}} = \sinh u_{xy}/\sinh u_{yt},$$

$$\coth u_{xy} = \coth u_{xt} \coth u_{yt},$$

etc. These equations arise as certain differential constraints which should be considered along with a full set of the WDVV equations.

Equations of the above type have been approached by a whole variety of modern techniques including symmetry analysis, differential-geometric and algebro-geometric methods, dispersionless $\bar{\partial}$-dressing, factorization techniques, Virasoro constraints, hydrodynamic reductions, etc. However, until recently there was no 'intrinsic' approach which would explain the integrability of these and other examples. Moreover, there was no satisfactory definition of the integrability which would (a) be algorithmically verifiable, (b) allow classification results and (c) provide a scheme for the construction of exact solutions. We emphasize that equations of the form (1) are not amenable to the inverse scattering transform, and require an alternative approach. Such approach, based on the method of hydrodynamic reductions and, primarily, the work [26], see also [14, 22, 23, 38], etc, was proposed in [20]. It was suggested to define the integrability of a multi-dimensional dispersionless system by requiring the existence of 'sufficiently many' hydrodynamic reductions which provide multi-phase solutions known as non-linear interactions of planar simple waves. Technically, one 'decouples' a three-dimensional PDE (1) into a pair of commuting n-component $(1+1)$-dimensional systems of hydrodynamic type,

$$R_t^i = \lambda^i(R) \ R_x^i, \qquad R_y^i = \mu^i(R) \ R_x^i, \tag{2}$$

where the characteristic speeds λ^i and μ^i satisfy the commutativity conditions

$$\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j, \tag{3}$$

$\partial_j = \partial_{R^j}$, see [50]. The best way to illustrate the method of hydrodynamic reductions is to discuss an example.

Example. Let us consider the dKP equation, $u_{xt} - \frac{1}{2}u_{xx}^2 = u_{yy}$, and introduce the notation $u_{xx} = a$, $u_{xy} = b$, $u_{xt} = p$, $u_{yy} = p - \frac{1}{2}a^2$; this results in the equivalent quasilinear representation of the dKP equation,

$$a_y = b_x, \quad a_t = p_x, \quad b_t = p_y, \quad b_y = (p - \frac{1}{2}a^2)_x. \tag{4}$$

Let us seek multi-phase solutions in the form $a = a(R^1, ..., R^n)$, $b = b(R^1, ..., R^n)$, $p = p(R^1, ..., R^n)$ where the 'phases' $R^i(x, y, t)$ satisfy the equations (2). The substitution of this ansatz into (4) implies the relations

$$\partial_i b = \mu^i \partial_i a, \quad \partial_i p = \lambda^i \partial_i a, \quad \lambda^i = a + (\mu^i)^2.$$

Calculating the compatibility conditions $\partial_i \partial_j b = \partial_j \partial_i b$, $\partial_i \partial_j p = \partial_j \partial_i p$, and substituting $\lambda^i = a + (\mu^i)^2$ into the commutativity conditions (3), one obtains the Gibbons-Tsarev system for $a(R)$ and $\mu^i(R)$,

$$\partial_j \mu^i = \frac{\partial_j a}{\mu^j - \mu^i}, \quad \partial_i \partial_j a = 2\frac{\partial_i a \partial_j a}{(\mu^j - \mu^i)^2},$$

3

$i \neq j$, which was first derived in [26] in the theory of hydrodynamic reductions of Benney's moment equations. It is remarkable that the Gibbons-Tsarev system is in involution, and its general solution depends, modulo reparametrizations $R^i \to \varphi^i(R^i)$, on n arbitrary functions of one variable. Thus, the dKP equation possesses infinitely many n-component reductions parametrized by n arbitrary functions of one variable. We point out that the compatibility conditions $\partial_k \partial_j \mu^i = \partial_j \partial_k \mu^i$ and $\partial_i \partial_j \partial_k a = \partial_i \partial_k \partial_j a$ involve *triples* of indices $i \neq j \neq k$ only. Thus, for $n = 2$ the Gibbons-Tsarev system is automatically consistent, while its consistency for $n = 3$ implies the consistency for arbitrary n. Based on this example, we give the following

Definition *An equation of the form (1) is said to be integrable if, for any n, it possesses infinitely many n-component hydrodynamic reductions parametrized by n arbitrary functions of one variable.*

We have verified that all examples presented above are indeed integrable in this sense, with the exception of the equations $Hess\ u = 1$ and $Hess\ u = \triangle u$, which do not pass the test (see Sect. 3.5). In Sect. 2 we derive the integrability conditions (as a system of third order differential relations for the function F in (1)), and prove our first main result (Theorem 1 of Sect. 2):

- **The moduli space of integrable equations of the dispersionless Hirota type is** 21-**dimensional.**

The class of equations (1) is form-invariant under the action of the contact group $Sp(6)$ generated by linear symplectic transformations of the variables x, y, t, u_x, u_y, u_t. These transformations map integrable equations to integrable. Taking into account that dim $Sp(6) = 21$, our second main result (Theorem 5 of Sect. 6) reads as follows:

- **The action of the equivalence group** $Sp(6)$ **on the moduli space of integrable equations has an open orbit.**

This result is, in a sense, surprising: it establishes the existence of a unique 'master-equation' which generates all other integrable examples via various (singular) limits. The existence of an open orbit does not mean, of course, that all integrable equations of the form (1) are $Sp(6)$-equivalent. The structure of the orbit space of this action is quite complicated, for instance, all examples listed above belong to various singular orbits of lower dimension, and are not $Sp(6)$-equivalent.

From differential-geometric point of view, the equation (1) can be viewed as defining a 5-dimensional hypersurface M^5 in the Lagrangian Grassmannian Λ which can (locally) be identified with the space of 3×3 symmetric matrices u_{ij}. In Sect. 4 we demonstrate that each hypersurface $M^5 \subset \Lambda$ inherits a special structure, namely, a field of rational normal curves γ of degree four specified in the tangent bundle TM^5. Bisecant surfaces $\Sigma^2 \subset M^5$ are defined as two-dimensional submanifolds whose projectivised tangent spaces are bisecant lines of γ. Similarly, trisecant 3-folds are three-dimensional submanifolds $\Sigma^3 \subset M^5$ whose projectivised tangent spaces are trisecant planes of γ. Our third main result is the following geometric characterization of integrable equations (Theorem 3 of Sect. 4):

4

- **Bisecant surfaces and trisecant 3-folds are geometric images of the two- and three-component hydrodynamic reductions, respectively. In particular, the equation is integrable if and only if the corresponding hypersurface M^5 possesses infinitely many trisecant 3-folds.**

The paper is organized as follows.

In Sect. 2 we derive the integrability conditions by transforming (1) to a quasilinear form, and calculating the compatibility conditions of the corresponding generalized Gibbons-Tsarev system which governs n-component hydrodynamic reductions. The result of this calculation is a complicated set of the integrability conditions, which are in involution. This establishes the 21-dimensionality of the moduli space.

Further examples of integrable equations and partial classification results are provided in Sect 3, based on the integrability conditions obtained in Sect. 2. In particular, we analyze the integrability of the symplectic Monge-Ampére equations.

Differential-geometric aspects of integrable equations of the form (1) are discussed in Sect. 4. We begin by introducing a flat generalized conformal structure on the Lagrangian Grassmannian Λ, whose conformal automorphism group is isomorphic to $Sp(6)$. Given a hypersurface $M^5 \subset \Lambda$, we describe geometric structures which are naturally induced on M^5, and provide a geometric characterization of the integrability conditions.

In Sect. 5 we briefly discuss multi-dimensional equations of the form (1).

The action of the equivalence group $Sp(6)$ on the moduli space of integrable equations is discussed in Sect. 6.

2 Derivation of the integrability conditions: proof of Theorem 1

In this section we derive the integrability conditions which follow from the requirement of the existence of n-component reductions, and prove the following

Theorem 1 *The moduli space of integrable equations of the dispersionless Hirota type is 21-dimensional.*

Proof:

To apply the method of hydrodynamic reductions we first rewrite the equation (1) in the evolutionary form,

$$u_{tt} = f(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}), \tag{5}$$

and introduce the notation

$$u_{xx} = a, \ u_{xy} = b, \ u_{yy} = c, \ u_{xt} = p, \ u_{yt} = q, \ u_{tt} = f(a, b, c, p, q).$$

This provides an equivalent quasilinear representation of (5),

$$a_y = b_x, \quad a_t = p_x, \quad b_y = c_x, \quad b_t = p_y = q_x, \quad c_t = q_y,$$

$$p_y = q_x, \quad p_t = f(a, b, c, p, q)_x, \quad q_t = f(a, b, c, p, q)_y. \tag{6}$$

Looking for solutions in the form $a = a(R^1, ..., R^n)$, $b = b(R^1, ..., R^n)$, $c = c(R^1, ..., R^n)$, $p = p(R^1, ..., R^n)$, $q = q(R^1, ..., R^n)$, where the Riemann invariants R^i satisfy the equations (2), and substituting this ansatz into (6), we obtain

$$\partial_i b = \mu^i \partial_i a, \quad \partial_i p = \lambda^i \partial_i a, \quad \partial_i q = \mu^i \lambda^i \partial_i a, \quad \partial_i c = (\mu^i)^2 \partial_i a, \tag{7}$$

$\partial_i = \partial/\partial_{R^i}$, along with the dispersion relation

$$D(\lambda_i, \mu_i) = f_a + f_b \mu^i + f_c(\mu^i)^2 + f_p \lambda^i + f_q \lambda^i \mu^i - (\lambda^i)^2 = 0; \tag{8}$$

in what follows we assume that the dispersion relation (8) defines an irreducible conic in the (λ, μ)-plane. This is equivalent to the requirement that the expression

$$\Delta = f_b^2 + f_b f_p f_q - f_a f_q^2 - f_c f_p^2 - 4 f_a f_c$$

does not vanish. Calculating the consistency conditions for (7) we obtain

$$\partial_i \partial_j a = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \partial_i a + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j} \partial_j a,$$

$$\partial_j \lambda^i \partial_i a + \partial_i \lambda^j \partial_j a = 0. \tag{9}$$

Applying the operator ∂_j to the dispersion relation (8) and using (7) and (3), we obtain $\partial_j \lambda^i$ and $\partial_j \mu^i$ in the form

$$\partial_j \lambda^i = (\lambda^i - \lambda^j) B_{ij} \partial_j a, \qquad \partial_j \mu^i = (\mu^i - \mu^j) B_{ij} \partial_j a \tag{10}$$

where B_{ij} are rational expressions in $\lambda^i, \lambda^j, \mu^i, \mu^j$ whose coefficients depend on the second order partial derivatives of the function $f(a, b, c, p, q)$. Explicitly, one has

$$B_{ij} = \frac{N_{ij}}{D_{ij}}$$

where

$$N_{ij} = f_{aa} + f_{ab}(\mu^i + \mu^j) + f_{ac}\left((\mu^i)^2 + (\mu^j)^2\right) + f_{ap}(\lambda^i + \lambda^j) + f_{aq}(\lambda^i \mu^i + \lambda^j \mu^j)$$

$$+ f_{bb}\mu^i \mu^j + f_{bc}\mu^i \mu^j(\mu^i + \mu^j) + f_{bp}(\lambda^i \mu^j + \lambda^j \mu^i) + f_{bq}\mu^i \mu^j(\lambda^i + \lambda^j)$$

$$+ f_{cc}(\mu^i)^2(\mu^j)^2 + f_{cp}\left(\lambda^i(\mu^j)^2 + \lambda^j(\mu^i)^2\right) + f_{cq}\mu^i \mu^j(\lambda^i \mu^j + \lambda^j \mu^i) \tag{11}$$

$$+ f_{pp}\lambda^i \lambda^j + f_{pq}\lambda^i \lambda^j(\mu^i + \mu^j) + f_{qq}\lambda^i \lambda^j \mu^i \mu^j$$

and

$$\begin{aligned} D_{ij} &= -2\lambda^i \lambda^j + 2f_a + f_b(\mu^i + \mu^j) + 2f_c \mu^i \mu^j + f_p(\lambda^i + \lambda^j) + f_q(\lambda^i \mu^j + \lambda^j \mu^i) \\ &= 4D\left(\frac{\lambda^i + \lambda^j}{2}, \frac{\mu^i + \mu^j}{2}\right), \end{aligned} \tag{12}$$

the last equality holding modulo the dispersion relation (8). Substituting (10) into (9)$_1$ one obtains

$$\partial_i \partial_j a = -2 B_{ij} \partial_i a \partial_j a; \tag{13}$$

the second condition (9)$_2$ holds identically by virtue of the symmetry $B_{ij} = B_{ji}$. The compatibility conditions $\partial_k \partial_j \lambda^i = \partial_j \partial_k \lambda^i$, $\partial_k \partial_j \mu^i = \partial_j \partial_k \mu^i$ and $\partial_k \partial_j \partial_i a = \partial_j \partial_k \partial_i a$ are equivalent to the relations

$$\partial_k B_{ij} = (B_{ij} B_{kj} + B_{ij} B_{ik} - B_{kj} B_{ik}) \partial_k a, \tag{14}$$

which must be satisfied identically by virtue of (7), (8) and (10).

Remark. Notice that each compatibility condition involves three distinct indices only. This observation immediately implies that

(i) any equation of the form (1) possesses infinitely many two-component reductions parametrized by two arbitrary functions of one variable. Indeed, these reductions are governed by the equations (10) and (13) where $i, j = 1, 2$ and λ^i, μ^i satisfy the dispersion relation (8). These equations are automatically consistent, and the general solution depends, modulo reparametrizations $R^i \to \varphi^i(R^i)$, on two arbitrary functions of one variable. Therefore, the existence of two-component reductions is a common phenomenon which is not related to the integrability.

(ii) the existence of three-component reductions implies the existence of n-component reductions for arbitrary n. Thus, one can define the integrability as the existence of infinitely many three-component reductions parametrized by three arbitrary functions of one variable. This property is reminiscent of the well-known three-soliton condition in the theory of integrable systems.

In order to simplify the derivation of the integrability conditions we first rewrite (14) as

$$\partial_k N_{ij} = N_{ij} \left(\frac{1}{D_{ij}} \partial_k D_{ij} + B_{kj} \partial_k a + B_{ik} \partial_k a \right) - D_{ij} B_{kj} B_{ik} \partial_k a. \tag{15}$$

Here, the third order derivatives of $f(a, b, c, p, q)$ are present only in the l.h.s. term $\partial_k N_{ij}$ (see the form of N_{ij} and D_{ij} above). Further reduction of the complexity of the expression in the r.h.s. is achieved by representing $1/D_{ij}$ in the form

$$\frac{1}{D_{ij}} = U_{ij} = [2 f_a + (\mu_1 + \mu_2) f_b + 2 \mu_1 \mu_2 f_c - (\lambda_1 + \lambda_2) f_p - (\lambda_1 \mu_2 + \lambda_2 \mu_1) f_q$$

$$+ f_p^2 + (\mu_1 + \mu_2) f_p f_q + \mu_1 \mu_2 f_q^2 + 2 \lambda_1 \lambda_2] / ((\mu_1 - \mu_2)^2 \Delta)$$

(which holds identically modulo the dispersion relation (8)), and the subsequent substitution $B_{st} = N_{st}/D_{st} = N_{st} U_{st}$. The denominators of the r.h.s. terms in (15) cancel out, producing a polynomial in $\lambda^i, \lambda^j, \lambda^k, \mu^i, \mu^j, \mu^k$ with coefficients depending on the derivatives of the density $f(a, b, c, p, q)$. This was the crucial simplification of the calculation: the starting expression for the r.h.s. of (15) has more than 450.000 terms with different denominators; after properly organized cancellations we get a polynomial expression with less than 6.000 terms. Using the dispersion relation (8), we simplify this polynomial by substituting the powers of $(\lambda^i)^s$, $(\lambda^j)^s$, $(\lambda^k)^s$, $s \geq 2$, arriving at a polynomial of degree

one in each of λ^i, λ^j, λ^k, and degree two in μ's. Equating similar coefficients in both sides of (15), we arrive at a set of 35 equations for the derivatives of the function $f(a, b, c, p, q)$, which are linear in the third order derivatives. Solving this linear system we obtain the closed form expressions for all third order derivatives of $f(a, b, c, p, q)$ in terms of its first and second order derivatives, which we represent symbolically in the form

$$d^3 f = R(df, d^2 f); \tag{16}$$

here R denotes complicated rational expressions in the first and second order derivatives of f (it is clear from the above that the function f itself will not enter these expressions explicitly). A straigtforward computer calculation shows that the overdetermined system (16) is in involution. Thus, the moduli space of integrable systems of the type (1) is 21-dimensional: one can arbitrarily specify the values of f, df and $d^2 f$ at any fixed point. This amounts to $1 + 5 + 15 = 21$ arbitrary constants. This finishes the proof of Theorem 1.

The right hand sides of (16) are not presented here because of their complexity (see Sect. 3 for a discussion of particular cases where these formulae become less cumbersome, and are presented explicitly). The integrability conditions (16) provide a straightforward computer test of integrability for any equation from the class under consideration, and allow one to obtain classification results. These conditions also allow computer checks of various properties of differential-geometric objects naturally associated with equations from the class (1).

Remark. In two dimensions, any second order PDE of the form $F(u_{xx}, u_{xy}, u_{yy}) = 0$ is automatically integrable. Indeed, introducing the parametrization $u_{xx} = a$, $u_{xy} = b$, $u_{yy} = f(a, b)$, so that $F(a, b, f(a, b)) = 0$, one obtains a two-component quasilinear system

$$a_y = b_x, \quad b_y = f(a, b)_x;$$

any system of this type linearises under a hodograph transformation which interchanges dependent and independent variables. This simple trick, however, does not work in more than two dimensions.

3 Examples and classification results

In this section we discuss examples (both known and new) and partial classification results of integrable equations of the form (1) which are obtained using the integrability conditions (16) derived in Sect. 2.

3.1 Integrable equations of the form $u_{tt} = f(u_{xx}, u_{yy})$

In this case the integrability conditions (16) simplify to

$$f_{aaa} = f_{aa} \left(\frac{f_{ac}}{f_c} + \frac{f_{aa}}{f_a} \right), \quad f_{aac} = f_{aa} \left(\frac{f_{cc}}{f_c} + \frac{f_{ac}}{f_a} \right),$$

$$f_{acc} = f_{cc} \left(\frac{f_{cc}}{f_c} + \frac{f_{ac}}{f_a} \right), \quad f_{ccc} = f_{cc} \left(\frac{f_{cc}}{f_c} + \frac{f_{ac}}{f_a} \right), \quad f_{aa} f_{cc} = (f_{ac})^2;$$

recall that, according to the notation introduced in Sect. 2, $a = u_{xx}$, $c = u_{yy}$. This system is in involution and its general solution depends on 5 integration constants. To solve it explicitly one notices that the first two equations imply $\frac{f_{aa}}{f_a f_c} = const$. Similarly, the next two equations imply $\frac{f_{cc}}{f_a f_c} = const$. Further elementary integration leads to the following general solution,

$$f(a, c) = s \ln(m e^{\mu a} + n e^{\nu c}) + const.$$

This corresponds to equations of the form

$$m e^{\mu u_{xx}} + n e^{\nu u_{yy}} + k e^{\kappa u_{tt}} = 0,$$

as well as degenerations thereof,

$$m u_{xx} + n u_{yy} + k e^{\kappa u_{tt}} = 0,$$

where the coefficients are arbitrary constants. In fact, all these coefficients can be eliminated by appropriate complex rescalings, leading to the two essentially different examples,

$$e^{u_{xx}} + e^{u_{yy}} = e^{u_{tt}} \quad \text{and} \quad u_{xx} + u_{yy} = e^{u_{tt}}.$$

The first equation is apparently new, while the second is the Boyer-Finley equation.

3.2 Integrable equations of the form $u_{tt} = f(u_{xx}, u_{xy}, u_{yy})$

This is a generalization of the previous case. Thus, we assume $f_b \neq 0$. The integrability conditions (16) take the form

$$f_{aa} f_{bb} = f_{ab}{}^2, \quad f_{aa} f_{cc} = f_{ac}{}^2, \quad f_{bb} f_{cc} = f_{bc}{}^2,$$

$$f_{aa} f_{bc} = f_{ab} f_{ac}, \quad f_{ab} f_{cc} = f_{ac} f_{bc}, \quad f_{ab} f_{bc} = f_{ac} f_{bb},$$

$$f_{aaa} = \frac{2 f_{aa} \left(f_b f_{ab} - 2 f_c f_{aa} - 2 f_a f_{ac}\right)}{f_b{}^2 - 4 f_a f_c}, \quad f_{aab} = \frac{2 f_{ab} \left(f_b f_{ab} - 2 f_c f_{aa} - 2 f_a f_{ac}\right)}{f_b{}^2 - 4 f_a f_c},$$

$$f_{aac} = \frac{2 f_{ac} \left(f_b f_{ab} - 2 f_c f_{aa} - 2 f_a f_{ac}\right)}{f_b{}^2 - 4 f_a f_c}, \quad f_{abb} = \frac{2 f_{ab} \left(f_b f_{bb} - 2 f_c f_{ab} - 2 f_a f_{bc}\right)}{f_b{}^2 - 4 f_a f_c},$$

$$f_{acc} = \frac{2 f_{ac} \left(f_b f_{bc} - 2 f_c f_{ac} - 2 f_a f_{cc}\right)}{f_b{}^2 - 4 f_a f_c}, \quad f_{abc} = \frac{2 f_{ab} \left(f_b f_{bc} - 2 f_c f_{ac} - 2 f_a f_{cc}\right)}{f_b{}^2 - 4 f_a f_c},$$

$$f_{bbb} = \frac{2 f_{bb} \left(f_b f_{bb} - 2 f_c f_{ab} - 2 f_a f_{bc}\right)}{f_b{}^2 - 4 f_a f_c}, \quad f_{bbc} = \frac{2 f_{bc} \left(f_b f_{bb} - 2 f_c f_{ab} - 2 f_a f_{bc}\right)}{f_b{}^2 - 4 f_a f_c},$$

$$f_{bcc} = \frac{2 f_{bc} \left(f_b f_{bc} - 2 f_c f_{ac} - 2 f_a f_{cc}\right)}{f_b{}^2 - 4 f_a f_c}, \quad f_{ccc} = \frac{2 f_{cc} \left(f_b f_{bc} - 2 f_c f_{ac} - 2 f_a f_{cc}\right)}{f_b{}^2 - 4 f_a f_c}.$$

Notice that the first six equations imply that the Hessian matrix of the function $f(a, b, c)$ has rank one, so that one can set $f_a = m(f_b)$, $f_c = n(f_b)$. The substitution into the remaining equations implies either $m'' = n'' = 0$, or $f_{bb} = 0$. Further elementary analysis leads, up to linear transformations of x and y, to the following canonical forms (here we only list those representatives which contain a nontrivial dependence on u_{xy}):

(a) $u_{tt} = \alpha u_{xx} + \beta u_{yy} + \varphi(u_{xy})$, here φ satisfies a third order ODE $\varphi'''((\varphi')^2 - 4\alpha\beta) = 2\varphi'(\varphi'')^2$. The integration leads to the three canonical forms,

$$u_{tt} = \alpha u_{xx} + \beta u_{yy} + \frac{2\sqrt{\alpha\beta}}{\gamma} \ln \cosh \gamma u_{xy},$$

$$u_{tt} = \alpha u_{xx} + \gamma \ln u_{xy}, \qquad \text{or} \qquad u_{tt} = \ln u_{xy},$$

the last two cases corresponding to $\beta = 0$ and $\alpha = \beta = 0$, respectively. The first two examples can be viewed as generalizations of the Boyer-Finley equation.

(b) $u_{tt} = u_{xy} + \beta(u_{xx} - u_{yy}) + \varphi(u_{xx} + u_{yy})$, here $\varphi'''(4(\varphi')^2 - 1 - 4\beta^2) = 8\varphi'(\varphi'')^2$. This leads to the equation

$$u_{tt} = u_{xy} + \beta(u_{xx} - u_{yy}) + \frac{\sqrt{1 + 4\beta^2}}{2\gamma} \ln \cosh \gamma(u_{xx} + u_{yy}).$$

(c) $u_{tt} = u_{xy} + \beta u_{xx} + \varphi(u_{yy})$, here φ satisfies a third order ODE $\varphi'''(4\beta\varphi' - 1) = 4\beta(\varphi'')^2$. This results in the equation

$$u_{tt} = u_{xy} + \beta u_{xx} + \frac{1}{4\beta} u_{yy} + \alpha e^{\gamma u_{yy}},$$

whose degeneration, corresponding to $\beta = 0$, is the dKP equation $u_{tt} = u_{xy} + u_{yy}^2$.

3.3 Integrable equations of the form $u_{xy} = f(u_{xt}, u_{yt})$

Formally, equations from this class are not of the form (5), however, they can readily be made 'evolutionary' by an appropriate linear change of the independent variables. The resulting set of integrability conditions looks as follows:

$$f_{ppp} = f_{pp} \left(\frac{f_{pq}}{f_q} + \frac{f_{pp}}{f_p} \right), \qquad f_{ppq} = f_{pp} \left(\frac{f_{qq}}{f_q} + \frac{f_{pq}}{f_p} \right),$$

$$f_{pqq} = f_{qq} \left(\frac{f_{pq}}{f_q} + \frac{f_{pp}}{f_p} \right), \qquad f_{qqq} = f_{qq} \left(\frac{f_{qq}}{f_q} + \frac{f_{pq}}{f_p} \right).$$

This system is in involution, and its general solution depends on 6 integration constants. To solve it explicitly one notices that the first two equations imply $\frac{f_{pp}}{f_p f_q} = const$. Similarly, the last two equations imply $\frac{f_{qq}}{f_p f_q} = const$. Further elementary integration gives, under the assumption that both f_{pp} and f_{qq} are nonzero, the general solution

$$f(p, q) = \frac{1}{\kappa} \ln \left(\frac{me^{\mu p} + ne^{\nu q}}{re^{\mu p + \nu q} - k} \right),$$

which leads to the dipersionless Hirota-type equation for the BKP hierarchy [7],

$$me^{\mu u_{xt}} + ne^{\nu u_{yt}} + ke^{\kappa u_{xy}} = re^{\mu u_{xt} + \nu u_{yt} + \kappa u_{xy}}. \tag{17}$$

Up to complex rescalings, it can be transformed to

$$e^{u_{xt}} + e^{u_{yt}} + e^{u_{xy}} = e^{u_{xt} + u_{yt} + u_{xy}}.$$

Degenerations of (17), corresponding to $f_{pp} = 0$ or $f_{pp} = f_{qq} = 0$, result in

$$u_{xy} = u_{xt} + e^{u_{yt}}, \qquad u_{xy} = u_{xt} \tan(u_{yt})$$

and

$$u_{xy} = u_{xt} u_{yt},$$

respectively.

3.4 Integrable equations of the form $u_{tt} = f(u_{xx}, u_{xt}, u_{xy})$

Equations of this type arise in the theory of integrable hydrodynamic chains satisfying the additional'Egorov' property. Here we reproduce the classification result from [41] (see also [11], [25]). The integrability conditions (16) take the form

$$f_{bbb} = \frac{2f_{bb}^2}{f_b}, \quad f_{abb} = \frac{2f_{ab}f_{bb}}{f_b}, \quad f_{pbb} = \frac{2f_{pb}f_{bb}}{f_b},$$

$$f_{aab} = \frac{2f_{ab}^2}{f_b}, \quad f_{apb} = \frac{2f_{ab}f_{pb}}{f_b}, \quad f_{ppb} = \frac{2f_{pb}^2}{f_b},$$

$$f_{ppp} = \frac{2}{f_b^2} \left(f_p f_{pb}^2 + f_{pb}(f_b f_{pp} + 2f_{ab}) - f_{bb}(f_p f_{pp} + 2f_{ap}) \right),$$

$$f_{app} = \frac{2}{f_b^2} \left(f_a f_{pb}^2 + f_{ab}(f_b f_{pp} + f_{ab}) - f_{bb}(f_a f_{pp} + f_{aa}) \right), \tag{18}$$

$$f_{aap} = \frac{2}{f_b^2} \left(f_{bb}(f_p f_{aa} - 2f_a f_{ap}) - f_{ab}(f_p f_{ab} - 2f_b f_{ap}) - f_{pb}(f_b f_{aa} - 2f_a f_{ab}) \right),$$

$$f_{aaa} = \frac{2}{f_b^2} \left((f_a + f_p^2)f_{ab}^2 + f_a^2 f_{pb}^2 + f_b^2(f_{ap}^2 - f_{aa}f_{pp}) - f_{pp}f_{bb}f_a^2 \right.$$
$$+ f_{ab}f_b(f_{aa} + 2(f_a f_{pp} - f_p f_{ap})) + 2f_{pb}(f_p(f_b f_{aa} - f_a f_{ab}) - f_a f_b f_{ap})$$
$$\left. - f_{bb}((f_a + f_p^2)f_{aa} - 2f_a f_p f_{ap}) \right);$$

this system is in involution and its general solution depends on 10 arbitrary constants.

The integration of these equations leads to the four essentially different canonical

forms,

$$u_{tt} = u_{xy} + \frac{1}{4A}(Au_{xt} + 2Bu_{xx})^2 + Ce^{-Au_{xx}},$$

$$u_{tt} = \frac{u_{xy}}{u_{xx}} + \left(\frac{1}{u_{xx}} + \frac{A}{4u_{xx}^2}\right)u_{xt}^2 + \frac{B}{u_{xx}^2}u_{xt} + \frac{B^2}{Au_{xx}^2} + Ce^{A/u_{xx}},$$

$$u_{tt} = \frac{u_{xy}}{u_{xt}} + \frac{1}{6}\eta(u_{xx})u_{xt}^2,$$

$$u_{tt} = \ln u_{xy} - \ln \theta_1(u_{xt}, u_{xx}) - \frac{1}{4}\int^{u_{xx}}\eta(\tau)d\tau,$$

see [41]. Here A, B, C are arbitrary constants, η is a solution to the Chazy equation [16],

$$\eta''' + 2\eta\eta'' = 3(\eta')^2,$$

and θ_1 is a theta-function in the variable u_{xt} whose dependence on u_{xx} is governed by the Chazy equation.

3.5   Symplectic Monge-Ampére equations

Let us consider a function $u(x^1, ...x^k)$ of k independent variables, and introduce the $k \times k$ Hessian matrix $U = [u_{ij}]$ of its second order partial derivatives. The symplectic Monge-Ampére equation is a PDE of the form

$$M_k + M_{k-1} + ... + M_1 + M_0 = 0$$

where M_l is a constant-coefficient linear combination of all $l \times l$ minors of the matrix U, $0 \leq l \leq k$. Thus, $M_k = det\ U = Hess\ u$, M_0 is a constant, etc. Equivalently, these PDEs can be obtained by equating to zero a constant-coefficient k-form in the $2k$ variables x^i, u_i. This class of equations is invariant under the natural action of the symplectic group $Sp(2k)$. In the case $k = 2$ one obtains a standard Monge-Ampére equation,

$$u_{11}u_{22} - u_{12}^2 + \alpha u_{11} + \beta u_{12} + \gamma u_{22} + \delta = 0, \tag{19}$$

which can be interpreted as the equation of a 'sphere' corresponding to the pseudo-Euclidean metric $du_{11}du_{22} - du_{12}^2$. Monge-Ampére equations (19) can be characterized as the only equations of the form $F(u_{11}, u_{12}, u_{22}) = 0$ which are linearizable by a transformation from the equivalence group $Sp(4)$.

The case $k = 3$ is also understood completely: one can show that, for $k = 3$, any symplectic Monge-Ampére equation is either linearizable (in this case it is automatically integrable), or $Sp(6)$-equivalent to either of the two essentially different canonical forms [37, 4],

$$Hess\ u = 1, \qquad Hess\ u = u_{11} + u_{22} + u_{33}. \tag{20}$$

Based on the integrability conditions (16), we have verified directly that both PDEs (20) are *not* integrable by the method of hydrodynamic reductions. Thus, a 3-dimensional

12

symplectic Monge-Ampére equation is integrable if and only if it is linearizable (this is no longer true in more than three dimensions - see Sect. 5). The linearizability condition constitutes a single relation among the coefficients of the equation: for a Monge-Ampére equation of the form

$$
\epsilon \ det \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{23} \\ u_{13} & u_{23} & u_{33} \end{bmatrix} + h_1(u_{22}u_{33} - u_{23}^2) + h_2(u_{11}u_{33} - u_{13}^2) + h_3(u_{11}u_{22} - u_{12}^2)
$$

$$
+ g_1(u_{11}u_{23} - u_{12}u_{13}) + g_2(u_{22}u_{13} - u_{12}u_{23}) + g_3(u_{33}u_{12} - u_{13}u_{23})
$$

$$
+ s_1u_{11} + s_2u_{22} + s_3u_{33} + \tau_1 u_{23} + \tau_2 u_{13} + \tau_3 u_{12} + \nu = 0,
$$

(21)

the linearizability condition defines a quartic hypersurface in the space of coefficients,

$$
h_1^2 s_1^2 + h_2^2 s_2^2 + h_3^2 s_3^2 + g_1^2 s_2 s_3 + g_2^2 s_1 s_3 + g_3^2 s_1 s_2
$$

$$
- 2(h_1 h_2 s_1 s_2 + h_1 h_3 s_1 s_3 + h_2 h_3 s_2 s_3) + 4\epsilon s_1 s_2 s_3 + 4\nu h_1 h_2 h_3
$$

$$
+ \epsilon \tau_1 \tau_2 \tau_3 - \nu g_1 g_2 g_3 - \epsilon^2 \nu^2 - \nu(g_1^2 h_1 + g_2^2 h_2 + g_3^2 h_3)
$$

$$
- (g_1 \tau_1 + g_2 \tau_2 + g_3 \tau_3 + 2\epsilon \nu)(h_1 s_1 + h_2 s_2 + h_3 s_3 - \epsilon \nu)
$$

(22)

$$
+ 2(g_1 h_1 s_1 \tau_1 + g_2 h_2 s_2 \tau_2 + g_3 h_3 s_3 \tau_3) + \tau_1^2 h_2 h_3 + \tau_2^2 h_1 h_3 + \tau_3^2 h_1 h_2
$$

$$
- \epsilon(\tau_1^2 s_1 + \tau_2^2 s_2 + \tau_3^2 s_3) + s_1 \tau_1 g_2 g_3 + s_2 \tau_2 g_1 g_3 + s_3 \tau_3 g_1 g_2
$$

$$
- (g_1 h_1 \tau_2 \tau_3 + g_2 h_2 \tau_1 \tau_3 + g_3 h_3 \tau_1 \tau_2) = 0.
$$

Notice that, if $\epsilon \neq 0$, one can always eliminate second order minors in (21) by adding to u an appropriate quadratic form. In this case the equation takes the form

$$
det \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} + s_1 u_{11} + s_2 u_{22} + s_3 u_{33} + \tau_1 u_{23} + \tau_2 u_{13} + \tau_3 u_{12} + \nu = 0,
$$

while the linearizability condition simplifies to

$$
4 s_1 s_2 s_3 + \nu^2 + \tau_1 \tau_2 \tau_3 - s_1 \tau_1^2 - s_2 \tau_2^2 - s_3 \tau_3^2 = 0.
$$

Remark 1. The condition (22) can be given an invariant formulation [37, 4]. Let $\Omega = du_i \wedge dx^i$ be the standard symplectic form, and $v = v_\Omega$ the associated canonical bivector. Let ω be the effective 3-form corresponding to the equation (21): recall that ω is effective if $\omega \wedge \Omega = 0$. Let us define the quadratic form

$$
\perp^2 (i_X \omega \wedge i_X \omega)
$$

(23)

where $i_X\omega$ is the inner product of ω with a vector X, and $\perp$ is the operator of inner multiplication by the canonical bivector. The linearizability condition (22) is equivalent to the requirement that the quadratic form (23) is degenerate.

Remark 2. The equation $(20)_1$ arises in the theory of improper affine spheres, while the equation $(20)_2$ describes special Lagrangian 3-folds in C^3 [13, 32]. Their contact-equivalent forms are

$$u_{tt} = u_{xx}u_{yy} - u_{xy}^2$$

and

$$u_{xx}u_{yy} - u_{xy}^2 + u_{xx}u_{tt} - u_{xt}^2 + u_{tt}u_{yy} - u_{ty}^2 = 1,$$

respectively (both non-integrable).

Remark 3. Equations of the form (21) and the linearizability condition (22) have a clear projective-geometric interpretation. Let us consider the Lagrangian Grassmannian Λ, which can be (locally) identified with the space of 3×3 symmetric matrices U (see Sect. 4 for more details). The minors of U define the Veronese embedding of Λ into the projective space P^{13} (we will identify Λ with its projective embedding). Thus, symplectic Monge-Ampére equations correspond to hyperplane sections of Λ. Linearizable equations correspond to hyperplanes which are tangential to Λ. Thus, the linearizability condition (22) coincides with the equation of the dual variety to Λ, which is known to be a hypersurface of degree four in $(P^{13})^*$ (see [35, 36, 40] for a general theory behind this example).

The results of this section answer in the negative the question formulated by Joyce in [32]: *do special Lagrangian m-folds in C^m for $m \geq 3$ constitute some kind of higher-dimensional integrable system?*

4 Differential-geometric aspects of integrable equations of the dispersionless Hirota type

In this section we adopt a geometric point of view, and consider (1) as the equation difining a 5-dimensional hypersurface M^5 in the 6-dimensional space with coordinates u_{ij} (which, as we will see shortly, is naturally identified with the Lagrangian Grassmannian Λ). Our aim is to reformulate the integrability conditions in the intrinsic geometric terms.

In Sect. 4.1 we discuss differential geometry of the Lagrangian Grassmannian Λ. We introduce the main object needed to develop differential geometry of hypersurfaces, namely, a canonically defined symmetric cubic form which is invariant under the natural action of the symplectic group $Sp(6)$. This form (to be precise, its conformal class), plays the role of an ambient flat conformal structure.

In Sect. 4.2 we develop differential geometry of hypersurfaces $M^5 \subset \Lambda$. The induced cubic form gives rise to a field of rational normal curves of degree four defined in the projectivization of the tangent bundle TM^5. Thus, a hypersurface of the Lagragian Grassmannian carries an intrinsic irreducible $GL(2)$-structure.

In Sect. 4.3 we show that two-component and three-component hydrodynamic reductions of the equation (1) correspond to bisecant and trisecant submanifolds of M^5, respectively. This allows one to reformulate the integrability as the existence of sufficiently many trisecant submanifolds parametrized by three arbitrary functions of one variable.

Some useful computational formulas are provided in Sect. 4.4.

General aspects of irreducible $GL(2)$-structures in dimension 5 are discussed in Sect. 4.5. We demonstrate that the requirement of the existence of bisecant surfaces imposes strong constraints on the torsion of the corresponding $GL(2)$-structure.

4.1 $Sp(6)$ as a symmetry group of the problem: differential geometry of the Lagrangian Grassmannian

Let us consider a 6-dimensional symplectic space with canonical coordinates $\mathbf{x} = (x^1, x^2, x^3)^t$ and $\mathbf{p} = (p_1, p_2, p_3)^t$, viewed as 3-component column vectors. Lagrangian submanifolds can be parametrized in terms of a generating function $u(x, y, t)$: $\mathbf{x} = (x, y, t)^t$ and $\mathbf{p} = (u_x, u_y, u_t)^t$. Lagrangian planes are defined by the equation $d\mathbf{p} = U d\mathbf{x}$ where U is a 3×3 symmetric matrix (the Hessian matrix of u). Thus, the Lagrangian Grassmannian Λ is 6-dimensional, and can (locally) be identified with the space of 3×3 symmetric matrices. The equation (1) defines a 5-dimensional hypersurface $M^5 \subset \Lambda$; the corresponding solutions $u(x, y, t)$ can be interpreted as Lagrangian submanifolds whose Gaussian images belong to the hypersurface M^5.

The action of the linear symplectic group $Sp(6)$,

$$\begin{pmatrix} d\tilde{\mathbf{p}} \\ d\tilde{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} d\mathbf{p} \\ d\mathbf{x} \end{pmatrix},$$

naturally extends to Λ:

$$\tilde{U} = (AU + B)(CU + D)^{-1}. \tag{24}$$

Here A, B, C, D are 3×3 matrices such that $A^t C = C^t A$, $B^t D = D^t B$, $A^t D - C^t B = id$; notice that the extended action is no longer linear. The transformation law (24) suggests that the action of $Sp(6)$ preserves the class of equations (1), indeed, second order derivatives transform through second order derivatives only. Moreover, since any changes of variables obviously preserve the integrability, the group $Sp(6)$ can be viewed as a natural *equivalence group* of the problem: it maps integrable equations to integrable. Thus, $Sp(6)$ is a point symmetry group of the integrability conditions (16) derived in Sect. 2. The classification of integrable equations of the form (1) has to be performed modulo this equivalence: two $Sp(6)$-related equations should be regarded as 'the same'.

Geometrically, our problem is reduced to the classification of hypersurfaces M^5 of the Lagrangian Grassmannian Λ (satisfying certain 'integrability' conditions to be specified later, see Sect. 4.3), up to the action of $Sp(6)$. Our next goal is to clarify which differential-geometric structures are induced on hypersurfaces of Λ. To do so one first needs to introduce differential-geometric objects which are naturally defined on the Lagrangian Grassmannian, and are invariant under the action of $Sp(6)$.

15

The condition of nontrivial intersection of two infinitesimally close Lagrangian planes defined by symmetric matrices U and $U + dU$, is $\det dU = 0$. This condition is manifestly invariant under the action of $Sp(6)$ as specified by (24). Thus, each tangent space to the Lagrangian Grassmannian Λ is equipped with a cubic cone C defined by the equation $\det dU = 0$. The projectivisation of this cone is known as the 'cubic symmetroid'. The singular locus $V \subset C$ of the cubic cone is specified by the condition $rank\, dU = 1$. Its projectivization is known as the Veronese variety, which is a non-singular algebraic surface of degree four.

The identity

$$\det d\tilde{U} = \frac{\det(A - (AU + B)(CU + D)^{-1}C)}{\det(CU + D)}\ \det dU,$$

which readily follows from (24), implies that the conformal class of the cubic form $\det dU$ is invariant under the action of $Sp(6)$. The converse is also true:

Theorem 2 *The group of conformal automorphisms of the symmetric cubic form* $\det dU$ *is isomorphic to* $Sp(6)$.

The proof consists of a direct calculation of conformal automorphisms of the cubic form $\det dU$. The corresponding infinitesimal generators are presented in Sect. 6. In fact, Theorem 2 can be traced back to the old paper by Cartan [15], Theorem XX. We refer to [5, 27] for further generalizations of this Liouville-type result.

Remark. The theorem is true in any dimension. Thus, for the Grassmannian of Lagrangian 2-planes in a 4-dimensional symplectic space (naturally identified with 2×2 symmetric matrices U), the expression $\det dU$ defines the Lorentzian metric $du_{11}du_{22} - du_{12}^2$ whose conformal automorphisms generate the group $SO(3,2)$. This establishes a well-known isomorphism between $Sp(4)$ and $SO(3,2)$.

4.2 Geometric structures on a hypersurface of the Lagrangian Grassmannian

Let M^5 be a hypersurface of the Lagrangian Grassmannian Λ. Taking a point $s \in M^5$ and intersecting the tangent space $T_s M^5$ with the singular locus V of the cubic cone C in $T_s\Lambda$, one obtains, after a projectivisation, a rational normal curve of degree four (it is well-known that a generic hyperplane section of the Veronese variety V is a rational normal curve). This curve can also be interpreted as a set of matrices of rank one in the tangent space $T_s M^5$ (recall that $T_s\Lambda$ is identified with a space of 3×3 symmetric matrices). Thus, the projectivised tangent bundle of the hypersurface M^5 is equipped with a field of rational normal curves. Since the group of conformal automorphisms of the rational normal curve is isomorphic to $GL(2)$, this specifies on M^5 an irreducible $GL(2)$-structure. Alternatively, one can say that each tangent space to M^5 is canonically identified with a five-dimensional space of binary quartics; in this picture the rational normal curve corresponds to quartics with a quadruple root. Similar structures were discussed in [19, 3] under the name 'paraconformal'.

Remark. Another natural source of irreducible $GL(2)$-structures in dimension five is provided by 5-th order ordinary differential equations with the vanishing Doubrov invariants [17, 19]. Here $GL(2)$-structures arise on the moduli spaces of solutions to the corresponding ODEs. The relation of these structures to equations of the dispersionless Hirota type is yet unclear.

Irreducible $GL(2)$-structures in dimension five possess a number of specific properties, allowing one to equip M^5 with more 'familiar' objects such as the conformal cubic form and the conformal metric. These are defined as follows:

The **conformal cubic form** C_{M^5} is a restriction to M^5 of the cubic form C defined on Λ. The knowledge of this cubic form allows one to reconstruct the $GL(2)$-structure, indeed, the singular locus of the cubic C_{M^5} is the rational normal curve introduced above. Conversely, given a rational normal curve, its bisecant variety is the cubic $C_{M^5} = 0$ [29], p.119. This specifies C_{M^5} uniquely up to a conformal factor.

The **conformal metric** Q_{M^5} is defined by the unique quadratic cone in $T_s M^5$ containing the tangent variety of the rational normal curve, see [29], p.119, for an alternative definition. The tangent variety of the rational normal curve is a complete intersection of Q_{M^5} and C_{M^5}.

One can show that the conformal metric Q_{M^5} and the cubic form C_{M^5} satisfy a system of remarkable relations which have appeared recently in [6] (notice that the signature of the quadratic form in [6] is Euclidean, while in our case the metric is pseudo-Euclidean of the signature $(3, 2)$). Let Q_{ij} and C_{ijk} be the components of Q_{M^5} and C_{M^5} in any coordinate system on M^5 (we point out that these objects are defined up to conformal factors). One can show that the conformal factors can be specified in such a way that the following relations are satisfied:

$$C_{ijk}Q^{kj} = 0 \quad \text{(apolarity condition)},$$

$$C_{jkr}Q^{rs}C_{lns} + C_{ljr}Q^{rs}C_{kns} + C_{klr}Q^{rs}C_{jns} = Q_{jk}Q_{ln} + Q_{lj}Q_{kn} + Q_{kl}Q_{jn};$$

$$(25)$$

these relations still allow simultaneous conformal reparametrizations of the form $C \to \varphi^3 C$, $Q \to \varphi^2 Q$. The explicit coordinate formulas for Q_{M^5} and C_{M^5} are provided in Sect. 4.4.

The normalised conformal metric Q_{M^5} and the cubic form C_{M^5} satisfying the equations (25) carry important geometric information about the hypersurface M^5. Their transformation laws suggest that one should consider the ratio $\sigma = (C_{M^5})^2/(Q_{M^5})^3$ which can be viewed as an analog of the Fubini projective element. By construction, this object is manifestly $Sp(6)$-invariant. We can formulate the following

Conjecture. *A generic five-dimensional hypersurface of the Lagrangian Grassmanian Λ is uniquely defined, modulo $Sp(6)$-equivalence, by its 'symplectic element' σ.*

4.3 Geometric interpretation of the integrability conditions: bisecant surfaces and trisecant 3-folds

As explained in Sect. 4.2, each projectivised tangent space $T_s M^5$ carries a rational normal curve γ of degree four. Among the most natural geometric objects associated with a rational normal curve are its bisecant lines (two-parameter family) and trisecant planes (three-parameter family).

Definition. A *bisecant surface* is a two-dimensional submanifold $\Sigma^2 \subset M^5$ whose projectivised tangent planes are bisecant lines. Similarly, a *trisecant 3-fold* is a three-dimensional submanifold $\Sigma^3 \subset M^5$ whose projectivised tangent spaces are trisecant planes. To be more precise, we will consider *holonomic* trisecant 3-folds which can be defined as follows. Notice first that each tangent space $T_s \Sigma^3$ carries three distinguished directions, namely, those corresponding to the three points of intersection of $PT_s \Sigma^3$ with γ. These directions define a net on Σ^3, which we will call the *characteristic* net. We require the characteristic net to be holonomic (that is, a coordinate net). A similar net exists on any bisecant surface, however, the requirement of holonomicity is superfluous: in two dimensions, any net is automatically holonomic.

Theorem 3 *Bisecant surfaces and holonomic trisecant 3-folds of a hypersurface M^5 correspond to two- and three-component hydrodynamic reductions of the associated equation of the dispersionless Hirota type. Moreover,*
(i) *Each hypersurface M^5 possesses infinitely many bisecant surfaces parametrized by two arbitrary functions of one variable;*
(ii) *A hypersuface M^5 corresponds to an integrable equation if and only if it possesses infinitely many holonomic trisecant 3-folds parametrized by three arbitrary functions of one variable. Thus, the existence of holonomic trisecant 3-folds is a geometric interpretation of the integrability property.*

Proof:

We follow the notation of Sect. 2. Consider an equation of the dispersionless Hirota type represented in the form (5), and take its n-component reduction specified by $a = a(R^1, ..., R^n)$, $b = b(R^1, ..., R^n)$, $c = c(R^1, ..., R^n)$, $p = p(R^1, ..., R^n)$, $q = q(R^1, ..., R^n)$, $f = f(R^1, ...R^n)$. The image of this solution in the Lagrangian Grassmannian is a submanifold $\Sigma^n \subset M^5$ (it would be sufficient for our purposes to restrict to $n = 2, 3$), represented by a symmetric matrix

$$U = \begin{pmatrix} a & b & p \\ b & c & q \\ p & q & f \end{pmatrix} \tag{26}$$

parametrized by $R^1, ..., R^n$. Using (7) and (8), one obtains the following expression for the derivative of U with respect to R^i,

$$\partial_i U = \begin{pmatrix} 1 & \mu^i & \lambda^i \\ \mu^i & (\mu^i)^2 & \lambda^i \mu^i \\ \lambda^i & \lambda^i \mu^i & (\lambda^i)^2 \end{pmatrix} \partial_i a; \tag{27}$$

18

thus, U_i is a matrix of rank one, so that the projectivization of $\partial_i U$ belongs to γ, and the coordinates R^i provide a characteristic net on Σ^n. For $n = 2$ we have a two-dimensional surface Σ^2 parametrized by R^1, R^2. Since both $\partial_1 U$ and $\partial_2 U$ have rank one, the surface Σ^2 is bisecant. As follows from Sect. 2, any equation of the dispersionless Hirota type (not necessarily integrable) possesses infinitely many two-component reductions parametrized by two arbitrary functions of one variable. Thus, any hypersurface M^5 of the Lagrangian Grassmannian possesses an infinity of bisecant surfaces. This establishes the first part (i) of the theorem.

Similarly, any three-component reduction corresponds to a holonomic trisecant 3-fold $\Sigma^3 \subset M^5$ parametrized by R^1, R^2, R^3. Since an integrable equation possesses (by definition) infinitely many three-component reductions parametrized by three arbitrary functions of one variable, the corresponding hypersurface M^5 possesses an infinity of holonomic trisecant 3-folds. This establishes the first part of (ii). To finish the proof one needs to show that, conversely, bisecant surfaces (holonomic trisecant 3-folds) of M^5 corresponds to two-component (three-component) reductions of the associated equation. This can be demonstrated as follows. Let Σ^2 be a bisecant surface represented in the form (26), referred to its characteristic net R^1, R^2. Thus, a, b, c, p, q, f are functions of R^1, R^2 such that the rank of $\partial_i U$ equals one, so that one can introduce the parametrization (27). The compatibility conditions for the equations $\partial_i b = \mu^i \partial_i a$ and $\partial_i p = \lambda^i \partial_i a$ imply

$$\partial_i \partial_j a = \frac{\partial_j \mu^i}{\mu^j - \mu^i} \partial_i a + \frac{\partial_i \mu^j}{\mu^i - \mu^j} \partial_j a \quad \text{and} \quad \partial_i \partial_j a = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \partial_i a + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j} \partial_j a, \qquad (28)$$

respectively. Using (28), the consistency conditions for $\partial_i c = (\mu^i)^2 \partial_i a$ and $\partial_i f = (\lambda^i)^2 \partial_i a$ simplify to $\partial_j \mu^i \partial_i a + \partial_i \mu^j \partial_j a = 0$ and $\partial_j \lambda^i \partial_i a + \partial_i \lambda^j \partial_j a = 0$, compare with (9). Substituting these relations into (28) one obtains

$$\partial_i \partial_j a = 2 \frac{\partial_j \mu^i}{\mu^j - \mu^i} \partial_i a \quad \text{and} \quad \partial_i \partial_j a = 2 \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \partial_i a,$$

which implies the commutativity condition (3). Finally, the last compatibility condition, $\partial_i q = \mu^i \lambda^i \partial_i a$, holds identically. The dispersion relation (8), which connects μ^i and λ^i, follows from the relation $\partial_i f = (\lambda^i)^2 \partial_i a$ after the substitution $f = f(a, b, c, p, q)$. Ultimately, we have recovered all equations defining hydrodynamic reductions, see Sect. 2. Thus, bisecant surfaces correspond to two-component reductions (for trisecant 3-folds, considerations are exactly the same).

We have characterized the integrability as the existence of 'sufficiently many' special submanifolds. This is very reminiscent of the characterization of self-dual 4-manifolds in terms of the existence of α-surfaces [1], or a similar characterization of semi-integrable almost Grassmann structures [2].

4.4 Computational formulas

Here we provide the explicit coordinate formulae for the conformal metric and the conformal cubic form as introduced in Sect. 4.2.

19

The Lagrangian Grassmannian Λ is (locally) parametrized by 3×3 symmetric matrices (26), so that the cubic form C is given by

$$C = \det dU = dadcdf - da(dq)^2 - (db)^2 df + 2dbdpdq - dc(dp)^2.$$

Each tangent space to the Lagrangian Grassmannian (which can also be identified with 3×3 symmetric matrices) carries a cubic cone $C = 0$ consisting of rank two symmetric matrices. Its singular locus coincides with rank one matrices, and can be parametrized in the form

$$\begin{pmatrix} 1 & \mu & \lambda \\ \mu & \mu^2 & \lambda\mu \\ \lambda & \lambda\mu & \lambda^2 \end{pmatrix} \tag{29}$$

where λ and μ are independent parameters. The restriction C_{M^5} of the cubic form C to the hypersurface M^5, defined by the equation $f = f(a, b, c, p, q)$, is obtained by setting

$$df = f_a da + f_b db + f_c dc + f_p dp + f_q dq. \tag{30}$$

This results in

$$C_{M^5} = f_a(da)^2 dc + f_b dadbdc + f_p dadcdp + f_c da(dc)^2 + f_q dadcdq - da(dq)^2 -$$
$$f_a da(db)^2 - f_b(db)^3 - f_p(db)^2 dp - f_c(db)^2 dc - f_q(db)^2 dq + 2dbdpdq - dc(dp)^2, \tag{31}$$

here a, b, c, p, q are local coordinates on M^5. The intersection of the singular locus (29) with the tangent space to M^5 is a rational normal curve with the affine parametrization

$$\gamma = (1, \ \mu, \ \mu^2, \ \lambda, \ \mu\lambda);$$

notice that λ and μ are no longer independent and satisfy, by virtue of (30), the quadratic relation

$$\lambda^2 = f_a + f_b\mu + f_c\mu^2 + f_p\lambda + f_q\lambda\mu,$$

which coincides with the dispersion relation (8). The tangent variety of γ is defined as $\gamma + t\gamma'$ where prime denotes differentiation with respect to μ, and λ is viewed as a function of μ specified by the dispersion relation. Explicitly, the tangent variety is given by

$$(1, \ \mu + t, \ \mu^2 + 2t\mu, \ \lambda + t\lambda', \ \mu\lambda + t(\lambda + \mu\lambda'));$$

here λ' is obtained by implicitly differentiating the dispersion relation with respect to μ,

$$\lambda' = (f_b + 2f_c\mu + f_q\lambda)/(2\lambda - f_q\mu - f_p).$$

One can verify that, up to a conformal factor, there exists a unique quadratic form Q_{M^5}

20

vanishing on the tangent variety of γ,

$$
\begin{aligned}
Q_{M^5} = {} & (4f_a^2 + f_a f_p^2)da^2 + (8f_a f_b + f_b f_p^2 + 2f_a f_p f_q)dadb + (4f_a f_p + f_p^3)dadp \\[4pt]
& + (f_b^2 + 4f_a f_c + f_c f_p^2 + f_a f_q^2)dadc + (2f_b f_p + f_p^2 f_q)dadq \\[4pt]
& + (3f_b^2 + 4f_a f_c + 2f_b f_p f_q)db^2 + (2f_b f_p + 4f_a f_q + 2f_p^2 f_q)dbdp \\[4pt]
& + (8f_b f_c + 2f_c f_p f_q + f_b f_q^2)dbdc + (4f_c f_p + 2f_b f_q + 2f_p f_q^2)dbdq \\[4pt]
& - (4f_a + f_p^2)dp^2 + (2f_b f_q + f_p f_q^2)dpdc - (4f_b + 2f_p f_q)dpdq \\[4pt]
& + (4f_c^2 + f_c f_q^2)dc^2 + (4f_c f_q + f_q^3)dcdq - (4f_c + f_q^2)dq^2;
\end{aligned}
\tag{32}
$$

(one needs to use the dispersion relation in order to verify the vanishing of Q_{M^5} on the tangent variety). Representing Q_{M^5} as a 5×5 symmetric matrix and calculating its determinant one obtains

$$
\det Q_{M^5} = 3(f_b^2 + f_b f_p f_q - f_a f_q^2 - f_c f_p^2 - 4f_a f_c)^4 = 3\triangle^4;
$$

we point out that $\triangle = 0$ if and only if the corresponding dispersion relation (8) defines a reducible conic. Thus, under the assumption of the irreducibility, the conformal metric Q_{M^5} is non-degenerate. The cubic form (31) and the conformal metric (32) satisfy the relations

$$
C_{ijk}Q^{kj} = 0,
$$

$$
C_{jkr}Q^{rs}C_{lns} + C_{ljr}Q^{rs}C_{kns} + C_{klr}Q^{rs}C_{jns} = \tfrac{1}{27\triangle^2}(Q_{jk}Q_{ln} + Q_{lj}Q_{kn} + Q_{kl}Q_{jn}).
$$

Thus, the normalization $C \to 3\sqrt{3}\,\triangle C$ results in the identities (25).

4.5   General aspects of irreducible $Gl(2)$-structures

Let P^4 be a projective space with coordinates $(x^0 : x^1 : x^2 : x^3 : x^4)$. A rational normal curve γ of degree four can be parametrized in the form $\gamma = (1 : t : t^2 : t^3 : t^4)$. Its bisecant variety is a cubic hypersurface defined by the equation

$$
\det \begin{pmatrix} x^0 & x^1 & x^2 \\ x^1 & x^2 & x^3 \\ x^2 & x^3 & x^4 \end{pmatrix} = 0;
$$

notice that γ can be recovered as a singular locus of its bisecant variety. The tangent variety of γ is contained in a unique quadric hypersurface

$$
x^0 x^4 - 4x^1 x^3 + 3(x^2)^2 = 0.
$$

The 'curved' analog of this picture is the following. Let M^5 be a 5-dimensional manifold with a field of rational normal curves of degree four specified in the projectivization

of each tangent space. One can choose a frame of 1-forms $\omega^1, \omega^2, \omega^3, \omega^4, \omega^5$ such that the equation of the bisecant variety of γ takes the canonical form $C = 0$ where

$$C = \det \begin{pmatrix} \omega^1 & \omega^2 & \omega^3 \\ \omega^2 & \omega^3 & \omega^4 \\ \omega^3 & \omega^4 & \omega^5 \end{pmatrix};$$

such frame is defined up to a natural $GL(2)$-equivalence, leading to the following structure equations:

$$
\begin{aligned}
d\omega^1 &= \Omega \wedge \omega^1 + 2\varphi \wedge \omega^1 + 4\eta \wedge \omega^2 + ..., \\
d\omega^2 &= \Omega \wedge \omega^2 + \varphi \wedge \omega^2 + \psi \wedge \omega^1 + 3\eta \wedge \omega^3 + ..., \\
d\omega^3 &= \Omega \wedge \omega^3 + 2\psi \wedge \omega^2 + 2\eta \wedge \omega^4 + ..., \\
d\omega^4 &= \Omega \wedge \omega^4 - \varphi \wedge \omega^4 + 3\psi \wedge \omega^3 + \eta \wedge \omega^5 + ..., \\
d\omega^5 &= \Omega \wedge \omega^5 - 2\varphi \wedge \omega^5 + 4\psi \wedge \omega^4 + ...
\end{aligned}
\tag{33}
$$

Here dots denote linear combinations of the terms $\omega^i \wedge \omega^j$ which are responsible for the 'torsion'. The structure equations for the secondary forms $\Omega, \varphi, \psi, \eta$, restricted to the fiber $\omega^i = 0$, take the standard $GL(2)$-form:

$$d\Omega = 0, \quad d\eta = \varphi \wedge \eta, \quad d\varphi = 2\eta \wedge \psi, \quad d\psi = -\varphi \wedge \psi.$$

Introducing $\omega = (\omega^1, \omega^2, \omega^3, \omega^4, \omega^5)^t$, one can represent (33) in a compact form

$$d\omega = (\Omega E + \psi E_1 + \varphi H + \eta E_2) \wedge \omega + T(\omega, \omega),\tag{34}$$

where T represents the torsion, E is the identity matrix, and E_1, H, E_2 define the 5-dimensional irreducible representation of $SL(2)$:

$$
E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}, \quad
H = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \quad
E_1 = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
$$

We point out that bisecant surfaces and trisecant three-folds make perfect sense for abstract $GL(2)$ structures. Recall that if the $GL(2)$ structure comes from a hypersurface of the Lagrangian Grassmanian, it automatically possesses infinitely many bisecant surfaces parametrized by two arbitrary functions of one variable. For abstract $GL(2)$ structures, the existence of bisecant surfaces imposes strong restrictions on the torsion T, which can be represented in a simple geometric form:

Theorem 4 *An abstract $GL(2)$ structure possesses infinitely many bisecant surfaces if and only if, for any two vectors X and Y which belong to the rational normal curve, one has $C(X, Y, T(X, Y)) = 0$.*

Proof:

Let us look for bisecant surfaces in the form $\omega = \alpha \, \mathbf{a} \, dR^1 + \beta \, \mathbf{b} \, dR^2$ where R^1, R^2 are the characteristic coordinates, $\mathbf{a}$ and $\mathbf{b}$ are (column) vectors which belong to the rational normal curve γ, that is,

$$\mathbf{a} = (1 : a : a^2 : a^3 : a^4)^t, \quad \mathbf{b} = (1 : b : b^2 : b^3 : b^4)^t,$$

and a, b, α, β are certain functions of R^1, R^2. Substituting this ansatz into (34) and using the identities

$$E_1\mathbf{a} = \mathbf{a}', \quad E_1\mathbf{b} = \mathbf{b}',$$
$$H\mathbf{a} = 2\mathbf{a} - a\mathbf{a}', \quad H\mathbf{b} = 2\mathbf{b} - b\mathbf{b}',$$
$$E_2\mathbf{a} = 4a\mathbf{a} - a^2\mathbf{a}', \quad E_2\mathbf{b} = 4b\mathbf{b} - b^2\mathbf{b}',$$

where

$$\mathbf{a}' = (0 : 1 : 2a : 3a^2 : 4a^3)^t, \quad \mathbf{b} = (0 : 1 : 2b : 3b^2 : 4b^3)^t,$$

one arrives at

$$\mathbf{a}(d\alpha - \alpha\Omega - 2\alpha\varphi - 4a\alpha\eta) \wedge dR^1 + \mathbf{b}(d\beta - \beta\Omega - 2\beta\varphi - 4b\beta\eta) \wedge dR^2 +$$

$$\mathbf{a}'(\alpha da - \alpha\psi + \alpha a\varphi + \alpha a^2\eta) \wedge dR^1 + \mathbf{b}'(\beta db - \beta\psi + \beta b\varphi + \beta b^2\eta) \wedge dR^2 = \tag{35}$$

$$2\alpha\beta \, T(\mathbf{a}, \mathbf{b}) \, dR^1 \wedge dR^2.$$

Thus, for the consistency of these equations one has to require

$$T(\mathbf{a}, \mathbf{b}) \in \text{span } \{\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}'\},$$

which is equivalent to the statement of the theorem. Once this condition is satisfied, one obtains a system of four first order PDEs for a, b, α, β by collecting similar terms in (35). Up to reparametrizations $R^i \to \varphi^i(R^i)$, the general solution of this system depends on two arbitrary functions of one variable.

Theorem 4 establishes necessary conditions for the realizability of a $GL(2)$ structure on a hypersurface of the Lagrangian Grassmannian. Further development of the general theory of abstract $GL(2)$ structures is beyond the scope of this paper, and will be addressed elsewhere.

5 Integrability in more than three dimensions

The integrability of three-dimensional equations of the form (1) was defined as the existence of n-component hydrodynamic reductions (2) parametrized by n arbitrary functions of a single variable. As demonstrated in [23, 24], this approach readily generalizes to any dimension: a d-dimensional PDE

$$F(u_{x^i x^j}) = 0 \tag{36}$$

for a function u of d independent variables $x^1, ..., x^d$ is said to be integrable if it possesses infinitely many n-component hydrodynamic reductions parametrized by $(d-2)n$ arbitrary functions of a single variable. In this case equations (2) are replaced by $d-1$ commuting $(1+1)$-dimensional systems of hydrodynamic type. Among the known four-dimensional integrable examples one should primarily mention the 'first heavenly equation',

$$u_{xy}u_{zt} - u_{xt}u_{zy} = 1,$$

as well as its equivalent forms,

$$u_{tx} + u_{zy} + u_{xx}u_{yy} - u_{xy}^2 = 0$$

and

$$u_{tt} = u_{xy}u_{zt} - u_{xt}u_{zy},$$

known as the 'second heavenly' and the 'Grant equation', respectively [47, 28]. It was demonstrated in [23, 24] that n-component reductions of these equations are parametrized by $2n$ arbitrary functions of a single variable.

An interesting six-dimensional integrable generalization of the heavenly equation,

$$u_{t\tilde{t}} + u_{z\tilde{z}} + u_{tx}u_{zy} - u_{ty}u_{zx} = 0,$$

arises in the context of $sdiff(\Sigma^2)$ self-dual Yang-Mills equations [48]. Its n-component reductions are parametrized by $4n$ arbitrary functions of a single variable [24]. Notice that all these examples belong to the class of symplectic Monge-Ampére equations as introduced in Sect. 3.

Although the general problem of classification of multi-dimensional integrable equations can be approached in a similar way, the method of Sect. 2 leads to quite complicated analysis. One way to bypass lengthy calculations is based on the following simple idea: suppose we want to classify four-dimensional integrable equations of the form (36) for a function $u(x, y, z, t)$. Let us look for travelling wave solutions in the form

$$u(X, Y, Z) = u(x + \alpha t, \ y + \beta t, \ z + \gamma t).$$

The substitution of this ansatz into (36) leads to a three-dimensional equation which must be integrable for *any* values of constants α, β, γ. Since, in three dimensions, the integrability conditions are explicitly known, this provides strong restrictions on the original function F, which are *necessary* for the integrability. The philosophy of this approach is well familiar from the soliton theory: symmetry reductions of integrable systems must be themselves integrable.

Thus, for the first heavenly equation, traveling wave solutions are governed by

$$\alpha(u_{XY}u_{XZ} - u_{XX}u_{YZ}) + \gamma(u_{XY}u_{ZZ} - u_{XZ}u_{YZ}) = 1,$$

which is a three-dimensional symplectic Monge-Ampere equation. One can show that it is indeed integrable (in fact, linearizable) for any values of constants.

We hope to return to the multi-dimensional case in subsequent publications.

6 $Sp(6)$-action on the moduli space of integrable equations

In this section we investigate the action of the equivalence group $Sp(6)$ on the moduli space of integrable equations of the dispersionless Hirota type (recall that both $Sp(6)$ and the moduli space have coinciding dimensions equal to 21). Our main result states that this action has an open orbit.

For a 3×3 symmetric matrix $U = [u_{ij}]$, the Lie algebra of the group of conformal automorphisms of the cubic form $det\ dU$ is spanned by 21 vector fields which generate the Lie algebra of the symplectic Lie group $Sp(6)$:

$$\mathbf{X_{11}} = \frac{\partial}{\partial u_{11}} \ , \ \mathbf{X_{12}} = \frac{\partial}{\partial u_{12}} \ , \ \mathbf{X_{13}} = \frac{\partial}{\partial u_{13}} \ , \ \mathbf{X_{22}} = \frac{\partial}{\partial u_{22}} \ , \ \mathbf{X_{23}} = \frac{\partial}{\partial u_{23}} \ , \ \mathbf{X_{33}} = \frac{\partial}{\partial u_{33}},$$

$$\mathbf{J_1} = 2u_{11}\frac{\partial}{\partial u_{11}} + u_{12}\frac{\partial}{\partial u_{12}} + u_{13}\frac{\partial}{\partial u_{13}},$$
$$\mathbf{J_2} = 2u_{22}\frac{\partial}{\partial u_{22}} + u_{21}\frac{\partial}{\partial u_{21}} + u_{23}\frac{\partial}{\partial u_{23}},$$
$$\mathbf{J_3} = 2u_{33}\frac{\partial}{\partial u_{33}} + u_{31}\frac{\partial}{\partial u_{31}} + u_{32}\frac{\partial}{\partial u_{32}},$$

$$\mathbf{L_{12}} = 2u_{12}\frac{\partial}{\partial u_{11}} + u_{22}\frac{\partial}{\partial u_{12}} + u_{23}\frac{\partial}{\partial u_{13}},$$
$$\mathbf{L_{13}} = 2u_{13}\frac{\partial}{\partial u_{11}} + u_{33}\frac{\partial}{\partial u_{13}} + u_{32}\frac{\partial}{\partial u_{12}},$$
$$\mathbf{L_{21}} = 2u_{21}\frac{\partial}{\partial u_{22}} + u_{11}\frac{\partial}{\partial u_{21}} + u_{13}\frac{\partial}{\partial u_{23}},$$
$$\mathbf{L_{23}} = 2u_{23}\frac{\partial}{\partial u_{22}} + u_{33}\frac{\partial}{\partial u_{23}} + u_{31}\frac{\partial}{\partial u_{21}},$$
$$\mathbf{L_{31}} = 2u_{31}\frac{\partial}{\partial u_{33}} + u_{11}\frac{\partial}{\partial u_{31}} + u_{12}\frac{\partial}{\partial u_{23}},$$
$$\mathbf{L_{32}} = 2u_{32}\frac{\partial}{\partial u_{33}} + u_{22}\frac{\partial}{\partial u_{32}} + u_{21}\frac{\partial}{\partial u_{13}},$$

$$\mathbf{H_1} = u_{11}{}^2\frac{\partial}{\partial u_{11}} + u_{11}u_{12}\frac{\partial}{\partial u_{12}} + u_{11}u_{13}\frac{\partial}{\partial u_{13}} + u_{12}{}^2\frac{\partial}{\partial u_{22}} + u_{12}u_{13}\frac{\partial}{\partial u_{23}} + u_{13}{}^2\frac{\partial}{\partial u_{33}},$$
$$\mathbf{H_2} = u_{22}{}^2\frac{\partial}{\partial u_{22}} + u_{22}u_{21}\frac{\partial}{\partial u_{21}} + u_{22}u_{23}\frac{\partial}{\partial u_{23}} + u_{21}{}^2\frac{\partial}{\partial u_{11}} + u_{21}u_{23}\frac{\partial}{\partial u_{13}} + u_{23}{}^2\frac{\partial}{\partial u_{33}},$$
$$\mathbf{H_3} = u_{33}{}^2\frac{\partial}{\partial u_{33}} + u_{33}u_{31}\frac{\partial}{\partial u_{31}} + u_{33}u_{32}\frac{\partial}{\partial u_{32}} + u_{31}{}^2\frac{\partial}{\partial u_{11}} + u_{31}u_{32}\frac{\partial}{\partial u_{21}} + u_{32}{}^2\frac{\partial}{\partial u_{22}},$$

$$\mathbf{P_1} = 2u_{12}u_{13}\frac{\partial}{\partial u_{11}} + (u_{12}u_{23} + u_{13}u_{22})\frac{\partial}{\partial u_{12}} + (u_{12}u_{33} + u_{13}u_{23})\frac{\partial}{\partial u_{13}} + 2u_{22}u_{23}\frac{\partial}{\partial u_{22}}$$

$$+(u_{22}u_{33} + u_{23}{}^2)\frac{\partial}{\partial u_{23}} + 2u_{23}u_{33}\frac{\partial}{\partial u_{33}},$$

$$\mathbf{P_2} = 2u_{21}u_{23}\frac{\partial}{\partial u_{22}} + (u_{21}u_{13} + u_{23}u_{11})\frac{\partial}{\partial u_{21}} + (u_{21}u_{33} + u_{23}u_{13})\frac{\partial}{\partial u_{23}} + 2u_{11}u_{13}\frac{\partial}{\partial u_{11}}$$

$$+(u_{11}u_{33} + u_{13}{}^2)\frac{\partial}{\partial u_{13}} + 2u_{13}u_{33}\frac{\partial}{\partial u_{33}},$$

$$\mathbf{P_3} = 2u_{31}u_{32}\frac{\partial}{\partial u_{33}} + (u_{31}u_{12} + u_{32}u_{11})\frac{\partial}{\partial u_{31}} + (u_{31}u_{22} + u_{32}u_{12})\frac{\partial}{\partial u_{32}} + 2u_{11}u_{12}\frac{\partial}{\partial u_{11}}$$

$$+(u_{11}u_{22} + u_{12}{}^2)\frac{\partial}{\partial u_{12}} + 2u_{12}u_{22}\frac{\partial}{\partial u_{22}}.$$

Given a PDE of the form $F(u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}) = 0$, we will look for its infinitesimal symmetries by solving the determining equation $L_X F|_{F=0} = 0$, where X is a linear combination of the 21 vector fields presented above (notice that the answer may not coincide with the full algebra of Lie-point symmetries: we consider only those symmetries which belong to the equivalence group $Sp(6)$). Below we list particular examples of integrable equations which possess symmetry algebras of different dimensions (it is worth noting that any two equations with different symmetry algebras are automatically non-equivalent).

Example 1. The 2-dimensional linear wave equation, $u_{11} + u_{22} - u_{33} = 0$, possesses nine infinitesimal symmetries:

$$\mathbf{X}_{12},\ \ \mathbf{X}_{13},\ \ \mathbf{X}_{23},\ \ \mathbf{X}_{11} + \mathbf{X}_{33},\ \ \mathbf{X}_{22} + \mathbf{X}_{33},\ \ \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3,\ \ \mathbf{L}_{12} - \mathbf{L}_{21},\ \ \mathbf{L}_{13} + \mathbf{L}_{31},\ \ \mathbf{L}_{23} + \mathbf{L}_{32}.$$

One can prove that the existence of nine infinitesimal symmetries is necessary and sufficient for the linearizability of a general equation of the form (1), [43].

Example 2. The dKP equation, $u_{22} - u_{13} + \frac{1}{2}u_{11}^2 = 0$, possesses seven infinitesimal symmetries:

$$\mathbf{X}_{12},\ \ \mathbf{X}_{23},\ \ \mathbf{X}_{33},\ \ \mathbf{X}_{13} + \mathbf{X}_{22},\ \ \mathbf{X}_{11} + \mathbf{L}_{31},\ \ \mathbf{J}_1 + 2\mathbf{J}_2 + 3\mathbf{J}_3,\ \ 2\mathbf{L}_{32} + \mathbf{L}_{21}.$$

It is likely that there exist no integrable equations with eight symmetries.

Example 3. The Boyer-Finley equation, $u_{11} + u_{22} - e^{u_{33}} = 0$, possesses six infinitesimal symmetries:

$$\mathbf{X}_{12},\ \ \mathbf{X}_{13},\ \ \mathbf{X}_{23},\ \ \mathbf{X}_{11} - \mathbf{X}_{22},\ \ \mathbf{J}_1 + \mathbf{J}_2 + 2\mathbf{X}_{33},\ \ \mathbf{L}_{12} - \mathbf{L}_{21}.$$

Example 4. The degeneration of the dispersionless Hirota equation, $u_{12} - u_{13} - e^{u_{23}} = 0$, possesses five infinitesimal symmetries:

$$\mathbf{X}_{11},\ \ \mathbf{X}_{22},\ \ \mathbf{X}_{33},\ \ \mathbf{J}_1 + \mathbf{X}_{23},\ \ \mathbf{X}_{12} + \mathbf{X}_{13}.$$

Example 5. The equation $e^{u_{11}} + e^{u_{22}} - e^{u_{33}} = 0$ possesses four infinitesimal symmetries:

$$\mathbf{X}_{12},\ \ \mathbf{X}_{13},\ \ \mathbf{X}_{23},\ \ \mathbf{X}_{11} + \mathbf{X}_{22} + \mathbf{X}_{33}.$$

Example 6. The dispersionless Hirota-type equation for the BKP hierarchy, $e^{u_{13}} + e^{u_{13}} + e^{u_{23}} = e^{u_{13}+u_{13}+u_{23}}$, possesses three infinitesimal symmetries:

$$\mathbf{X}_{11}, \quad \mathbf{X}_{22}, \quad \mathbf{X}_{33}.$$

Remark. We point out that the existence of 'many' symmetries is not related to the integrability: for instance, both equations (20), which are not integrable, possess 8-dimensional symmetry algebras isomorphic to $SL(3,R)$ and $SU(3,R)$, respectively [4]. Thus, the equation $Hess\ u = 1$ possesses eight infinitesimal symmetries:

$$\mathbf{L}_{12}, \quad \mathbf{L}_{13}, \quad \mathbf{L}_{21}, \quad \mathbf{L}_{23}, \quad \mathbf{L}_{31}, \quad \mathbf{L}_{32}, \quad \mathbf{J}_1 - \mathbf{J}_2, \quad \mathbf{J}_1 - \mathbf{J}_3.$$

The equation $Hess\ u = u_{11} + u_{22} + u_{33}$ also possesses eight infinitesimal symmetries:

$$\mathbf{X}_{22} - \mathbf{X}_{11}, \quad \mathbf{X}_{33} - \mathbf{X}_{11}, \quad \mathbf{P}_1 + \mathbf{X}_{23} \quad \mathbf{P}_2 + \mathbf{X}_{13}, \quad \mathbf{P}_3 + \mathbf{X}_{12}, \quad \mathbf{L}_{12} + \mathbf{L}_{21}, \quad \mathbf{L}_{13} + \mathbf{L}_{31}, \quad \mathbf{L}_{23} + \mathbf{L}_{32}.$$

The main result of this section is the following

Theorem 5 *The action of the equivalence group $Sp(6)$ on the moduli space of integrable equations of the dispersionless Hirota type has an open orbit.*

This fact is, in a sense, surprising: it establishes the existence of a 'universal' equation with no symmetries, which generates an open part of the moduli space under the action of $Sp(6)$. In particular, one should be able to obtain all equations with non-trivial symmetries by taking appropriate degenerations of this universal equation.

Proof of Theorem 2

The main idea of the proof is to prolong the 21 infinitesimal generators $\mathbf{X}_{11} - \mathbf{P}_3$ to the moduli space of solutions of the involutive system (16). We point out that, since third order derivatives of f are explicitly known, this moduli space can be identified with the values of f and its partial derivatives f_i, f_{ij} up to second order (21 parameters altogether). The prolongation can be calculated as follows:

(1) Following the standard notation adopted in the symmetry analysis of differential equations [31, 44], we introduce the variables

$$x^1 = u_{11}, \quad x^2 = u_{12}, \quad x^3 = u_{13}, \quad x^4 = u_{22}, \quad x^5 = u_{23}, \quad u = u_{33},$$

and rewrite the above 21 generators in the form

$$\xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u};$$

here ξ^i and η are certain functions of x and u. In this notation, a dispersionless Hirota-type equation is represented in the form $u = u(x^1, ..., x^5)$ (the function u is denoted by f in Sect. 2).

(2) Prolong infinitesimal generators to the second order jet space with coordinates u, u_i, u_{ij},

$$\xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_i} + \zeta_{ij} \frac{\partial}{\partial u_{ij}},$$

where ζ_i and ζ_{ij} are calculated according to the standard prolongation formulae

$$\zeta_i = D_i(\eta) - u_k D_i(\xi^k), \quad \zeta_{ij} = D_j \zeta_i - u_{ik} D_j(\xi^k);$$

here D_i are the operators of total differentiation.

(3) To eliminate the $\frac{\partial}{\partial x^i}$-terms, subtract the linear combination of total derivatives $\xi^i D_i$ from the prolonged operators. It is sufficient to keep only the following terms in D_i:

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + u_{ijk} \frac{\partial}{\partial u_{jk}};$$

notice that, since u_{ijk} are explicit functions of lower order derivatives, the resulting operators will be well-defined on the 21-dimensional space with coordinates u, u_i, u_{ij}. Although these operators will depend on the variables x^i as parameters (indeed, the isomorphism of the moduli space with the space u, u_i, u_{ij} depends on the choice of a point in the x-space), all algebraic properties of these operators will be x-independent.

(4) Finally, the dimension of the maximal $Sp(6)$-orbit equals the rank of the 21×21 matrix of coefficients of these operators. It remains to point out that this rank equals 21 for any 'random' choice of numerical values for x^i, u, u_i, u_{ij} (however, it equals $21 - r$ for any example with r symmetries).

7 Concluding remarks

We would like to formulate a list of natural questions which are left beyond the scope of this paper.

— It was demonstrated that the integrability of equations of the form (1) can be interpreted as the existence of 'sufficiently many' trisecant submanifolds of the corresponding hypersurfaces in the Lagrangian Grassmannian. It would be desirable to have a tensor characterization of this condition in terms of the metric Q_{ij} and the cubic form C_{ijk} introduced in Sect. 4.2. Presumably, these conditions will involve the Weyl tensor of the conformal metric Q_{ij} (which is not conformally flat in general). Alternatively, one may try to express the integrability conditions using the structure equations of the corresponding $GL(2)$-structure.

— We have shown that any hypersurface of the Lagrangian Grassmannian possesses an intrinsic $GL(2)$-structure. In would be desirable to develop a general theory of hypersurfaces of the Lagrangian Grassmannian, as well as the theory of abstract $GL(2)$-structures, and to establish the embedding theorem. Notice that the theory of curves in the Lagrangian Grassmannian received some attention, see [45, 42, 55].

— One of our main observations is the existence of a universal 'master-equation' which generates all other integrable equations of the dispersionless Hirota type via various (singular) limits. It seems to be very important to understand the reduction procedure geometrically, and to obtain the master-equation explicitly in terms of the appropriate special functions (one shouldn't expect simple formulas since even particular examples contain transcendental functions such as the elliptic theta-functions and solutions to the Chazy equation).

To some extent, the picture resembles the situation with the classification of (non-holonomic) cyclids of Dupin in E^4: as demonstrated in [46], up to the action of the Lie sphere group $SO(5,2)$, there exists a unique representative, namely, the stereographic projection of the isoparametric hypersurface of Cartan. Coinciding dimensions of the equivalence groups, dim $Sp(6)$=dim $SO(5,2)$=21, add to this similarity. Notice, however, that any nonholonomic Dupin hypersurface in E^4 is Lie-homogeneous, while our generic example possesses no continuous $Sp(6)$-symmetries.

Acknowledgements

References

[1] M.F. Atiyah, N.J. Hitchin and I.M. Singer, Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. London Ser. A **362**, no. 1711 (1978) 425-461.

[2] M.A. Akivis and V.V. Goldberg, Semiintegrable almost Grassmann structures, Diff. Geom. Appl. **10** (1999) 257-294.

[3] T.N. Bailey and M.G. Eastwood, Complex paraconformal manifolds – their differential geometry and twistor theory, Forum Math. **3** (1991) 61-103.

[4] B. Banos, On symplectic classification of effective 3-forms and Monge-Ampére equations, Diff. Geom. Appl. **19** (2003) 147-166.

[5] W. Bertram, Un theoréme de Liouville pour les algébres de Jordan, Bull. Soc. Math. France **124**, no. 2 (1996) 299-327.

[6] M. Bobienski and P. Nurowski, Irreducible SO(3) geometry in dimension five, math.DG/0507152.

[7] L.V. Bogdanov and B.G. Konopelchenko, On dispersionless BKP hierarchy and its reductions, J. Nonlinear Math. Phys. **12**, suppl. 1 (2005) 64-73.

[8] A. Boyarsky, A. Marshakov, O. Ruchayskiy, P. Wiegmann and A. Zabrodin, Associativity equations in dispersionless integrable hierarchies, Phys. Lett. B **515** (2001) 483-492.

[9] C.P. Boyer and J.D. Finley, Killing vectors in self-dual Euclidean Einstein spaces, J. Math. Phys. **23** 1126-1130 (1982)

[10] H.W. Braden and A. Marshakov, WDVV equations as functional relations, Phys. Lett. B **451** (2002) 376-383.

[11] V.M. Buchstaber, D.V. Leikin and M.V. Pavlov, The Egorov hydrodynamic chains, the Chazy equation and $SL(2, C)$ group, Func. Anal. Appl. **37** (2003) 13-26.

[12] C.P. Boyer and J.D. Finley, Killing vectors in self-dual Euclidean Einstein spaces, J. Math. Phys. **23** (1982) 1126-1130.

[13] E. Calabi, Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jorgens, Michigan Math. J. **5** (1958) 105-126.

[14] R. Carroll and Y. Kodama, Solution of the dispersionless Hirota equations, J. Phys. A **28**, no. 22 (1995) 6373-6387.

[15] E. Cartan, Les groupes de transformations continus, infinis, simples, Ann. Sci. cole Norm. Sup. (3) **26** (1909) 93-161.

[16] J. Chazy, Sur les équations différentiellles dont l'intégrale générale possède un coupure essentielle mobile, C.R. Acad. Sc. Paris, **150** (1910) 456-458.

[17] B. Doubrov, Contact trivialization of ordinary differential equations, Differential geometry and its applications (Opava, 2001), 73-84, Math. Publ., 3, Silesian Univ. Opava, Opava, 2001.

[18] M. Dunajski, L.J. Mason and P. Tod, Einstein-Weyl geometry, the dKP equation and twistor theory, J. Geom. Phys. **37**, no. 1-2 (2001) 63-93.

[19] M. Dunajski and P. Tod, Paraconformal geometry of nth-order ODEs, and exotic holonomy in dimension four, J. Geom. Phys. **56** (2006) 1790-1809.

[20] E.V. Ferapontov and K.R. Khusnutdinova, On the integrability of (2+1)-dimensional quasilinear systems, Comm. Math. Phys. **248** (2004) 187-206.

[21] E.V. Ferapontov and K.R. Khusnutdinova, The characterization of two-component (2+1)-dimensional integrable systems of hydrodynamic type, J. Phys. A: Math. Gen. **37** (2004) 2949-2963.

[22] E.V. Ferapontov, D. A. Korotkin and V.A. Shramchenko, Boyer-Finley equation and systems of hydrodynamic type, Class. Quantum Grav. **19**, no. 24 (2002) L205-L210.

[23] E.V. Ferapontov and M.V. Pavlov, Hydrodynamic reductions of the heavenly equation, Class. Quantum Grav. **20** (2003) 2429-2441.

[24] E.V. Ferapontov and K.R. Khusnutdinova, Hydrodynamic reductions of multidimensional dispersionless PDEs: the test for integrability, J. Math. Phys. **45** (2004) 2365-2377.

[25] E.V. Ferapontov, K.R. Khusnutdinova and M.V. Pavlov, Classification of integrable (2+1)-dimensional quasilinear hierarchies, Theor. Math. Phys. **144** (2005) 35-43.

[26] J. Gibbons and S.P. Tsarev, Reductions of the Benney equations, Phys. Lett. A **211** (1996) 19-24; J. Gibbons and S.P. Tsarev, Conformal maps and reductions of the Benney equations, Phys. Lett. A **258** (1999) 263-270.

[27] S. Gindikin and S. Kaneyuki, On the automorphism group of the generalized conformal structure of a symmetric R-space, Diff. Geom. Appl. **8** (1998) 21-33.

[28] J.D.E. Grant, On self-dual gravity, Phys. Rev. D **47** (1993) 2606-2612.

[29] J. Harris, Algebraic geometry: a first course, Springer-Verlag, New York (1992), 328pp.

[30] L.K. Hoevenaars and R. Martini, Second order reductions of the WDVV equations related to classical Lie algebras, Lett. Math. Phys. **71** (2005) 83-88.

[31] N. Kh. Ibragimov, Transformation groups applied to mathematical physics. Translated from the Russian. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht (1985) 394 pp.

[32] D. Joyce, Lectures on special Lagrangian geometry, in: Global theory of minimal surfaces, Clay Math. Proc., 2, Amer. Math. Soc., Providence, RI (2005) 667-695.

[33] I.M. Krichever, The τ-function of the universal Whitham hierarchy, matrix models and topological field theories, Comm. Pure Appl. Math. **47**, no. 4 (1994) 437-475.

[34] I.M. Krichever, A. Marshakov and A. Zabrodin, Integrable structure of the Dirichlet boundary problem in multiply-connected domains, Comm. Math. Phys. **259**, no. 1 (2005) 1-44.

[35] J.M. Landsberg and L. Manivel, The projective geometry of Freudental's magic square, Journal of Algebra **239** (2001) 477-512.

[36] J.M. Landsberg and L. Manivel, Construction and classification of complex simple Lie algebras via projective geometry, Sel. Math., New ser. **8** (2002) 137-159.

[37] V.V. Lychagin, V.N. Rubtsov and I.V. Chekalov, A classification of Monge-Ampre equations, Ann. Sci. cole Norm. Sup. **26**, no. 4 (1993) 281-308.

[38] M. Mañas, L. Martnez Alonso and E. Medina, Reductions and hodograph solutions of the dispersionless KP hierarchy. J. Phys. A **35**, no. 2 (2002) 401-417.

[39] A. Marshakov, P. Wiegmann and A. Zabrodin, Integrable structure of the Dirichlet boundary problem in two dimensions, Comm. Math. Phys. **227** (2002) 131-153.

[40] S. Mukai, Simple Lie algebras and Legendre variety, Preprint, translation of the article in Nagoya Suri Forum, **3** (1996) 1-12.

[41] M.V. Pavlov, New integrable (2+1)-equations of hydrodynamic type, Russian Math. Surv., **58**, no. 2 (2003) 384-385; M.V. Pavlov, Classifying integrable Egoroff hydrodynamic chains, Theor. Math. Phys. **138** (2004) 45-58.

[42] G. Mari Beffa, On completely integrable geometric evolutions of curves of Lagrangian planes, to appear in the Bulletin of the Royal Acad. of Edinburg.

[43] P. Nurowski, private communication.

[44] P. Olver, Applications of Lie groups to differential equations, Graduate Texts in Mathematics, 107. Springer-Verlag, New York (1986) 497 pp.

[45] V. Ovsienko, Lagrange Schwarzian derivative and symplectic Sturm theory, Annales de la Faculté des Sciences de Toulouse, **2**, no. 1 (1993) 73-96.

[46] U. Pinkall, Dupinishe Hyperflachen in E^4, Manuscripta Math. **51** (1985) 89-119.

[47] J.F. Plebański, Some solutions of complex Einstein equations, J. Math. Phys. **16** (1975) 2395-2402.

[48] J.F. Plebański and M. Przanowski, The Lagrangian for a self-dual gravitational field as a limit of the SDYM Lagrangian, Phys. lett. A **212** (1996) 22-28.

[49] K. Takasaki and T. Takebe, Integrable hierarchies and dispersionless limit, Rev. Math. Phys. **7**, no. 5 (1995) 743-808.

[50] S.P. Tsarev, Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, Soviet Math. Dokl. **31** (1985) 488-491; S.P. Tsarev, The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method, Math. USSR Izvestiya **37** (1991) 397-419.

[51] P.B. Wiegmann and A. Zabrodin, Conformal maps and integrable hierarchies, Comm. Math. Phys. **213**, no. 3 (2000) 523-538.

[52] E.A. Zabolotskaya and R.V. Khokhlov, Quasi-plane waves in the nonlinear acoustics of confined beams, Sov. Phys. Acoust. **15** (1969) 35-40.

[53] A.V. Zabrodin, The dispersionless limit of the Hirota equations in some problems of complex analysis, Theor. Math. Phys. **129**, no. 2 (2001) 1511-1525.

[54] E.V. Zakharov, Dispersionless limit of integrable systems in $2 + 1$ dimensions, in: Singular Limits of Dispersive Waves, Ed. N.M. Ercolani et al., Plenum Press, NY (1994) 165-174.

[55] I. Zelenko, Complete systems of invariants for rank 1 curves in Lagrange Grassmannians, Differential geometry and its applications, 367-382, Matfyzpress, Prague, 2005.