

Integrable Lagrangians and modular forms

E.V. Ferapontov and A.V. Odesskii

Department of Mathematical Sciences
Loughborough University
Loughborough, Leicestershire LE11 3TU, UK

and

School of Mathematics
University of Manchester
Oxford Road, Manchester M13 9PL, UK

e-mails:

`E.V.Ferapontov@lboro.ac.uk`
`Alexander.Odesskii@manchester.ac.uk`

Abstract

We investigate non-degenerate Lagrangians of the form

$$\int f(u_x, u_y, u_t) dx dy dt$$

such that the corresponding Euler-Lagrange equations $(f_{u_x})_x + (f_{u_y})_y + (f_{u_t})_t = 0$ are integrable by the method of hydrodynamic reductions. We demonstrate that the integrability conditions, which constitute an involutive over-determined system of fourth order PDEs for the Lagrangian density f , are invariant under a 20-parameter group of Lie-point symmetries whose action on the moduli space of integrable Lagrangians has an open orbit. The density of the ‘master-Lagrangian’ corresponding to this orbit is shown to be a modular form in three variables defined on a complex hyperbolic ball. We demonstrate how the knowledge of the symmetry group allows one to linearise the integrability conditions.

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1 Introduction

In this paper we investigate integrable three-dimensional Euler-Lagrange equations,

$$(f_{u_x})_x + (f_{u_y})_y + (f_{u_t})_t = 0, \quad (1.1)$$

corresponding to Lagrangian densities of the form $f(u_x, u_y, u_t)$. Familiar examples include the dispersionless KP equation $u_{xt} - u_x u_{xx} = u_{yy}$ with the Lagrangian density $f = \frac{1}{3}u_x^3 + u_y^2 - u_x u_t$; this equation, also known as the Khokhlov-Zabolotskaya equation, arises in non-linear acoustics [19]. Another example, $u_{xx} + u_{yy} = e^{u_t} u_{tt}$, is known as the Boyer-Finley equation [3]: it appears as a symmetry reduction of the self-duality equations, and corresponds to the Lagrangian density $f = u_x^2 + u_y^2 - 2e^{u_t}$.

The paper [8] provides a system of partial differential equations for the Lagrangian density $f(a, b, c)$ (we set $a = u_x$, $b = u_y$, $c = u_t$) which are necessary and sufficient for the integrability of the equation (1.1) by the method of hydrodynamic reductions as proposed in [7]. These conditions can be represented in a remarkable compact form:

Theorem 1 [8]. *For a non-degenerate Lagrangian, the Euler-Lagrange equation (1.1) is integrable by the method of hydrodynamic reductions if and only if the density f satisfies the relation*

$$d^4 f = d^3 f \frac{dH}{H} + \frac{3}{H} \det(dM); \quad (1.2)$$

here $d^3 f$ and $d^4 f$ are the symmetric differentials of f . The Hessian H and the 4×4 matrix M are defined as follows:

$$H = \det \begin{pmatrix} f_{aa} & f_{ab} & f_{ac} \\ f_{ab} & f_{bb} & f_{bc} \\ f_{ac} & f_{bc} & f_{cc} \end{pmatrix}, \quad M = \begin{pmatrix} 0 & f_a & f_b & f_c \\ f_a & f_{aa} & f_{ab} & f_{ac} \\ f_b & f_{ab} & f_{bb} & f_{bc} \\ f_c & f_{ac} & f_{bc} & f_{cc} \end{pmatrix}. \quad (1.3)$$

The differential $dM = M_a da + M_b db + M_c dc$ is a matrix-valued form

$$\begin{pmatrix} 0 & f_{aa} & f_{ab} & f_{ac} \\ f_{aa} & f_{aaa} & f_{aab} & f_{aac} \\ f_{ab} & f_{aab} & f_{abb} & f_{abc} \\ f_{ac} & f_{aac} & f_{abc} & f_{acc} \end{pmatrix} da + \begin{pmatrix} 0 & f_{ab} & f_{bb} & f_{bc} \\ f_{ab} & f_{aab} & f_{abb} & f_{abc} \\ f_{bb} & f_{abb} & f_{bbb} & f_{bbc} \\ f_{bc} & f_{abc} & f_{bbc} & f_{bcc} \end{pmatrix} db + \begin{pmatrix} 0 & f_{ac} & f_{bc} & f_{cc} \\ f_{ac} & f_{aac} & f_{abc} & f_{acc} \\ f_{bc} & f_{abc} & f_{bbc} & f_{bcc} \\ f_{cc} & f_{acc} & f_{bcc} & f_{ccc} \end{pmatrix} dc.$$

A Lagrangian is said to be non-degenerate iff $H \neq 0$ (we point out that the equations $H = 0$ and $\det M = 0$ have been discussed in the literature, see [6] and references therein).

Both sides of the relation (1.2) are homogeneous symmetric quartics in da, db, dc . Equating similar terms we obtain expressions for *all* fourth order partial derivatives of the density f in terms of its second and third order derivatives (15 equations altogether). The resulting over-determined system for f is in involution, and its solution space is 20-dimensional: indeed, the values of partial derivatives of f up to order 3 at a point (a_0, b_0, c_0) amount to 20 arbitrary constants. Thus, we are dealing with a 20-dimensional moduli space of integrable Lagrangians.

In Sect. 2 we prove that the integrability conditions (1.2) are invariant under a 20-parameter group of Lie-point symmetries whose action on the moduli space of integrable Lagrangians possesses an open orbit.

Explicit formulae for integrable Lagrangians in terms of modular forms are constructed in Sect. 3. We first consider Lagrangian densities of the form $f = u_x u_y g(u_t)$, which can be viewed as a deformation of the integrable density $f = u_x u_y u_t$ found in [8]. By virtue of the integrability conditions (1.2), the function g has to satisfy the fourth order ODE

$$g''''(g^2 g'' - 2g(g')^2) - 9(g')^2(g'')^2 + 2gg'g''g''' + 8(g')^3 g''' - g^2(g''')^2 = 0,$$

which inherits a remarkable $Gl(2, \mathbb{R})$ -invariance. We prove that the ‘generic’ solution of this ODE is given by the series

$$g(u_t) = \sum_{(\alpha, \beta) \in \mathbb{Z}^2} e^{(\alpha^2 - \alpha\beta + \beta^2)u_t} = 1 + 6e^{u_t} + 6e^{3u_t} + 6e^{4u_t} + 12e^{7u_t} + \dots;$$

notice that under the substitution $u_t = 2\pi iz$ the right hand side of this formula becomes a special modular form of weight one and level three, known as the Eisenstein series $E_{1,3}(z)$. We point out that modular forms and non-linear ODEs related to them appear in a variety of problems in mathematical physics, see e.g. [1, 2, 4, 5, 9, 10, 14, 15, 18] and references therein.

Lagrangian densities of the form $g(u_x, u_y)u_t$ and the general case $f(u_x, u_y, u_t)$ are discussed in Sect. 3.2 and 3.3, respectively. Here the ‘generic’ solution is an automorphic form of two (three) variables.

2 Symmetry group of the problem

The first main observation, overlooked in [8], is the invariance of the integrability conditions (1.2) under projective transformations of the form

$$\tilde{a} = \frac{l_1(a, b, c)}{l(a, b, c)}, \quad \tilde{b} = \frac{l_2(a, b, c)}{l(a, b, c)}, \quad \tilde{c} = \frac{l_3(a, b, c)}{l(a, b, c)}, \quad \tilde{f} = \frac{f}{l(a, b, c)}; \quad (2.4)$$

here l, l_1, l_2, l_3 are arbitrary (inhomogeneous) linear forms in a, b, c . Introducing the quartic form

$$F = Hd^4f - d^3fdH - 3det(dM),$$

one can verify that $\tilde{F} = l^4F$, which establishes the $SL(4, \mathbb{R})$ -invariance of the integrability conditions (1.2). Combined with obvious symmetries of the form

$$\tilde{f} = sf + \alpha a + \beta b + \gamma c + \delta, \quad (2.5)$$

this provides a 20-dimensional symmetry group of the problem.

Remark. The class of Euler-Lagrange equations (1.1) is form-invariant under a point group generated by arbitrary linear transformations of the variables x, y, t and u . Obviously, point

transformations preserve the integrability. Since the prolongation of these transformations to the variables a, b, c and f is given by (2.4), this explains the $SL(4, \mathbb{R})$ -invariance of the integrability conditions (1.2).

The main result of this section is the following

Theorem 2. *The action of the symmetry group on the 20-dimensional moduli space of integrable Lagrangians possesses an open orbit.*

Proof:

The infinitesimal generators of the symmetry group (2.4), (2.5) are the following vector fields:

3 translations in a, b, c :

$$\frac{\partial}{\partial a}, \quad \frac{\partial}{\partial b}, \quad \frac{\partial}{\partial c};$$

9 linear transformations of a, b, c :

$$a \frac{\partial}{\partial a}, \quad b \frac{\partial}{\partial a}, \quad c \frac{\partial}{\partial a}, \quad a \frac{\partial}{\partial b}, \quad b \frac{\partial}{\partial b}, \quad c \frac{\partial}{\partial b}, \quad a \frac{\partial}{\partial c}, \quad b \frac{\partial}{\partial c}, \quad c \frac{\partial}{\partial c};$$

3 projective transformations of a, b, c, f :

$$a^2 \frac{\partial}{\partial a} + ab \frac{\partial}{\partial b} + ac \frac{\partial}{\partial c} + af \frac{\partial}{\partial f}, \quad ab \frac{\partial}{\partial a} + b^2 \frac{\partial}{\partial b} + bc \frac{\partial}{\partial c} + bf \frac{\partial}{\partial f}, \quad ac \frac{\partial}{\partial a} + bc \frac{\partial}{\partial b} + c^2 \frac{\partial}{\partial c} + cf \frac{\partial}{\partial f};$$

moreover, we have 5 extra generators corresponding to the transformations (2.5):

$$\frac{\partial}{\partial f}, \quad a \frac{\partial}{\partial f}, \quad b \frac{\partial}{\partial f}, \quad c \frac{\partial}{\partial f}, \quad f \frac{\partial}{\partial f}.$$

The main idea of the proof is to prolong these infinitesimal generators to the 20-dimensional moduli space of solutions of the involutive system (1.2). We point out that, since all fourth order derivatives of f are explicitly known, this moduli space can be identified with the values of f and its partial derivatives f_i, f_{ij}, f_{ijk} up to order three (20 parameters altogether). The prolongation can be calculated as follows:

(1) Following the standard notation adopted in the symmetry analysis of differential equations [11, 16], we introduce the variables $x^1 = a, x^2 = b, x^3 = c$ and represent each of the above generators in the form

$$\xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial f};$$

here ξ^i and η are functions of x^i and f .

(2) Prolong infinitesimal generators to the third order jet space with coordinates $x^i, f, f_i, f_{ij}, f_{ijk}$,

$$\xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial f} + \zeta_i \frac{\partial}{\partial f_i} + \zeta_{ij} \frac{\partial}{\partial f_{ij}} + \zeta_{ijk} \frac{\partial}{\partial f_{ijk}},$$

where ζ_i, ζ_{ij} and ζ_{ijk} are calculated according to the standard prolongation formulae

$$\zeta_i = D_i(\eta) - f_s D_i(\xi^s), \quad \zeta_{ij} = D_j \zeta_i - f_{is} D_j(\xi^s), \quad \zeta_{ijk} = D_k \zeta_{ij} - f_{ijs} D_k(\xi^s); \quad (2.6)$$

here D_i denotes the operator of total differentiation with respect to x^i .

(3) To eliminate the $\frac{\partial}{\partial x^i}$ -terms we subtract the linear combination of total derivatives $\xi^i D_i$ from the prolonged operators where, in D_i , it is sufficient to keep only the following terms:

$$D_i = \frac{\partial}{\partial x^i} + f_i \frac{\partial}{\partial f} + f_{ij} \frac{\partial}{\partial f_j} + f_{ijk} \frac{\partial}{\partial f_{jk}} + f_{ijkl} \frac{\partial}{\partial f_{jkl}};$$

notice that, since f_{ijkl} are explicit functions of lower order derivatives, the resulting operators will be well-defined vector fields on the 20-dimensional moduli space with coordinates f, f_i, f_{ij}, f_{ijk} . Although these operators will depend on the variables x^i as on parameters (indeed, the isomorphism of the moduli space with the space f, f_i, f_{ij}, f_{ijk} depends on the choice of a point in the x -space), all algebraic properties of these operators will be x -independent.

(4) Finally, the dimension of the maximal orbit equals the rank of the 20×20 matrix of coefficients of these operators. It remains to point out that this rank equals 20 for any ‘random’ numerical choice of the values for $x^i, f, f_i, f_{ij}, f_{ijk}$.

3 Lagrangian densities in terms of modular forms

In this section we provide explicit formulae for integrable Lagrangians in terms of modular forms. We start with the case $f(u_x, u_y, u_t) = u_x u_y g(u_t)$ (Sect. 3.1), where g is shown to be an Eisenstein series $E_{1,3}$: a special modular form of weight 1 and level three. The case $f(u_x, u_y, u_t) = g(u_x, u_y) u_t$ and the general case $f(u_x, u_y, u_t)$ are discussed in Sect. 3.2 and 3.3, respectively. We demonstrate how the knowledge of the symmetry group of the problem allows one to linearize the complicated nonlinear equations for Lagrangian densities resulting from the integrability conditions.

3.1 Lagrangian densities of the form $f = u_x u_y g(u_t)$

The integrability conditions (1.2) imply a single fourth order ODE for $g(z)$,

$$g''''(g^2 g'' - 2g(g')^2) - 9(g')^2 (g'')^2 + 2g g' g'' g''' + 8(g')^3 g''' - g^2 (g''')^2 = 0; \quad (3.7)$$

to comply with the standard notation, the argument of g is now denoted by z . This equation enjoys a remarkable $SL(2, \mathbb{R})$ -invariance inherited from (2.4):

$$\tilde{z} = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \tilde{g} = (\gamma z + \delta)g; \quad (3.8)$$

here $\alpha, \beta, \gamma, \delta$ are arbitrary constants such that $\alpha\delta - \beta\gamma = 1$. Moreover, there is an obvious scaling symmetry $g \rightarrow \lambda g$. The equation (3.7) can be linearized as follows. Introducing $h = g'/g$, we first rewrite it in the form

$$h'''(h' - h^2) = h^6 - 3h^4 h' + 9h^2 (h')^2 - 3(h')^3 - 4h^3 h'' + (h'')^2; \quad (3.9)$$

the corresponding symmetry group modifies to

$$\tilde{z} = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \tilde{h} = (\gamma z + \delta)^2 h + \gamma(\gamma z + \delta). \quad (3.10)$$

We point out that the same symmetry occurs in the case of the Chazy equation ([1], p. 342), as well as its analogue discussed recently in [2]. The presence of the $SL(2, \mathbb{R})$ -symmetry of this type implies the linearizability of the equation under study. One can formulate the following general statement which is, in fact, contained in [5]:

Theorem 3. *Any third order ODE of the form $F(z, h, h', h'', h''') = 0$, which is invariant under the action of $SL(2, \mathbb{R})$ as specified by (3.10), can be linearized by a substitution*

$$z = \frac{w_1}{w_2}, \quad h = \frac{d}{dz} \ln w_2 \quad (3.11)$$

where $w_1(t)$ and $w_2(t)$ are two linearly independent solutions of a linear equation $d^2w/dt^2 = V(t)w$ with the Wronskian W normalized as $W = w_2 dw_1/dt - w_1 dw_2/dt = 1$ (the potential $V(t)$ depends on the given third order ODE, and can be effectively reconstructed).

In particular, the general solution of the equation (3.9) is given by parametric formulae (3.11) where $w_1(t)$ and $w_2(t)$ are two linearly independent solutions of the linear equation $d^2w/dt^2 = \frac{2}{9}(\cosh^{-2} t)w$ with $W = 1$.

Proof:

To establish the first part of the theorem we essentially reproduce the calculation from Sect. 5 in [5]. Let us consider a linear ODE $d^2w/dt^2 = V(t)w$, take two linearly independent solutions $w_1(t)$, $w_2(t)$ with $W = 1$, and introduce the new dependent and independent variables h, z by parametric relations

$$z = \frac{w_1}{w_2}, \quad h = \frac{d}{dz} \ln w_2.$$

Using the readily verifiable formulae $dt/dz = w_2^2$ and $h = w_2 dw_2/dt$, one obtains the identities

$$h' - h^2 = w_2^4 V,$$

$$h'' - 6hh' + 4h^3 = w_2^6 dV/dt$$

and

$$h''' - 12hh'' - 6(h')^2 + 48h^2h' - 24h^4 = w_2^8 d^2V/dt^2;$$

here prime denotes differentiation with respect to z . Thus, one arrives at the relations

$$I_1 = \frac{(h'' - 6hh' + 4h^3)^2}{(h' - h^2)^3} = \frac{(dV/dt)^2}{V^3},$$

$$I_2 = \frac{h''' - 12hh'' - 6(h')^2 + 48h^2h' - 24h^4}{(h' - h^2)^2} = \frac{d^2V/dt^2}{V^2}.$$

We point out that I_1 and I_2 are the simplest second- and third-order differential invariants of the action (3.10) whose infinitesimal generators, prolonged to the third jets z, h, h', h'', h''' , are of the form

$$X_1 = \partial_z, \quad X_2 = z\partial_z - h\partial_h - 2h'\partial_{h'} - 3h''\partial_{h''} - 4h'''\partial_{h'''},$$

$$X_3 = z^2\partial_z - (2zh + 1)\partial_h - (2h + 4zh')\partial_{h'} - (6h' + 6zh'')\partial_{h''} - (12h'' + 8zh''')\partial_{h'''};$$

notice the standard commutation relations $[X_1, X_2] = X_1$, $[X_1, X_3] = 2X_2$, $[X_2, X_3] = X_3$. One can verify that the Lie derivatives of I_1, I_2 with respect to X_1, X_2, X_3 are indeed zero. Thus, any third order ODE which is invariant under the $SL(2, \mathbb{R})$ -action (3.10), can be represented in the form $I_2 = F(I_1)$ where F is an arbitrary function of one variable. The corresponding potential $V(t)$ has to satisfy the relation $\frac{d^2V/dt^2}{V^2} = F\left(\frac{(dV/dt)^2}{V^3}\right)$.

This simple scheme produces some of the well-known equations, for instance, the relation $I_2 = -24$ implies the Chazy equation for h , that is, $h''' - 12hh'' + 18(h')^2 = 0$. The corresponding potential satisfies the equation $d^2V/dt^2 = -24V^2$.

Similarly, the choice $I_2 = I_1 - 8$ results in the ODE $h''' = 4hh'' - 2(h')^2 + \frac{(h'' - 2hh')^2}{h' - h^2}$ which, under the substitution $h = y/2$, coincides with the equation (4.7) from [2]. The potential V satisfies the equation $Vd^2V/dt^2 = (dV/dt)^2 - 8V^3$.

Finally, the relation $I_2 = I_1 - 9$ coincides with (3.9). The corresponding potential V satisfies the equation $Vd^2V/dt^2 = (dV/dt)^2 - 9V^3$. It remains to point out that, up to elementary equivalence transformations, the general solution of the last equation for V is given by $V = \frac{2}{9} \cosh^{-2} t$.

Since the equation $d^2w/dt^2 = \frac{2}{9} \cosh^{-2} t w$ is related to the hypergeometric equation $s(1-s)w_{ss} + (1-2s)w_s - \frac{2}{9}w = 0$, corresponding to the parameter values $a = 1/3$, $b = 2/3$, $c = 1$, by a change of variables $s/(1-s) = e^{2t}$, we can reformulate the above Theorem as follows:

Proposition 1. *The general solution of the equation (3.9) is given by parametric formulae (3.11) where $w_1(s)$ and $w_2(s)$ are two linearly independent solutions of the hypergeometric equation $s(1-s)w_{ss} + (1-2s)w_s - \frac{2}{9}w = 0$ with the Wronskian normalized as $w_2dw_1/ds - w_1dw_2/ds = 1/(2s(1-s))$.*

As $h = g'/g$, this immediately implies the following formula for the general solution of (3.7):

Proposition 2. *The general solution of the equation (3.7) is given by parametric formulae*

$$z = \frac{w_1}{w_2}, \quad g = w_2,$$

where $w_1(s)$ and $w_2(s)$ are two linearly independent solutions to the hypergeometric equation $s(1-s)w_{ss} + (1-2s)w_s - \frac{2}{9}w = 0$ with the Wronskian normalized as $w_2dw_1/ds - w_1dw_2/ds = 1/(2s(1-s))$.

One can construct the following explicit solution of the equation (3.7),

$$g(z) = \sum_{(\alpha, \beta) \in \mathbb{Z}^2} q^{(\alpha^2 - \alpha\beta + \beta^2)} = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + \dots; \quad (3.12)$$

here $q = e^{2\pi iz}$. To get a real-valued solution, one has to restrict z to the imaginary axis. This function is known as the Eisenstein series $E_{1,3}(z)$. Equivalently, it can be defined by the formula

$$g(z) = E_{1,3}(z) = 1 + 6 \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi_3(d) \right) q^n$$

where χ_3 denotes the Legendre symbol mod 3 (that is, $\chi_3(d) = 0$ if $d \equiv 0 \pmod{3}$, $\chi_3(d) = 1$ if $d \equiv 1 \pmod{3}$, and $\chi_3(d) = -1$ if $d \equiv 2 \pmod{3}$). The Eisenstein series transforms as $g\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = \chi_3(\delta)(\gamma z + \delta)g(z)$ under the Hecke congruence subgroup $\Gamma_0(3)$ defined as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(3) \subset SL(2, \mathbb{Z}) \quad \text{if} \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \pmod{3}.$$

It follows that $g(z)$ is a modular form of weight one and level 3, namely, $g\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = (\gamma z + \delta)g(z)$ where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z}) \quad \text{and} \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \pmod{3}.$$

The function $g(z)$ can also be written in the form involving summation over \mathbb{N} only,

$$g(z) = 1 - 6 \sum_{k \in \mathbb{N}} \left(\frac{q^{3k-1}}{1 - q^{3k-1}} - \frac{q^{3k-2}}{1 - q^{3k-2}} \right).$$

Theorem 4. *The function $g(z)$ is a solution of the differential equation (3.7).*

Proof:

Recall that, given a modular form $g(z)$ of weight k , its Rankin-Cohen brackets $[g, g]_2$ and $[g, g]_4$ are defined as follows:

$$[g, g]_2 = (k+1)(k g g'' - (k+1)(g')^2),$$

and

$$[g, g]_4 = (k+2)(k+3) \left(\frac{k(k+1)}{12} g g'''' - \frac{(k+1)(k+3)}{3} g' g''' + \frac{(k+2)(k+3)}{4} (g'')^2 \right);$$

these are known to be modular forms of weights $2k+4$ and $2k+8$, respectively (we use the normalization of [20]). In our case $k=1$, so that we get

$$G = [g, g]_2 = 2(g g'' - 2(g')^2), \quad [g, g]_4 = 2(g g'''' - 16g' g''' + 18(g'')^2),$$

weights 6 and 10, respectively. One can verify that, up to a constant multiple, the left hand side of the equation (3.7) can be represented in the form

$$7[g, g]_4[g, g]_2 + [G, G]_2,$$

which shows that it is a modular form (in fact, a cusp form) of weight 16 with respect to the same group. To show that this form vanishes identically we recall that the dimension of the space of cusp forms of weight 16 and level 3 equals 4, and the order of zero cannot exceed 5. Thus, it is sufficient to verify the vanishing of the first five coefficients in the decomposition of this form as a power series in $q = \exp(2\pi iz)$. This can be done by a direct calculation.

Remark. The relation between the modular form (3.12) and the hypergeometric equation from the Proposition 2 can be summarized as follows. Choosing a basis of solutions of the hypergeometric equation in the form

$$w_2 = 1 + \frac{2}{9}s + \frac{10}{81}s^2 + \dots, \quad w_1 = w_2 \ln s + \frac{5}{9}s + \frac{57}{162}s^2 + \dots,$$

one obtains parametric equations

$$z = \frac{w_1}{w_2} = \ln s + \frac{5}{9}s + \frac{37}{162}s^2 + \dots, \quad g = w_2 = 1 + \frac{2}{9}s + \frac{10}{81}s^2 + \dots$$

Solving the first equation for s in the form $s = e^z + ae^{2z} + be^{3z} + \dots$ one arrives at $s = e^z - \frac{5}{9}e^{2z} + \frac{19}{81}e^{3z} + \dots$. The substitution into the second equation implies the expression for $g(z)$ in the form $g(z) = 1 + pe^z + qe^{3z} + \dots$ which, up to an appropriate affine transformation of z , coincides with (3.12).

3.2 Lagrangian densities of the form $f = g(u_x, u_y)u_t$

As follows from [8], the integrability conditions (1.2) result in a system of five equations expressing all fourth order partial derivatives of $g(a, b)$ in terms of its lower order derivatives. In symbolic form, this system can be represented as follows:

$$d^4g = d^3g \frac{dh}{h} + 6 \frac{dg}{h} \det(dm) + 3 \frac{(dg)^2}{h} \det(dn). \quad (3.13)$$

Here $d^s g$ are symmetric differentials of g , the matrices m and n are defined as

$$m = \begin{pmatrix} 0 & g_a & g_b \\ g_a & g_{aa} & g_{ab} \\ g_b & g_{ab} & g_{bb} \end{pmatrix}, \quad n = \begin{pmatrix} g_{aa} & g_{ab} \\ g_{ab} & g_{bb} \end{pmatrix},$$

and

$$h = -\det(m) = g_b^2 g_{aa} - 2g_a g_b g_{ab} + g_a^2 g_{bb}.$$

The non-degeneracy of the Lagrangian density $f(a, b, c) = g(a, b)c$ is equivalent to the condition $h \neq 0$. One can show that the over-determined system (3.13) is in involution, and

its solution space is 10-dimensional (indeed, the values of partial derivatives of g up to order 3 at a point (a_0, b_0) amount to 10 arbitrary constants). The system (3.13) is invariant under a 10-dimensional group of Lie-point symmetries which consists of arbitrary projective transformations of a and b , isomorphic to $SL(3, \mathbb{R})$, along with transformations of the form $g \rightarrow \alpha g + \beta$, $\alpha, \beta = \text{const}$. The corresponding infinitesimal generators include

$$\begin{aligned}
& 2 \text{ translations : } \quad \frac{\partial}{\partial a}, \quad \frac{\partial}{\partial b}; \\
& 4 \text{ linear transformations : } \quad a \frac{\partial}{\partial a}, \quad b \frac{\partial}{\partial a}, \quad a \frac{\partial}{\partial b}, \quad b \frac{\partial}{\partial b}; \\
& 2 \text{ projective transformations : } \quad a^2 \frac{\partial}{\partial a} + ab \frac{\partial}{\partial b}, \quad ab \frac{\partial}{\partial a} + b^2 \frac{\partial}{\partial b}; \\
& 2 \text{ affine transformations of } g : \quad \frac{\partial}{\partial g}, \quad g \frac{\partial}{\partial g}.
\end{aligned} \tag{3.14}$$

We will demonstrate that the existence of this symmetry group allows one to linearize the integrability conditions (3.13). For this purpose we consider a linear system of the form

$$\begin{aligned}
z_{xx} &= Mz_x - Iz_y + Az, \\
z_{xy} &= -Nz_x - Mz_y + Bz, \\
z_{yy} &= -Jz_x + Nz_y + Cz,
\end{aligned} \tag{3.15}$$

where the coefficients I, J, M, N, A, B, C are certain functions of x, y which have to satisfy the compatibility conditions resulting from the requirement of consistency of the equations (3.15):

$$A = 2(M^2 + IN) + I_y - M_x, \quad B = M_y + N_x + IJ - MN, \quad C = 2(N^2 + JM) + J_x - N_y,$$

along with two extra relations involving I, J, M, N only:

$$\begin{aligned}
J_{xx} &= 2N_{xy} + M_{yy} - 3(N^2 + JM)_x - 3NM_y + 2JI_y + IJ_y, \\
I_{yy} &= 2M_{xy} + N_{xx} - 3(M^2 + IN)_y - 3MN_x + 2IJ_x + JI_x.
\end{aligned}$$

These relations imply that the space of solutions of the linear system (3.15) is three-dimensional. Notice that any involutive second order linear system of the form $z_{ij} = \Gamma_{ij}^k z_k + g_{ij} z$ with two independent variables x, y can be reduced to the form (3.15) by a gauge transformation $z \rightarrow \varphi(x, y)z$. This is a standard normalization in the theory of multi-dimensional Schwarzian derivatives [17]. It implies that the Wronskian W of any three linearly independent solutions,

$$W = \det \begin{bmatrix} z_1 & z_2 & z_3 \\ (z_1)_x & (z_2)_x & (z_3)_x \\ (z_1)_y & (z_2)_y & (z_3)_y \end{bmatrix},$$

is constant: $W_x = W_y = 0$. Let us choose three linearly independent solutions $z_i(x, y)$ with the Wronskian normalized as $W = 1$, and introduce the new independent variables

$$a = \frac{z_1}{z_3}, \quad b = \frac{z_2}{z_3}.$$

Let us consider x and y as functions of a, b :

$$x = x(a, b), \quad y = y(a, b).$$

A direct calculation shows that these functions satisfy the following nonlinear system:

$$\begin{aligned} x_a^2 x_{bb} - 2x_a x_b x_{ab} + x_b^2 x_{aa} &= (x_a y_b - y_a x_b)^2 J, \\ y_a^2 y_{bb} - 2y_a y_b y_{ab} + y_b^2 y_{aa} &= (x_a y_b - y_a x_b)^2 I, \\ y_a x_{aa} - x_a y_{aa} &= 3N x_a y_a^2 + J y_a^3 - I x_a^3 - 3M x_a^2 y_a, \\ x_b y_{bb} - y_b x_{bb} &= 3M x_b^2 y_b + I x_b^3 - J y_b^3 - 3N x_b y_b^2; \end{aligned} \tag{3.16}$$

here I, J, M, N are the same functions of x, y as in (3.15). The system (3.16) is in involution, and its solution space, which is 8-dimensional, possesses a transitive action of $SL(3, \mathbb{R})$: indeed, there is an $SL(3, \mathbb{R})$ -freedom in the choice of a basis z_1, z_2, z_3 . Conversely, one can show that any involutive system of four second order PDEs for two functions $x(a, b)$ and $y(a, b)$ which is invariant under a transitive projective action of $SL(3, \mathbb{R})$, has the form (3.16), and comes from a linear system (3.15). This immediately follows from the equivalent representation of the system (3.16),

$$\begin{aligned} \frac{x_a^2 x_{bb} - 2x_a x_b x_{ab} + x_b^2 x_{aa}}{(x_a y_b - y_a x_b)^2} &= J(x, y), \\ \frac{y_a^2 y_{bb} - 2y_a y_b y_{ab} + y_b^2 y_{aa}}{(x_a y_b - y_a x_b)^2} &= I(x, y), \\ \frac{y_b^2 x_{aa} + y_a^2 x_{bb} + 2x_b y_b y_{aa} + 2x_a y_a y_{bb} - 2y_a y_b x_{ab} - 2(x_a y_b + x_b y_a) y_{ab}}{(x_a y_b - y_a x_b)^2} &= -3M(x, y), \\ \frac{x_a^2 y_{bb} + x_b^2 y_{aa} + 2x_a y_a x_{bb} + 2x_b y_b x_{aa} - 2x_a x_b y_{ab} - 2(x_a y_b + x_b y_a) x_{ab}}{(x_a y_b - y_a x_b)^2} &= -3N(x, y). \end{aligned} \tag{3.17}$$

The variables x, y and the left hand sides of (3.17) form a basis of differential invariants of the $SL(3, \mathbb{R})$ -action extended to second order jet space with coordinates $a, b, x, y, x_a, x_b, y_a, y_b, x_{aa}, x_{ab}, x_{bb}, y_{aa}, y_{ab}, y_{bb}$. Thus, any system with the required symmetry properties can be obtained by expressing the four second order differential invariants as functions of x and y . The expressions in the left hand sides of (3.17) are related to the two-dimensional Schwarzian derivatives [17].

Let us return to the integrability conditions (3.13). Extending the action of the symmetry generators (3.14) to the third order jet space with coordinates $a, b, g, g_a, g_b, g_{aa}, g_{ab}, g_{bb}, g_{aaa}, g_{aab}, g_{abb}, g_{bbb}$ according to the prolongation formulae (2.6), one obtains 10 vector fields on a 12-dimensional space; thus, there exist two differential invariants, which we will denote by

x and y , respectively (the explicit formulae for x and y in terms of g and its derivatives are provided below). We claim that x and y , viewed as functions of a and b , satisfy a system of the form (3.16) (equivalently, (3.17)). Indeed, the action of the $SL(3, \mathbb{R})$ part of the symmetry group on the variables a, b, x, y is exactly the same as for the system (3.16). The passage from a, b, g to a, b, x, y can be viewed as a factorization of the integrability equations (3.13) by the two-dimensional affine group corresponding to the last two generators (3.14), which respects the action of $SL(3, \mathbb{R})$. To write down the expressions for the differential invariants x and y we introduce the following notation:

$$Z^1 = \frac{\frac{g_{aaa}}{g_a} - \frac{3}{2} \frac{g_{aa}^2}{g_a^2}}{(g_a^2 g_{bb} - 2g_a g_b g_{ab} + g_b^2 g_{aa})^2} g_a^2 g_b^4, \quad Z^2 = \frac{\frac{g_{bbb}}{g_b} - \frac{3}{2} \frac{g_{bb}^2}{g_b^2}}{(g_a^2 g_{bb} - 2g_a g_b g_{ab} + g_b^2 g_{aa})^2} g_a^4 g_b^2,$$

$$V^1 = \frac{\frac{g_{aab}}{g_a} - 2 \frac{g_{aa} g_{ab}}{g_a^2} + \frac{1}{2} \frac{g_b}{g_a^3} g_{aa}^2}{(g_a^2 g_{bb} - 2g_a g_b g_{ab} + g_b^2 g_{aa})^2} g_a^3 g_b^3, \quad V^2 = \frac{\frac{g_{abb}}{g_b} - 2 \frac{g_{bb} g_{ab}}{g_b^2} + \frac{1}{2} \frac{g_a}{g_b^3} g_{bb}^2}{(g_a^2 g_{bb} - 2g_a g_b g_{ab} + g_b^2 g_{aa})^2} g_a^3 g_b^3.$$

Moreover, let

$$\Sigma = Z^1 - 3V^1 - (Z^2 - 3V^2), \quad S = Z^1 + Z^2 - V^1 - V^2.$$

Then the invariants x and y can be chosen in the following form:

$$x = \frac{[\Sigma^3 + 9\Sigma S + 18(Z^1 - Z^2)]^2}{[\Sigma^2 + 6S + 3]^3}, \quad y = \frac{\Sigma^2 - S^2 + 4\Sigma(V^1 - V^2) + 4(V^1 + V^2) - 1}{[\Sigma^2 + 6S + 3]^2};$$

notice that the expressions for x and y are manifestly symmetric under the interchange of indices $1 \leftrightarrow 2$. The functions $x(a, b)$ and $y(a, b)$ satisfy a system of the form (3.16) which is invariant under the action of $SL(3, \mathbb{R})$ as specified above. The explicit formulae for the coefficients I, J, M, N , as well as the properties of the corresponding linear system (3.15), will be discussed elsewhere.

3.3 General case: Lagrangian densities of the form $f(u_x, u_y, u_t)$

Let us begin by introducing special functions which will appear in the general formula for f . The first one is a theta-function of order one [12] (with modular parameter $\tau = \varepsilon$),

$$\theta(z) = \sum_{k \in \mathbb{Z}} (-1)^k \exp \left(2\pi i \left(kz + \frac{k(k-1)}{2} \varepsilon \right) \right), \quad (3.18)$$

which is known to satisfy the relations

$$\theta(z+1) = \theta(z), \quad \theta(z+\varepsilon) = -\exp(-2\pi iz)\theta(z);$$

here $\varepsilon = \frac{1}{2}(1 + \sqrt{3}i)$. Note that ε is a primitive 6th root of unity. In particular, $\varepsilon^3 = -1$ and $\varepsilon^2 = \varepsilon - 1$. It is known that $\theta(0) = 0$, and this is the only zero of the function $\theta(z)$ modulo 1, ε .

Moreover, $\theta'(0) \neq 0$. Next we define a function $\tilde{\theta}(z)$ which differs from $\theta(z)$ by an exponential factor:

$$\tilde{\theta}(z) = \frac{1}{\theta'(0)} \exp\left(-2\pi i\left(\frac{\varepsilon}{3} - \frac{1}{6}\right)z^2 - \pi i z\right) \theta(z). \quad (3.19)$$

It can be readily verified that

$$\begin{aligned} \tilde{\theta}(z+1) &= \exp\left(-\frac{2\pi i}{3}((2\varepsilon-1)z + \varepsilon - 2)\right) \tilde{\theta}(z), \\ \tilde{\theta}(z+\varepsilon) &= \exp\left(-\frac{2\pi i}{3}((\varepsilon+1)z + \varepsilon - 2)\right) \tilde{\theta}(z). \end{aligned} \quad (3.20)$$

These relations imply

$$\tilde{\theta}(z + \alpha + \beta\varepsilon) = \exp\left(-2\pi i\left(\left(\frac{2\varepsilon-1}{3}\alpha + \frac{\varepsilon+1}{3}\beta\right)z + \frac{\varepsilon+1}{3}\alpha^2 + \frac{\varepsilon+1}{3}\alpha\beta + \frac{\varepsilon+1}{3}\beta^2\right)\right) \tilde{\theta}(z) \quad (3.21)$$

where $\alpha, \beta \in \mathbb{Z}$. One can also show that

$$\tilde{\theta}(\varepsilon z) = \varepsilon \tilde{\theta}(z),$$

which implies

$$\tilde{\theta}(z) = \sum_{j \geq 0} a_j z^{6j+1}; \quad (3.22)$$

note that $a_0 = \tilde{\theta}'(0) = 1$.

Finally, let us define the ‘master-density’ f by the formula

$$f(x, y, z) = xy + \sum_{(k,l) \in \mathbb{Z}^2 \setminus \{0\}} \frac{\tilde{\theta}((k-\varepsilon l)x) \tilde{\theta}((k-\varepsilon l)y)}{(k-\varepsilon l)^2} \exp\left(\frac{2\pi i}{3}(k^2 - kl + l^2)z\right), \quad (3.23)$$

which we claim to be a ‘generic’ solution of (1.2)¹. For convenience, the arguments of f are now denoted by x, y, z . From (3.21) one can derive the following modular properties of f :

$$\begin{aligned} f(x+1, y, z + (2\varepsilon-1)x + \varepsilon + 1) &= f(x, y, z) + y, \\ f(x+\varepsilon, y, z + (\varepsilon+1)x + \varepsilon + 1) &= f(x, y, z) + \varepsilon y, \\ f(x, y+1, z + (2\varepsilon-1)y + \varepsilon + 1) &= f(x, y, z) + x, \\ f(x, y+\varepsilon, z + (\varepsilon+1)y + \varepsilon + 1) &= f(x, y, z) + \varepsilon x. \end{aligned} \quad (3.24)$$

Substituting (3.22) into (3.23) we obtain an alternative representation for f ,

$$f(x, y, z) = xy + \sum_{(k,l) \in \mathbb{Z}^2 \setminus \{0\}, m, n \geq 0} a_m a_n x^{6m+1} y^{6n+1} \left((k-\varepsilon l)^{6(m+n)} \exp\left(\frac{2\pi i}{3}(k^2 - kl + l^2)z\right) \right) =$$

¹We did not prove that f satisfies the system (1.2), although computer calculations support this claim.

$$\sum_{m,n \geq 0} a_m a_n x^{6m+1} y^{6n+1} g_{m+n}(z)$$

where

$$g_n(z) = \sum_{k,l \in \mathbb{Z}^2} \frac{1}{2} ((k - \varepsilon l)^{6n} + (\varepsilon k - l)^{6n}) \exp\left(\frac{2\pi i}{3}(k^2 - kl + l^2)z\right).$$

It is known [13] that $g_n(z)$ is a modular form, namely, $g_n\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = (\gamma z + \delta)^{6n+1} g_n(z)$ where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z})$ and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \pmod{3}$. Indeed, coefficients at the exponents are harmonic polynomials with respect to the quadratic form $k^2 - kl + l^2$. This gives us the following modular property of $f(x, y, z)$ with respect to the same group:

$$f\left(\frac{x}{\gamma z + \delta}, \frac{y}{\gamma z + \delta}, \frac{\alpha z + \beta}{\gamma z + \delta}\right) = f(x, y, z) \frac{1}{\gamma z + \delta}. \quad (3.25)$$

Note that $g(z) = g_0(z)$ is a solution of the differential equation (3.7) and is given by (3.12) (up to a rescaling of z). Functions $g_n(z)$ can be represented as rational differential functions in $g(z)$, for example, $g_1(z) = g(z)^2 g''(z) - 2g(z)g'(z)^2 = \frac{1}{2}g[g, g]_2$ where $[g, g]_2$ is the Rankin-Cohen bracket (see the proof of Theorem 4). Transformations (3.24) and (3.25) generate a discrete subgroup $\Gamma \subset SL(4, \mathbb{C})$ which plays the role of the modular group for $f(x, y, z)$. The subgroup Γ has the following algebraic description. Let $\mathbb{D} \subset \mathbb{C}^3$ be a domain in \mathbb{C}^3 defined by

$$\mathbb{D} = \{(x, y, z) \in \mathbb{C}^3; \quad |x|^2 + |y|^2 < \frac{2}{\sqrt{3}}\Im z\}.$$

Note that \mathbb{D} is a complex hyperbolic ball. One can check that the series (3.23) converges exactly in the domain \mathbb{D} . Let $G \subset SL(4, \mathbb{C})$ be a group defined by

$$G = \{A \in SL(4, \mathbb{C}); \quad AJA^* = J\}$$

where

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3}i \\ 0 & 0 & \sqrt{3}i & 0 \end{pmatrix},$$

and A^* stands for the Hermitian conjugate of A . One can check that the complex hyperbolic ball \mathbb{D} is an orbit of G under its standard projective action on $\mathbb{C}P^3$: if $A = (a_{ij}) \in G$ and $(x, y, z) = (x : y : z : 1) \in \mathbb{C}P^3$, then

$$A(x, y, z) = \left(\frac{a_{11}x + a_{12}y + a_{13}z + a_{14}}{a_{41}x + a_{42}y + a_{43}z + a_{44}}, \frac{a_{21}x + a_{22}y + a_{23}z + a_{24}}{a_{41}x + a_{42}y + a_{43}z + a_{44}}, \frac{a_{31}x + a_{32}y + a_{33}z + a_{34}}{a_{41}x + a_{42}y + a_{43}z + a_{44}} \right).$$

In our context, the domain \mathbb{D} and the group G play a role similar to that of the upper half plane and its automorphism group $SL(2, \mathbb{R})$ in the classical theory of modular forms. Let $\Gamma \subset G$ be

a discrete subgroup of G consisting of matrices $A = (a_{ij}) \in G$ with the following properties: $a_{ij} \in \mathbb{Z}[\varepsilon]$ and

$$A \equiv \begin{pmatrix} 1 & 0 & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{43} & 1 \end{pmatrix} \pmod{(1 + \varepsilon)}.$$

Here $\mathbb{Z}[\varepsilon] = \{m + \varepsilon n; \quad n, m \in \mathbb{Z}\}$, and $a \equiv b \pmod{(1 + \varepsilon)}$ for $a, b \in \mathbb{Z}[\varepsilon]$ means $\frac{a-b}{1+\varepsilon} \in \mathbb{Z}[\varepsilon]$. Note that all matrices corresponding to the transformations (3.24) and (3.25) belong to Γ . We conjecture that the group Γ is generated by these transformations.

Remark 1: limiting cases. Lagrangian densities of the form $xg(y, z)$ can be obtained as $\lim_{t \rightarrow 0} \frac{f(tx, y, z)}{t}$ from $f(x, y, z)$ as defined by (3.23). This gives the function $g(y, z)$ in the form

$$g(y, z) = y + \sum_{(k, l) \in \mathbb{Z}^2 \setminus 0} \frac{\tilde{\theta}((k - \varepsilon l)y)}{k - \varepsilon l} \exp\left(\frac{2\pi i}{3}(k^2 - kl + l^2)z\right) = \sum_{n \geq 0} a_n y^{6n+1} g_n(z). \quad (3.26)$$

This function is defined on the domain $\{(y, z) \in \mathbb{C}^2; \quad |y|^2 < \frac{2}{\sqrt{3}}\Im z\}$ and satisfies the following modular properties:

$$\begin{aligned} g(y + 1, z + (2\varepsilon - 1)y + \varepsilon + 1) &= g(y, z) + 1, \\ g(y + \varepsilon, z + (\varepsilon + 1)y + \varepsilon + 1) &= g(y, z) + \varepsilon, \\ g\left(\frac{y}{\gamma z + \delta}, \frac{\alpha z + \beta}{\gamma z + \delta}\right) &= g(y, z). \end{aligned} \quad (3.27)$$

Similarly, Lagrangian densities of the form $xyg(z)$ can be obtained as $\lim_{t \rightarrow 0} \frac{f(tx, ty, z)}{t^2}$. This brings us back to the modular form $g(z)$ discussed in Sect. 3.1.

Remark 2. Computer experiments show that solutions of the system (1.2) can also be sought in the form of a power series,

$$f(x, y, z) = \sum_{i, j, k \geq 0} c_{ijk} x^{6i+1} y^{6j+1} z^{6k+1}.$$

Moreover, $c_{ijk} = a_i a_j a_k b_{i+j+k}$ where a_i are the same as in (3.22), and b_i is yet another sequence of complex numbers. Taking a limit we obtain

$$g(y, z) = \sum_{j, k \geq 0} a_j a_k b_{j+k} y^{6j+1} z^{6k+1}$$

for densities of the form $f(x, y, z) = xg(y, z)$, and

$$g(z) = \sum_{k \geq 0} a_k b_k z^{6k+1}$$

for densities of the form $f(x, y, z) = xyg(z)$. We point out that integrable densities of this type differ from the solutions constructed above by appropriate transformations from the equivalence group.

Remark 3. The action of the 20-dimensional equivalence group on the 23-dimensional space of third order jets of the function $f(a, b, c)$ possesses three differential invariants which we denote by x, y, z . Viewed as functions of a, b, c , these invariants satisfy a nonlinear system which possesses a transitive action of $SL(4, \mathbb{R})$. This system can be linearised following the procedure outlined in Sect. 3.2. We plan to report the details elsewhere.

Remark 4. There exist a number of examples of integrable Lagrangian densities expressible in terms of elementary functions. One can mention, e.g., the four polynomial Lagrangians classified in [7]:

$$f = u_x u_y u_t, \quad f = u_x^2 u_y + u_y u_t, \quad f = u_x^3/3 + u_y^2 - u_x u_t$$

and

$$f = u_x^4 + 2u_x^2 u_t - u_x u_y - u_t^2.$$

It would be interesting to explicitly demonstrate how these (and other) examples can be obtained as degenerations of the ‘master-Lagrangian’ constructed in Section 3.2, and to describe singular orbits of lower dimensions.

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