Integrable equations in $2+1$ dimensions: deformations of dispersionless limits

E.V. Ferapontov, A. Moro and V.S. Novikov

Department of Mathematical Sciences
Loughborough University
Loughborough, Leicestershire LE11 3TU
United Kingdom

e-mails:

E.V.Ferapontov@lboro.ac.uk
A.Moro@lboro.ac.uk
V.Novikov@lboro.ac.uk

Abstract

We classify integrable third order equations in $2+1$ dimensions which generalize the examples of Kadomtsev-Petviashvili, Veselov-Novikov and Harry Dym equations. Our approach is based on the observation that dispersionless limits of integrable systems in $2+1$ dimensions possess infinitely many multi-phase solutions coming from the so-called hydrodynamic reductions. In this paper we adopt a novel perturbative approach to the classification problem. Based on the method of hydrodynamic reductions, we first classify integrable quasilinear systems which may (potentially) occur as dispersionless limits of soliton equations in $2+1$ dimensions. To reconstruct dispersive deformations, we require that all hydrodynamic reductions of the dispersionless limit are inherited by the corresponding dispersive counterpart. This procedure leads to a complete list of integrable third order equations, some of which are apparently new.

MSC: 35L40, 35Q51, 35Q58, 37K10, 37K55.

Keywords: dispersionless equations, hydrodynamic reductions, dispersive deformations, integrability.

1 Introduction

The classification of integrable systems has been a topic of active research from the very beginning of soliton theory. In $1+1$ dimensions, this resulted in extensive lists of integrable equations within particularly important subclasses [23], which were obtained by means of the symmetry approach. Although this technique generalizes to $2+1$ dimensions, one encounters additional difficulties due to the appearance of non-local variables [24]. A way to bypass the problem of non-locality, known as the perturbative symmetry approach [25], provides an efficient way to classify soliton equations in $2+1$ dimensions. In this framework, one starts with a linear equation having degenerate dispersion law [39], and reconstructs the allowed nonlinearity. However, few classification results have been obtained so far. In fact, most of the $(2+1)$-dimensional examples known to date were derived by postulating a special structure of the corresponding Lax pair, see e.g. [20], [33].

In this paper we adopt a novel approach to the problem of classification of scalar third order soliton equations in $2+1$ dimensions with the 'simplest' possible non-localities,

$$u_t = F(u, w, Du, Dw),$$

and

$$u_t = F(u, v, w, Du, Dv, Dw),$$

respectively. Here $u(x, y, t)$ is a scalar field, and the non-local variables $v(x, y, t)$ and $w(x, y, t)$ are defined via $w_x = u_y$ and $v_y = u_x$, equivalently, $w = D_x^{-1} D_y u$, $v = D_y^{-1} D_x u$. The symbols Du, Dv, Dw denote the collection of all partial derivatives of u, v, w with respect to x and y up to the third order. In fact, it is sufficient to allow only y-derivatives of w and x-derivatives of v. We will refer to the above equations as the 'non-symmetric' and 'symmetric' cases, respectively. We assume that in both cases the dependence of the right hand side F on the derivatives of u and w (resp, u, v, w) is *polynomial*, where the coefficients are allowed to be arbitrary functions of u and w (resp, u, v, w). Explicitly, in the non-symmetric case we have

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(...) + \epsilon^2(...), \qquad w_x = u_y, \tag{1}$$

where φ, ψ, η are functions of u and w, while the terms at ϵ and ϵ^2 are assumed to be homogeneous differential polynomials of the order two and three in the derivatives of u and w, whose coefficients can be arbitrary functions of u and w. We use the following weighting scheme: u and w are assumed to have order zero, their derivatives u_x, u_y, w_x, w_y are of order one, the expressions $u_{xx}, u_{xy}, u_{yy}, w_{yy}, u_x^2, u_x u_y, u_y^2, u_x w_y, u_y w_y, w_y^2$ are of order two, etc. Thus, the term at ϵ is a linear combination of the ten second order expressions whose coefficients can be arbitrary functions of u and w. The most familiar example within the class (1) is the Kadomtsev-Petviashvili (KP) equation,

$$u_t = u u_x + w_y + \epsilon^2 u_{xxx}, \qquad w_x = u_y.$$

Similarly, in the symmetric case we consider equations of the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_x + \epsilon(...) + \epsilon^2(...), \qquad w_x = u_y, \ v_y = u_x, \tag{2}$$

here $\varphi, \psi, \eta, \tau$ are functions of u, v and w. A canonical example of the form (2) is the Veselov-Novikov (VN) equation,

$$u_t = (uv)_x + (uw)_y + \epsilon^2(u_{xxx} + u_{yyy}), \qquad w_x = u_y, \ v_y = u_x.$$

In Sect. 2 we bring together other known examples of the form (1) and (2) which include the KP, VN, Harry Dym equations and their modifications.

Our approach to the classification problem is based on the following key observations.

- **Dispersionless limits of integrable soliton equations in $2+1$ dimensions possess infinitely many hydrodynamic reductions.**

In particular, dispersionless limits of Eqs. (1) and (2),

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \qquad w_x = u_y, \tag{3}$$

and

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_x, \quad w_x = u_y, \ v_y = u_x, \tag{4}$$

should possess infinitely many hydrodynamic reductions and, thus, must be integrable in the sense of [10]. It was observed in [10] that the method of hydrodynamic reductions provides an efficient classification criterion. Thus, as a first step, in Sect. 3 we classify integrable first order equations of the form (3) and (4) which may (potentially) occur as dispersionless limits of integrable equations of the form (1) and (2). We emphasize that the requirement of being a dispersionless limit of a *third order* soliton equation imposes further severe constraints, so that very few particular cases obtained in Sect. 3 do actually survive.

Given an integrable dispersionless equation, one needs to reconstruct dispersive deformations. In $1+1$ dimensions, this problem has been a subject of extensive research in [7, 8, 9, 21], see also [1]. In $2+1$ dimensions, the reconstruction procedure is based on the following key observation [14]:

- **Hydrodynamic reductions of dispersionless limits of integrable soliton equations can be deformed into reductions of the corresponding dispersive counterparts (strictly speaking, this is only true if the dispersionless limit is linearly non-degenerate, see Sect. 4). Furthermore, the requirement of the inheritance of all hydrodynamic reductions allows one to efficiently reconstruct dispersive terms in $2+1$ dimensions.**

This suggests the following alternative **definition** of the integrability:

A $(2+1)$-dimensional system is said to be integrable if all hydrodynamic reductions of its dispersionless limit (which is assumed to be linearly non-degenerate) can be deformed into reductions of the corresponding dispersive counterpart.

Although this property is satisfied for all known integrable equations whose dispersionless limit is not totally linearly degenerate, it would be important to formulate more precise statements about the equivalence of our definition with more 'conventional' approaches to the integrability.

The procedure of the reconstruction of dispersive terms is thoroughly illustrated in Sect. 4, where we examine case-by-case all integrable dispersionless limits from Sect. 3. Our calculations result in a complete list of integrable $(2+1)$-dimensional equations, some of which are apparently new. It is important to emphasize that, although our approach is based on the requirement of the inheritance of hydrodynamic reductions, all examples from the final list do actually possess conventional Lax pairs. Altogether, we found three new equations. One of them is

$$u_t = (\beta w + \beta^2 u^2)u_x - 3\beta u u_y + w_y + \epsilon^2[B^3(u) - \beta u_x B^2(u)], \quad w_x = u_y, \tag{5}$$

here $B = \beta u D_x - D_y$, β=const. It possesses the Lax pair

$$\psi_{xy} = \beta u \psi_{xx} + \frac{1}{3\epsilon^2}\psi,$$
$$\psi_t = \epsilon^2\beta^3 u^3\psi_{xxx} - \epsilon^2\psi_{yyy} + 3\epsilon^2\beta^2 u u_y \psi_{xx} + \beta w \psi_x.$$

The second example is

$$u_t = \frac{4}{3}\beta^2 u^3 u_x + (w - 3\beta u^2)u_y + u w_y + \epsilon^2[B^3(u) - \beta u_x B^2(u)], \qquad w_x = u_y, \tag{6}$$

here again $B = \beta u D_x - D_y$, β=const. The corresponding Lax pair is

$$\psi_{xy} = \beta u \psi_{xx} + \frac{1}{3\epsilon^2}u\psi,$$
$$\psi_t = \epsilon^2\beta^3 u^3\psi_{xxx} - \epsilon^2\psi_{yyy} + 3\epsilon^2\beta^2 u u_y \psi_{xx} + \frac{\beta^2}{3}u^3\psi_x + w\psi_y + \beta u u_y \psi.$$

We point out that similar Lax operators appeared in the context of the $(2+1)$-dimensional Camassa-Holm equation [40]. Our last example is a deformation of the Harry Dym (HD) equation,

$$u_t = \frac{\delta}{u^3}u_x - 2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx}, \qquad w_x = u_y, \tag{7}$$

for $\delta = 0$ it reduces to the standard HD equation (Example 6 of Sect. 2.1). It has the Lax pair $L_t = [A, L]$ where

$$L = \frac{\epsilon^2}{u^2}D_x^2 + \frac{\epsilon}{\sqrt{3}}D_y + \frac{\delta}{4u^2},$$
$$A = \frac{4\epsilon^2}{u^3}D_x^3 + \left(-\frac{6\epsilon^2 u_x}{u^4} + \frac{2\sqrt{3}\epsilon w}{u^2}\right)D_x^2 + \frac{\delta}{u^3}D_x + \left(-\frac{3\delta u_x}{2u^4} + \frac{\sqrt{3}\delta w}{2\epsilon u^2}\right).$$

All three examples belong to the non-symmetric case. In the symmetric case we have no new equations apart from those listed in Sect. 2.2. This leads to the following main result:

Theorem 1 *Equations (5) – (7) along with the known examples of KP, non-symmetric VN, HD equations and their modifications provide a complete list of integrable equations of the form*

(1) with $\eta \neq 0$ whose dispersionless limit is linearly nondegenerate:

KP equation	$u_t = uu_x + w_y + \epsilon^2 u_{xxx}$,
mKP equation	$u_t = (w - u^2/2)u_x + w_y + \epsilon^2 u_{xxx}$,
Gardner equation	$u_t = (\beta w - \frac{\beta^2}{2}u^2 + \delta u)u_x + w_y + \epsilon^2 u_{xxx}$,
VN equation	$u_t = (uw)_y + \epsilon^2 u_{yyy}$,
mVN equation	$u_t = (uw)_y + \epsilon^2 \left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u}\right)_y$,
HD equation	$u_t = -2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx}$,
deformed HD equation	$u_t = \frac{\delta}{u^3}u_x - 2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx}$,
Equation (5)	$u_t = (\beta w + \beta^2 u^2)u_x - 3\beta u u_y + w_y + \epsilon^2[B^3(u) - \beta u_x B^2(u)]$,
Equation (6)	$u_t = \frac{4}{3}\beta^2 u^3 u_x + (w - 3\beta u^2)u_y + uw_y + \epsilon^2[B^3(u) - \beta u_x B^2(u)]$.

In the symmetric case there exist only two examples of integrable equations of the form (2) with $\eta, \tau \neq 0$:

VN equation	$u_t = (uv)_x + (uw)_y + \epsilon^2 u_{xxx} + \epsilon^2 u_{yyy}$,
mVN equation	$u_t = (uv)_x + (uw)_y + \epsilon^2 \left(u_{xx} - \frac{3}{4}\frac{u_x^2}{u}\right)_x + \epsilon^2 \left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u}\right)_y$.

The proof is summarised in Sect. 4. Under the substitution $w = 0,\ u_y = 0$ the equations (5), (6) reduce to

$$u_t = \epsilon^2 \beta^3 \left(u^3 u_{xxx} + 3u^2 u_x u_{xx}\right) + \beta^2 u^2 u_x$$

and

$$u_t = \epsilon^2 \beta^3 \left(u^3 u_{xxx} + 3u^2 u_x u_{xx}\right) + \frac{4}{3}\beta^2 u^3 u_x,$$

respectively. In this form, they have appeared in [34], see also [35] and references therein. It was pointed out (see e.g. [36, 35, 28]) that there exist differential substitutions bringing these equations to a constant separant form. It would be interesting to find out whether Eqs. (5) – (7) are related to any of the known soliton hierarchies: the main problem here is that the above differential substitutions do not extend to $2+1$ dimensions in any obvious way.

Remark 1. The examples of VN and mVN equations show that different $(2+1)$-dimensional equations may have one and the same dispersionless limit.

Remark 2. Our approach to the classification problem does not apply to non-symmetric equations with $\eta = 0$ (or symmetric equations with $\eta = \tau = 0$). As we explain in Sect. 3, these conditions are equivalent to the reducibility of the dispersion relations of the corresponding systems (3), (4). A familiar example within this class is the so-called 'breaking soliton' equation,

$$u_t = 2wu_x + 4uu_y - \epsilon^2 u_{xxy}, \qquad w_x = u_y,$$

see e.g. [3]. Here $\varphi = 2w,\ \psi = 4u,\ \eta = 0$. Equations of this type are not amenable to the method of hydrodynamic reductions, and require an alternative approach.

2 Known Examples

2.1 Non-symmetric case

Here we bring together known examples of soliton equations whose dispersionless limit is of the form (3). The relation $w_x = u_y$ will be automatically assumed whenever w appears explicitly in the equation. Examples 1-6 list third order equations. Examples 7-10 correspond to equations of order five, or differential-difference equations.

Example 1. The Kadomtsev-Petviashvili (KP) equation,

$$u_t = uu_x + w_y + \epsilon^2 u_{xxx}, \tag{8}$$

arises in mathematical physics as a two-dimensional generalization of the KdV equation. Its dispersionless limit (dKP equation),

$$u_t = uu_x + w_y, \tag{9}$$

also known as the Khokhlov-Zabolotskaya equation [37], is of interest in its own, playing important role in non-linear acoustics, gas dynamics and differential geometry.

Example 2. The modified KP (mKP) equation,

$$u_t = (w - u^2/2)u_x + w_y + \epsilon^2 u_{xxx}, \tag{10}$$

has the dispersionless limit

$$u_t = (w - u^2/2)u_x + w_y. \tag{11}$$

Example 3. The $(2+1)$-dimensional version of the Gardner equation is of the form [20],

$$u_t = (\beta w - \frac{\beta^2}{2}u^2 + \delta u)u_x + w_y + \epsilon^2 u_{xxx}, \tag{12}$$

which reduces to the KP or mKP equations upon setting $\beta = 0$ or $\delta = 0$, respectively. Its dispersionless limit has the form

$$u_t = (\beta w - \frac{\beta^2}{2}u^2 + \delta u)u_x + w_y. \tag{13}$$

Example 4. The non-symmetric version of the Veselov-Novikov equation [32, 27, 4],

$$u_t = (uw)_y + \epsilon^2 u_{yyy}, \tag{14}$$

has the dispersionless limit

$$u_t = (uw)_y. \tag{15}$$

Example 5. The non-symmetric version of the modified Veselov-Novikov equation [2],

$$u_t = (uw)_y + \epsilon^2 \left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u}\right)_y, \tag{16}$$

has the same dispersionless limit as in the previous example,

$$u_t = (uw)_y. \tag{17}$$

Example 6. The Harry Dym equation [20],

$$u_t = -2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx}, \tag{18}$$

(set $\tilde{u} = 1/u$ to obtain the equation from [20]), has the dispersionless limit

$$u_t = -2wu_y + uw_y. \tag{19}$$

Example 7. The fifth order version of the Harry Dym equation is

$$u_t = -3wu_y + uw_y - \frac{\epsilon^2}{u^4}(u^2 u_{xxy} - 3u(u_x u_y)_x + 6u_x^2 u_y) + \frac{\epsilon^4}{u^2}\left(\frac{1}{u^2}\right)_{xxxxx}, \tag{20}$$

see [20]. Its dispersionless limit has the form

$$u_t = -3wu_y + uw_y. \tag{21}$$

Example 8. The Toda lattice is a system of two differential-difference equations

$$\begin{aligned} \epsilon u_t &= u\,(w(y) - w(y-\epsilon)), \\ \epsilon w_x &= u(y+\epsilon) - u(y), \end{aligned} \tag{22}$$

or

$$\begin{aligned} u_t/u &= w_y - \frac{\epsilon}{2}w_{yy} + \frac{\epsilon^2}{6}w_{yyy} + \dots + (-1)^{n+1}\frac{\epsilon^n}{n!}w_{ny} + \dots, \\ w_x &= u_y + \frac{\epsilon}{2}u_{yy} + \frac{\epsilon^2}{6}u_{yyy} + \dots + \frac{\epsilon^n}{n!}u_{ny} + \dots. \end{aligned} \tag{23}$$

Its dispersionless limit is

$$u_t = uw_y. \tag{24}$$

Example 9. The nonlocal Toda lattice equation is

$$\epsilon\sigma_{xt} = e^{\frac{\sigma(x+\epsilon,y+\epsilon)-\sigma}{\epsilon}} - e^{\frac{\sigma-\sigma(x-\epsilon,y-\epsilon)}{\epsilon}}, \tag{25}$$

see [31]. Its dispersionless limit is

$$\sigma_{xt} = e^{\sigma_x+\sigma_y}(\sigma_{xx} + 2\sigma_{xy} + \sigma_{yy}), \tag{26}$$

or, setting $\sigma_x = u,\ \sigma_y = w$,

$$u_t = e^{u+w}(u_x + 2u_y + w_y).$$

Example 10. The BKP and CKP equations are of the form

$$u_t - 5(u^2 + w)u_x - 5uw_x + 5w_y + \epsilon^2(uu_{xxx} + w_{xxx} + u_{xxx}) - \frac{\epsilon^4}{25}\,u_{xxxxx} = 0, \tag{27}$$

and

$$u_t - 5(u^2 + w)u_x - 5uw_x + 5w_y + \epsilon^2(uu_{xxx} + w_{xxx} + \frac{5}{2}u_{xxx}) - \frac{\epsilon^4}{25}\,u_{xxxxx} = 0, \tag{28}$$

respectively [20]. Their dispersionless limits coincide:

$$u_t = 5(u^2 + w)u_x + 5uu_y - 5w_y. \tag{29}$$

2.2 Symmetric case

Here we list known examples of the form (2). The relations $v_y = u_x$ and $w_x = u_y$ will be automatically assumed whenever v and w appear explicitly in the equation. It is quite remarkable that the 'symmetric' list is very restrictive, and contains only two examples.

Example 1. The Veselov-Novikov equation,

$$u_t = (uv)_x + (uw)_y + \epsilon^2 u_{xxx} + \epsilon^2 u_{yyy}, \tag{30}$$

was introduced in [32], [27]. It has the dispersionless limit

$$u_t = (uv)_x + (uw)_y. \tag{31}$$

Example 2. The modified Veselov-Novikov equation,

$$u_t = (uv)_x + (uw)_y + \epsilon^2 \left(u_{xx} - \frac{3}{4}\frac{u_x^2}{u} \right)_x + \epsilon^2 \left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u} \right)_y, \tag{32}$$

was first introduced in [2] (in a somewhat different form). It has the same dispersionless limit as in the previous example,

$$u_t = (uv)_x + (uw)_y. \tag{33}$$

3 Classification of integrable dispersionless limits

In this section we classify integrable dispersionless equations of the form (3) and (4) which may potentially occur as dispersionless limits of integrable soliton equations of the form (1) and (2), respectively. The integrability conditions are derived based on the method of hydrodynamic reductions. For the convenience of the reader, we briefly recall the main steps of this construction. As proposed in [10], the method of hydrodynamic reductions applies to quasilinear equations of the following general form:

$$A(\mathbf{u})\mathbf{u}_t + B(\mathbf{u})\mathbf{u}_x + C(\mathbf{u})\mathbf{u}_y = 0; \tag{34}$$

here $\mathbf{u} = (u^1, ..., u^m)^t$ is an m-component column vector of the dependent variables, and A, B, C are $m \times m$ matrices. The method of hydrodynamic reductions consists of seeking multi-phase solutions in the form

$$\mathbf{u} = \mathbf{u}(R^1, ..., R^N) \tag{35}$$

where the 'phases' $R^i(x, y, t)$ are required to satisfy a pair of consistent equations of hydrodynamic type,

$$R^i_y = \mu^i(R) R^i_x, \qquad R^i_t = \lambda^i(R) R^i_x.$$

We recall that the consistency conditions, $R^i_{yt} = R^i_{ty}$, imply the following restrictions for the characteristic speeds μ^i and λ^i:

$$\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i},$$

$i \neq j$, $\partial_i = \partial/\partial_{R^i}$, see [30]. The substitution of the ansatz (35) into (34) leads to a complicated over-determined system of PDEs for the functions $\mathbf{u}(R)$, $\mu^i(R)$ and $\lambda^i(R)$ whose coefficients depend on the matrix elements of A, B, C, and their derivatives. In particular, the characteristic speeds $\mu^i(R)$ and $\lambda^i(R)$ satisfy an algebraic relation $\det(\lambda A + B + \mu C) = 0$ which is nothing but the dispersion relation of the system (34). We will assume that the dispersion relation defines an irreducible algebraic curve of degree m.

Definition [10]. *System (34) is said to be integrable if, for any number of phases N, it possesses infinitely many N-phase solutions parametrized by $2N$ arbitrary functions of one variable.*

The requirement of the existence of such solutions imposes strong constraints on the matrices A, B, C, which can be effectively computed. Although these constraints are quite formidable in general, there exists a simple necessary condition for the integrability which can be expressed in an invariant differential geometric form as follows. Let us first introduce the $m \times m$ matrix

$$V = (\alpha A + \beta B + \gamma C)^{-1}(\tilde{\alpha} A + \tilde{\beta} B + \tilde{\gamma} C)$$

where α, β, γ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ are arbitrary constants. Given a $(1,1)$-tensor $V = [v^i_j]$, let us introduce the following objects:
Nijenhuis tensor

$$\mathcal{N}^i_{jk} = v^p_j \partial_{u^p} v^i_k - v^p_k \partial_{u^p} v^i_j - v^i_p(\partial_{u^j} v^p_k - \partial_{u^k} v^p_j),$$

Haantjes tensor

$$\mathcal{H}^i_{jk} = \mathcal{N}^i_{pr} v^p_j v^r_k - \mathcal{N}^p_{jr} v^i_p v^r_k - \mathcal{N}^p_{rk} v^i_p v^r_j + \mathcal{N}^p_{jk} v^i_r v^r_p.$$

One has the following result.

Theorem 2 [12] *The vanishing of the Haantjes tensor is a necessary condition for the integrability of the system (34).*

Since the Haantjes tensor can be obtained using computer algebra, one gets an efficient integrability test (notice that all components of the Haantjes tensor have to vanish for *any* values of the constants α, β, γ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$). These necessary conditions are very strong indeed, and in many cases turn out to be sufficient. We point out that, for $m = 2$, the Haantjes tensor vanishes identically and does not produce any non-trivial integrability conditions. In this case one proceeds as follows: let us multiply (34) by A^{-1}, and diagonalize B (this is always possible in the 2-component case). Thus, without any loss of generality one can assume

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

In this particular normalization, the integrability conditions for 2×2 systems were obtained in [11]. These conditions constitute a system of second order constraints for the coefficients a, b, p, q, r, s which can be easily tested. Let us now apply this approach to the classification of integrable systems of the form (3) and (4).

3.1 Non-symmetric dispersionless limits

Given an equation of the form (3),

$$u_t = \varphi u_x + \psi u_y + \eta w_y,$$
$$w_x = u_y,$$

let us first rewrite it in matrix form (34) as follows:

$$\begin{pmatrix} -1/\varphi & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}_x + \begin{pmatrix} \psi/\varphi & \eta/\varphi \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}_y = 0.$$

This system is now in the form as studied in [11]. The integrability conditions reduce to a system of second order partial differential equations for the coefficients φ, ψ and η, which can be derived from the general integrability conditions for 2×2 systems of hydrodynamic type in $2+1$ dimensions as obtained in [11]:

$$\begin{aligned}
\varphi_{uu} &= -\frac{\varphi_w^2 + \psi_u \varphi_w - 2\psi_w \varphi_u}{\eta}, \\
\varphi_{uw} &= \frac{\eta_w \varphi_u}{\eta}, \\
\varphi_{ww} &= \frac{\eta_w \varphi_w}{\eta}, \\
\\
\psi_{uu} &= \frac{-\varphi_w \psi_w + \psi_u \psi_w - 2\varphi_w \eta_u + 2\eta_w \varphi_u}{\eta}, \\
\psi_{uw} &= \frac{\eta_w \psi_u}{\eta}, \\
\psi_{ww} &= \frac{\eta_w \psi_w}{\eta}, \\
\\
\eta_{uu} &= -\frac{\eta_w (\varphi_w - \psi_u)}{\eta}, \\
\eta_{uw} &= \frac{\eta_w \eta_u}{\eta}, \\
\eta_{ww} &= \frac{\eta_w^2}{\eta};
\end{aligned} \tag{36}$$

we assume $\eta \neq 0$: this is equivalent to the requirement that the dispersion relation of the system (3) defines an irreducible conic (indeed, the condition $\det(\lambda A + B + \mu C) = 0$ is equivalent to $\lambda = \varphi + \psi\mu + \eta\mu^2$). We have verified that the system (36) is in involution, and all dispersionless limits appearing in Sect. 2.1 indeed satisfy these integrability conditions. Eqs. (36) are straightforward to solve. First of all, the equations for η imply that, up to translations and rescalings, $\eta = 1$, $\eta = u$ or $\eta = e^w h(u)$. We will consider all three possibilities case-by-case below. Notice that φ and ψ are defined up to additive constants which can always be set equal to zero via the Galilean transformations of the initial equation (3). Moreover, the system (36) is form-invariant under transformations of the form

$$\tilde{\varphi} = \varphi - s\psi + s^2\eta, \quad \tilde{\psi} = \psi - 2s\eta, \quad \tilde{\eta} = \eta, \quad \tilde{u} = u, \quad \tilde{w} = w + su, \tag{37}$$

s=const, which correspond to the following transformations preserving the structure of equations (3):

$$\tilde{x} = x - sy, \quad \tilde{y} = y, \quad \tilde{u} = u, \quad \tilde{w} = w + su.$$

All our classification results are formulated modulo this equivalence.

Case 1: $\eta = 1$. Then the remaining equations imply $\psi = \alpha w + f(u),\ \varphi = \beta w + g(u)$, where f and g satisfy the linear ODEs

$$f'' = \alpha(f' - \beta), \quad g'' = 2\alpha g' - \beta f' - \beta^2.$$

The subcase $\alpha = 0$ leads to polynomial solutions of the form

$$\psi = \gamma u, \qquad \varphi = \beta w - \frac{1}{2}\beta(\beta + \gamma)u^2 + \delta u. \tag{38}$$

Up to equivalence transformations, the case $\alpha \neq 0$ leads to exponential solutions,

$$\psi = \alpha w + \gamma e^{\alpha u}, \qquad \varphi = \delta e^{2\alpha u}; \tag{39}$$

here $\alpha, \beta, \gamma, \delta$ are arbitrary constants.

Case 2: $\eta = u$. Then the remaining equations imply $\psi = \alpha w + f(u),\ \varphi = \beta w + g(u)$, where f and g satisfy the linear ODEs

$$uf'' = \alpha(f' - \beta) - 2\beta, \quad ug'' = 2\alpha g' - \beta f' - \beta^2.$$

The case $\alpha \notin \{0, -1, -1/2\}$ leads to power-like solutions of the form

$$\psi = \alpha w + \gamma u^{\alpha+1}, \qquad \varphi = \delta u^{2\alpha+1}. \tag{40}$$

The subcase $\alpha = 0$ leads to logarithmic solutions,

$$\psi = -2\beta u \ln u - \beta u, \qquad \varphi = \beta w + \beta^2 u \ln^2 u + \delta u. \tag{41}$$

The subcase $\alpha = -1$ gives

$$\psi = -w + \gamma \ln u, \qquad \varphi = \delta/u. \tag{42}$$

Finally, the subcase $\alpha = -1/2$ gives

$$\psi = -\frac{1}{2}w + \gamma\sqrt{u}, \qquad \varphi = \delta \ln u. \tag{43}$$

Case 3: $\eta = e^w h(u)$. Then the remaining equations imply $\psi = e^w f(u),\ \varphi = e^w g(u)$ where f, g and h satisfy the nonlinear system of ODEs

$$h'' = f' - g, \qquad g''h = 2fg' - gf' - g^2, \qquad f''h = 2hg' - 2gh' + ff' - fg.$$

Setting $g = p',\ f = h' + p$, we can rewrite this system as a pair of third order ODEs

$$hp''' = 2h'p'' - p'h'' + 2pp'' - 2p'^2, \quad hh''' = h'h'' - 2h'p' + hp'' + ph'',$$

which, up to a change of sign $p \to -p$, identically coincides with a system arising in the classification of integrable conservative hydrodynamic chains (subcase I_1 of Sect. 3.1 in [13]). Setting $p = h'$, the second equation will be satisfied identically, while the first one implies a fourth order ODE for h, $h''''h + 3(h'')^2 - 4h'h''' = 0$, whose general solution is an elliptic sigma-function: $h = \sigma(u)$, here $(\ln\sigma)'' = -\wp$, $(\wp')^2 = 4\wp^3 - c$ (notice that $g_2 = 0,\ g_3 = c$). Thus, as a particular case we have

$$h = \sigma(u), \quad f = 2\sigma'(u), \quad g = \sigma''(u).$$

Another subclass of solutions can be obtained by setting $p = ch$ which implies

$$h'''h - h''h' = 2c(h''h - h'^2)$$

with the general solution

$$h = \alpha e^{(c+\gamma)u} + \beta e^{(c-\gamma)u};$$

here α, β, γ are arbitrary constants. Although the structure of the general solution is quite complicated, one can show that Case 3 cannot arise as a dispersionless limit of an integrable third order soliton equation.

3.2 Symmetric dispersionless limits

In this section we consider first order equations of the form (4),

$$\begin{aligned} u_t &= \varphi u_x + \psi u_y + \eta w_y + \tau v_x, \\ w_x &= u_y, \\ v_y &= u_x, \end{aligned}$$

where the coefficients $\varphi, \psi, \eta, \tau$ are functions of u, v, w. We assume that the dispersion relation of this system defines an irreducible cubic, which is equivalent to the requirement $\eta \neq 0$ and $\tau \neq 0$ (indeed, the dispersion relation has the form $\lambda\mu = \tau + \varphi\mu + \psi\mu^2 + \eta\mu^3$). In this case the integrability conditions reduce to a system of first order partial differential equations for the coefficients φ, ψ, η and τ which can be obtained from the requirement of the vanishing of the Haantjes tensor [12] as outlined in Sect. 3. The details are as follows: first we rewrite Eq. (4) in matrix form,

$$A\mathbf{u}_t + B\mathbf{u}_x + C\mathbf{u}_y = 0,$$

where $\mathbf{u}$ is a three-component column vector $\mathbf{u} = (u, v, w)^t$, and A, B, C are 3×3 matrices,

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \varphi & \tau & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \psi & 0 & \eta \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The necessary conditions for integrability can be obtained from the requirement of the vanishing of the Haantjes tensor of the following family of matrices,

$$(\alpha A + \beta B + \gamma C)^{-1}(\tilde{\alpha} A + \tilde{\beta} B + \tilde{\gamma} C).$$

In fact, it is sufficient to require the vanishing of the Haantjes tensor for a two-parameter family $(\alpha A + B)^{-1}(\tilde{\alpha} A + C)$. This condition turns out to be very restrictive, and leads to the following constraints for the coefficients φ, ψ, η and τ:

$$\begin{aligned} \tau_u &= \varphi_v, & \eta_u &= \psi_w, \\ \tau_v &= \frac{\tau}{\eta}\psi_u, & \eta_v &= 0, \\ \tau_w &= 0, & \eta_w &= \frac{\eta}{\tau}\varphi_u, \\ \psi_v &= \varphi_w = 0, & \tau\psi_w &= \eta\varphi_v. \end{aligned}$$

The integration of this system is straightforward. First of all, one can set $\psi = f_u,\ \eta = f_w$ and $\varphi = g_u,\ \tau = g_v$ where $f = f(u, w)$ and $g = g(u, v)$. The separation of variables leads to the relations

$$f_w = a(w)k(u), \qquad g_v = b(v)k(u),$$
$$f_{uu} = \beta a(w)k(u), \qquad g_{uu} = \alpha b(v)k(u),$$

where the functions $a(w),\ b(v)$ and $k(u)$ satisfy the ODEs $a' = \alpha a,\ b' = \beta b$ and $k'' = \alpha\beta k$; here α and β are arbitrary constants. Up to elementary translations, rescalings and Galilean transformations, this leads to the following subcases:

Case 1. $\alpha = \beta = 0$. This leads to equations of the form

$$u_t = \nu(uv)_x + \mu(uw)_y,$$

where μ, ν are arbitrary constants. These correspond to the Veselov-Novikov cases from Sect. 2.2.

Case 2. $\alpha \neq 0,\ \beta = 0$. This leads to equations of the form

$$u_t = \nu(uv + \alpha u^3/6)_x + \mu(e^{\alpha w}u)_y,$$

and

$$u_t = \nu(v + \alpha u^2/2)_x + \mu(e^{\alpha w})_y,$$

here μ, ν, α are arbitrary constants.

Case 3. $\alpha \neq 0,\ \beta \neq 0$. This leads to equations of the form

$$u_t = \nu(e^{\beta v}k(u))_x + \mu(e^{\alpha w}k(u))_y,$$

where ν, μ, α, β are arbitrary constants, and $k'' = \alpha\beta k$.

4 Classification of integrable 3rd order dispersive equations

Given an integrable dispersionless limit, one has to reconstruct dispersive terms. This can be done by requiring that all hydrodynamic reductions of the dipersionless system are inherited by its dispersive counterpart. We will illustrate this procedure using the KP equation,

$$u_t = uu_x + w_y + \epsilon^2 u_{xxx}, \qquad w_x = u_y.$$

Its dispersionless limit, the dKP equation,

$$u_t = uu_x + w_y, \qquad w_x = u_y,$$

possesses one-phase solutions of the form $u = R$, $w = w(R)$ where the phase $R(x, y, t)$ satisfies a pair of Hopf-type equations

$$R_y = \mu R_x, \qquad R_t = (\mu^2 + R)R_x; \tag{44}$$

here $\mu(R)$ is an arbitrary function, and $w' = \mu$. Equivalently, one can say that Eqs. (44) constitute a one-component hydrodynamic reduction of the dKP equation. Although the dKP equation is known to possess infinitely many N-component reductions for arbitrary N [15, 16, 17, 18], one-component reductions will be sufficient for our purposes. The main observation of [14] is that *all* one-component reductions (44) can be deformed into reductions of the full KP equation by adding appropriate dispersive terms which are *polynomial* in the x-derivatives of R. Explicitly, one has the following formulae for the deformed one-phase solutions,

$$u = R, \quad w = w(R) + \epsilon^2 \left(\mu' R_{xx} + \frac{1}{2}(\mu'' - (\mu')^3) R_x^2 \right) + O(\epsilon^4), \tag{45}$$

notice that one can always assume that u remains undeformed modulo the Miura group [7]. The deformed equations (44) take the form

$$\begin{aligned} R_y =& \mu R_x \\ &+ \epsilon^2 \left(\mu' R_{xx} + \frac{1}{2}(\mu'' - (\mu')^3) R_x^2 \right)_x + O(\epsilon^4), \\ R_t =& (\mu^2 + R) R_x \\ &+ \epsilon^2 \left((2\mu\mu' + 1) R_{xx} + (\mu\mu'' - \mu(\mu')^3 + (\mu')^2/2) R_x^2 \right)_x + O(\epsilon^4). \end{aligned} \tag{46}$$

In other words, the KP equation can be 'decoupled' into a pair of $(1+1)$-dimensional equations (46) in infinitely many ways, indeed, $\mu(R)$ is an arbitrary function. The series in (45) and (46) contain only even powers of ϵ, and do not terminate in general.

Conversely, the requirement of the inheritance of all one-component reductions allows one to reconstruct dispersive terms: given the dKP equation, let us look for a third order dispersive extension in the form

$$u_t = u u_x + w_y + \epsilon(...) + \epsilon^2(...), \qquad w_x = u_y, \tag{47}$$

where the terms at ϵ and ϵ^2 are homogeneous differential polynomials in the x- and y-derivatives of u and w of the order two and three, respectively, whose coefficients are allowed to be arbitrary functions of u and w. We require that all one-component reductions (44) can be deformed accordingly, so that we have the following analogues of Eqs. (45) and (46),

$$u = R, \quad w = w(R) + \epsilon(...) + \epsilon^2(...) + O(\epsilon^3), \tag{48}$$

and

$$R_y = \mu R_x + \epsilon(...) + \epsilon^2(...) + O(\epsilon^3), \qquad R_t = (\mu^2 + R) R_x + \epsilon(...) + \epsilon^2(...) + O(\epsilon^3), \tag{49}$$

respectively. In Eqs. (48) and (49), dots denote terms which are polynomial in the derivatives of R. Substituting Eqs. (48) into (47), and using (49) along with the consistency conditions $R_{ty} = R_{yt}$, one arrives at a complicated set of relations allowing one to uniquely reconstruct dispersive terms in (47): not surprisingly, we obtain that all terms at ϵ vanish, while the terms at ϵ^2 result in the familiar KP equation. Moreover, one only needs to perform calculations up to the order ϵ^4 to arrive at this result! It is important to emphasize that the above procedure is required to work for *arbitrary* μ: whenever one obtains a differential polynomial in μ which has to vanish due to the consistency conditions, all its coefficients have to be set equal to zero independently. Another observation is that the reconstruction procedure does not necessarily lead to a unique dispersive extension as in the dKP case: one and the same dispersionless system may possess essentially non-equivalent dispersive extensions. In most of the cases one can get the necessary classification results working with one-component reductions only. There is however

one particular situation where one-component reductions are not sufficient. This is explained in the remark below.
Remark 1. Let us consider the dKP equation,

$$u_t = uu_x + w_y, \qquad w_x = u_y;$$

its one-component reductions (44) can be shown to satisfy a pair of additional first order constraints,

$$u_y^2 - u_x w_y = 0, \qquad (w_t - uu_y)u_x - u_y w_y = 0.$$

Conversely, any solution satisfying these constraints comes from one-component reductions. Similarly, one can show that two-component reductions of dKP are characterised by a pair of second order differential constraints, etc. Let us introduce an extension of dKP in the form

$$u_t = uu_x + w_y + \epsilon(u_y^2 - u_x w_y), \qquad w_x = u_y;$$

by construction, it inherits all *undeformed* one-component reductions: the ϵ-term vanishes on one-component reductions identically. This extension is, however, not integrable: one can show that it is not consistent with the requirement of the inheritance of N-component reductions for $N \geq 2$. Thus, in what follows we eliminate deformations which inherit undeformed one-component reductions.

In general, we proceed as follows. For definiteness, we will outline the algorithm for integrable dispersionless equations of the form (3),

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \qquad w_x = u_y.$$

Its one-component reductions are of the form $u = R$, $w = w(R)$ where $R(x, y, t)$ satisfies a pair of Hopf-type equations

$$R_y = \mu R_x, \qquad R_t = (\varphi + \psi\mu + \eta\mu^2)R_x;$$

here $\mu(R)$ is an arbitrary function, and $w' = \mu$. We seek a third order dispersive deformation of Eq. (3) in the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(...) + \epsilon^2(...), \qquad w_x = u_y,$$

and postulate that one-phase solutions can be deformed accordingly,

$$u = R, \quad w = w(R) + \epsilon(...) + \epsilon^2(...) + O(\epsilon^3),$$

where

$$R_y = \mu R_x + \epsilon(...) + \epsilon^2(...) + O(\epsilon^3), \qquad R_t = (\varphi + \psi\mu + \eta\mu^2)R_x + \epsilon(...) + \epsilon^2(...) + O(\epsilon^3).$$

Proceeding as outlined above we reconstruct possible dispersive terms. In fact, one can start with arbitrary φ, ψ, η: our procedure will eventually recover the constraints obtained in Sect. 3. However, using the classification results of Sect. 3 from the very beginning considerably simplifies the calculations.
Remark 2. We point out that the formulae for dispersive deformations contain the expression

$$\eta_w\mu^3 + (\psi_w + \eta_u)\mu^2 + (\varphi_w + \psi_u)\mu + \varphi_u$$

in the denominator. Since μ is assumed to be arbitrary, this expression is nonzero unless φ, ψ, η satisfy the relations

$$\eta_w = 0, \quad \psi_w + \eta_u = 0, \quad \varphi_w + \psi_u = 0, \quad \varphi_u = 0. \tag{50}$$

These relations characterize the so-called *totally linearly degenerate systems*, which are known to be quite special from the point of view of the global existence of classical solutions: it was conjectured in [22] that smooth initial data for totally linearly degenerate systems do not break down in finite time. Modulo the integrability conditions (36), the relations (50) lead to equations of the form

$$u_t = \alpha(wu_x - uw_x) + \beta(wu_y - uw_y) + \gamma w_y, \quad w_x = u_y,$$

which have been discussed before in the context of the so-called 'universal hierarchy' [26]. For totally linearly degenerate systems (in particular, for linear systems), the procedure based on deformations of hydrodynamic reductions does not work, as the following simple example shows. Let us consider the KP equation,

$$u_t = \alpha u u_x + w_y + \epsilon^2 u_{xxx}, \quad w_x = u_y,$$

where we introduced a parameter α: for $\alpha = 0$ the equation becomes linear. Looking for deformed one-phase solutions in the form

$$u = R, \quad w = w(R) + \epsilon^2(...) + O(\epsilon^4),$$

where

$$R_y = \mu R_x + \epsilon^2(...) + O(\epsilon^4), \quad R_t = (\mu^2 + \alpha R)R_x + \epsilon^2(...) + O(\epsilon^4),$$

one can obtain the relation $\alpha b(R) - \mu' = 0$ where $b(R)$ is the coefficient at R_{xxx} in the ϵ^2-term in the expansion of R_y. For $\alpha = 0$ one cannot solve for $b(R)$, and obtains a relation $\mu' = 0$. Thus, the linear equation $u_t = w_y + \epsilon^2 u_{xxx}$ does not inherit generic hydrodynamic reductions of its dispersionless limit. Another example of this kind is provided by the potential KP equation,

$$u_t = w_y + \frac{\epsilon}{2} u_x^2 + \epsilon^2 u_{xxx}. \tag{51}$$

One can show that this equation does not inherit hydrodynamic reductions of its dispersionless limit. However, some particular reductions can be inherited, for instance, those with μ=const.

Thus, we exclude totally linearly degenerate systems from the further considerations: dispersive deformations of such systems do not inherit hydrodynamic reductions, and require a different approach.

4.1 Non-symmetric dispersive equations

In this Section we summarize the classification results for integrable non-symmetric third order equations (1),

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(...) + \epsilon^2(...), \quad w_x = u_y,$$

which are obtained by adding dispersive terms to integrable dispersionless candidates from Sect. 3.1. Thus, we follow the classification of Sect. 3.1.

Case 1: We have verified that the exponential solutions (39) do not survive, so that all non-trivial examples come from the polynomial case (38),

$$\eta = 1, \qquad \psi = \gamma u, \qquad \varphi = \beta w - \frac{1}{2}\beta(\beta+\gamma)u^2 + \delta u.$$

We point out that the corresponding dispersionless system possesses the Lax pair

$$\begin{aligned} S_y &= \beta u S_x + r(S_x), \\ S_t &= \left(\beta w + \frac{1}{2}\beta(\beta+\gamma)u^2\right) S_x + \beta u S_x r'(S_x) + z(S_x), \end{aligned} \tag{52}$$

where

$$r(S_x) = -\frac{\delta}{\beta+\gamma} S_x + S_x^{\frac{2\beta+\gamma}{\beta}}, \quad z' = r'^2.$$

Lax pairs of this kind, consisting of two compatible Hamilton-Jacobi type equations, were first introduced by Zakharov in [38]. A detailed analysis of dispersive deformations leads to the two branches: $\gamma = 0$, which corresponds to the $(2+1)$-dimensional Gardner equation (Example 3 of Sect. 2.1), and the case $\gamma = -3\beta$. In the latter case one can set $\delta = 0$, which leads to the apparently new equation (5),

$$u_t = (\beta w + \beta^2 u^2)u_x - 3\beta u u_y + w_y + \epsilon^2[B^3(u) - \beta B^2(u)u_x],$$

where $B = \beta u D_x - D_y$. The dispersionless limit of this equation possesses the Lax pair

$$\begin{aligned} S_x S_y &= \beta u S_x^2 + \frac{1}{3}, \\ S_t &= \beta^3 u^3 S_x^3 - S_y^3 + \beta w S_x, \end{aligned} \tag{53}$$

which follows from (52) when $\gamma = -3\beta$. Its dispersive extension is

$$\begin{aligned} \psi_{xy} &= \beta u \psi_{xx} + \frac{1}{3\epsilon^2}\psi, \\ \psi_t &= \beta^3 \epsilon^2 u^3 \psi_{xxx} - \epsilon^2 \psi_{yyy} + 3\beta^2\epsilon^2 u u_y \psi_{xx} + \beta w \psi_x. \end{aligned} \tag{54}$$

This is case (5) from the Introduction.

Case 2: One can prove that none of the logarithmic cases (41), (42) and (43) survive, so that all non-trivial examples come from the power case (40),

$$\eta = u, \qquad \psi = \alpha w + \gamma u^{\alpha+1}, \qquad \varphi = \delta u^{2\alpha+1}.$$

Further analysis leads to the following branches.
Subcase 2.1: $\alpha = 1$. In this case

$$\eta = u, \qquad \psi = w + \gamma u^2, \qquad \varphi = \delta u^3.$$

The corresponding dispersionless Lax pair is of the form

$$\begin{aligned} S_y &= ua, \\ S_t &= uwa + \frac{1}{3}a(\gamma + a')u^3, \end{aligned} \tag{55}$$

where the function $a(S_x)$ solves the ODE $aa'' - 2a'^2 = 3\delta + 2\gamma a'$. The further analysis gives either $\gamma = \delta = 0$, which leads to the non-symmetric Veselov-Novikov cases (Examples 4 and 5 of Sect. 2.1, in this case one can take $a = 1/S_x$), or $\delta = \frac{4}{27}\gamma^2$, in which case one arrives at the apparently new dispersive equation (6),

$$u_t = \frac{4}{27}\gamma^2 u^3 u_x + (w + \gamma u^2)u_y + uw_y + \epsilon^2[B^3(u) - \frac{1}{3}\gamma u_x B^2(u)],$$

where $B = \frac{1}{3}\gamma u D_x + D_y$. This corresponds to the choice $a = 1/S_x - \frac{\gamma}{3}S_x$ in the dispersionless Lax pair (55), which gives

$$\begin{aligned} S_x S_y &= -\frac{\gamma}{3}uS_x^2 - \frac{u}{3}, \\ S_t &= \frac{\gamma^3}{27}u^3 S_x^3 + S_y^3 + \frac{\gamma^2}{27}u^3 S_x + wS_y. \end{aligned} \tag{56}$$

The dispersive extension of this Lax pair is

$$\begin{aligned} \psi_{xy} &= -\frac{\gamma}{3}u\psi_{xx} - \frac{1}{3\epsilon^2}u\psi, \\ \psi_t &= \frac{\epsilon^2\gamma^3}{27}u^3\psi_{xxx} + \epsilon^2\psi_{yyy} - \frac{\epsilon^2\gamma^2}{3}uu_y\psi_{xx} + \frac{\gamma^2}{27}u^3\psi_x + w\psi_y - \frac{\gamma}{3}uu_y\psi. \end{aligned} \tag{57}$$

The transformation $\gamma \to 3\beta,\ y \to -y,\ w \to -w$ reduces this case to Eq. (6) from the Introduction.

Subcase 2.2: $\alpha = -2$. In this case one obtains $\gamma = 0$, while δ can be an arbitrary constant. The corresponding dispersive extension takes the form (7),

$$u_t = \frac{\delta}{u^3}u_x - 2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx},$$

for $\delta = 0$ it reduces to the Harry Dym equation (Example 6 of Sect. 2.1). The dispersionless limit of this equation possesses the Lax pair

$$\begin{aligned} S_y &= \frac{S_x^2 + \tau}{u^2}, \\ S_t &= -2w\frac{S_x^2 + \tau}{u^2} + \frac{4}{3}\frac{S_x^3 + \tau S_x}{u^3}; \end{aligned} \tag{58}$$

here $\tau = 3\delta/4$. Its dispersive extension is of the form $L_t = [A, L]$ where

$$\begin{aligned} L &= \frac{\epsilon^2}{u^2}D_x^2 + \frac{\epsilon}{\sqrt{3}}D_y + \frac{\delta}{4u^2}, \\ A &= \frac{4\epsilon^2}{u^3}D_x^3 + \left(-\frac{6\epsilon^2 u_x}{u^4} + \frac{2\sqrt{3}\epsilon w}{u^2}\right)D_x^2 + \frac{\delta}{u^3}D_x + \left(-\frac{3\delta u_x}{2u^4} + \frac{\sqrt{3}\delta w}{2\epsilon u^2}\right). \end{aligned} \tag{59}$$

Case 3: One can show that none of the examples from this class possess third order dispersive extensions.

4.2 Symmetric dispersive equations

A detailed analysis of dispersive extensions of the form (2),

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_x + \epsilon(...) + \epsilon^2(...), \quad w_x = u_y,\ v_y = u_x,$$

does not give any new examples: everything reduces to the two cases of Sect. 2.2. Notice that both symmetric VN and mVN equations can be viewed as linear combinations of the two commuting non-symmetric counterparts thereof.

5 Concluding remarks

We have proposed a new approach to the classification of integrable equations in 2+1 dimensions based on the concept of hydrodynamic reductions and their dispersive deformations. It consists of the two steps:
— Classification of dispersionless systems which may (potentially) arise as dispersionless limits of soliton equations. This can be efficiently achieved using the method of hydrodynamic reductions as outlined in [10];
— Classification of possible dispersive deformations based on the requirement that hydrodynamic reductions of the dispersionless limit are inherited by the dispersive equation [14].
This procedure was applied to the classification of third order soliton equations with 'simplest' nonlocalities. Further research in this direction may include the following topics:
(a) Classification of more general (in particular, higher order) soliton equations/systems with more complicated structure of nonlocal terms. Thus, one may allow 'nested' nonlocalities of the type $w = D_x^{-1}D_y u$, $v = D_x^{-1}D_y F(u, w)$, etc.
(b) Construction of dispersive deformations via an appropriate quantization of the corresponding dispersionless Lax pairs [38].
(c) Investigation of the structure of multi-soliton solutions of the new equations (5) – (7) in the spirit of [5, 6].

Acknowledgements

We thank B. Dubrovin, A. Hone, B. Konopelchenko, A. Mikhailov, M. Pavlov and J.P. Wang for clarifying discussions. We also thank the referees for useful comments. The research of EVF and AM was supported by the EPSRC grant EP/D036178/1, the European Union through the FP6 Marie Curie RTN project ENIGMA (Contract number MRTN-CT-2004-5652), and the ESF programme MISGAM. The research of VSN was supported by the EPSRC Postdoctoral Fellowship grant EP/C527747/1.

References

[1] V.A. Baikov, R.K Gazizov and N.Kh. Ibragimov, Approximate symmetries and formal linearization, J. Appl. Mech. Tech. Phys. **30**, no. 2 (1989) 204–212.

[2] L.V. Bogdanov, Veselov-Novikov equation as a natural two-dimensional generalization of the Korteweg-de Vries equation, Theor. and Math. Phys. **70** (1987) 309–314.

[3] O.I. Bogoyavlenskii, Overturning solitons in two-dimensional integrable equations, Russian Math. Surveys **45**, no. 4 (1990) 1–86.

[4] M. Boiti, J. Jp. Leon, M. Manna, F. Pempinelli, On the spectral transform of a Korteweg de Vries equation in two spatial dimensions, Inverse Problems **2**, no. 3 (1986) 271–279.

[5] S. Chakravarty and Y Kodama, Classification of the line-soliton solutions of KPII, J. Phys. A **41**, no. 27 (2008) 275209, 33 pp.

[6] S. Chakravarty and Y. Kodama, Soliton solutions of the KP equation and application to shallow water waves, arXiv:0902.4433.

[7] B.A. Dubrovin and Youjin Zhang, Bi-Hamiltonian hierarchies in 2D topological field theory at one-loop approximation, Comm. Math. Phys. **198** (1998) no. 2, 311–361.

[8] B.A. Dubrovin, Si-Qi Liu and Youjin Zhang, On Hamiltonian perturbations of hyperbolic systems of conservation laws. I. Quasi-triviality of bi-Hamiltonian perturbations, Comm. Pure Appl. Math. **59**, no. 4 (2006) 559–615.

[9] B.A. Dubrovin, On Hamiltonian perturbations of hyperbolic systems of conservation laws. II. Universality of critical behaviour, Comm. Math. Phys. **267**, no. 1 (2006) 117–139.

[10] E.V. Ferapontov and K.R. Khusnutdinova, On integrability of (2+1)-dimensional quasilinear systems, Comm. Math. Phys. **248** (2004) 187–206.

[11] E.V. Ferapontov and K.R. Khusnutdinova, The characterization of 2-component (2+1)-dimensional integrable systems of hydrodynamic type, J. Phys. A: Math. Gen. **37**, no. 8 (2004) 2949–2963.

[12] E.V. Ferapontov and K.R. Khusnutdinova, Double waves in multi-dimensional systems of hydrodynamic type: the necessary condition for integrability, Proc. Royal Soc. A **462** (2006) 1197–1219.

[13] E.V. Ferapontov and D.G. Marshall, Differential-geometric approach to the integrability of hydrodynamic chains: the Haantjes tensor, Math. Ann. **339**, no. 1 (2007) 61–99.

[14] E.V. Ferapontov and A. Moro, Dispersive deformations of hydrodynamic reductions of 2D dispersionless integrable systems, J. Phys. A: Math. Theor. 42 (2009) 035211, 15pp.

[15] J. Gibbons and Y. Kodama, A method for solving the dispersionless KP hierarchy and its exact solutions. II, Phys. Lett. A **135** (1989) 167–170.

[16] J. Gibbons and S.P. Tsarev, Reductions of the Benney equations, Phys. Lett. A **211** (1996) 19–24.

[17] J. Gibbons and S.P. Tsarev, Conformal maps and reductions of the Benney equations, Phys. Lett. A **258** (1999) 263–271.

[18] Yu. Kodama, A method for solving the dispersionless KP equation and its exact solutions, Phys. Lett. A **129**, no. 4 (1988) 223–226.

[19] Yu. Kodama and A.V. Mikhailov, Obstacles to asymptotic integrability. Algebraic aspects of integrable systems, 173–204, Progr. Nonlinear Differential Equations Appl., 26, Birkh?user Boston, Boston, MA, 1997.

[20] B.G. Konopelchenko and V.G. Dubrovsky, Some new integrable nonlinear evolution equations in 2+1 dimensions, Phys. Letters A **102**, N 1, 2 (1984) 15–17.

[21] Si-Qi Liu and Youjin Zhang, On quasi-triviality and integrability of a class of scalar evolutionary PDEs, J. Geom. Phys. **57**, no. 1 (2006) 101–119.

[22] A. Majda, Compressible fluid flow and systems of conservation laws in several space variables, Applied Mathematical Sciences, 53, Springer-Verlag, New York (1984) 159 pp.

[23] A.V. Mikhailov, A. B Shabat and V.V. Sokolov, The symmetry approach to classification of integrable equations. What is integrability?, 115–184, Springer Ser. Nonlinear Dynam., Springer, Berlin, 1991.

[24] A.V. Mikhailov and R.I. Yamilov, Towards classification of $(2+1)$-dimensional integrable equations. Integrability conditions, I. J. Phys. A **31**, no. 31 (1998) 6707–6715.

[25] A.V. Mikhailov and V.S. Novikov, Perturbative symmetry approach, J. Phys. A **35**, no. 22 (2002) 4775–4790.

[26] L. Martinez Alonso and A.B. Shabat, Hydrodynamic reductions and solutions of a universal hierarchy, Teoret. Mat. Fiz. **140** (2004), 216–229.

[27] L.P. Nizhnik, Integration of multidimensional nonlinear equations by the method of inverse problem, DAN SSSR, **254** (1980) 332.

[28] S.Yu. Sakovich, Fujimoto-Watanabe equations and differential substitutions, J. Phys. A **24**, no. 10 (1991) L519–L521.

[29] B.M. Szablikowski and M. Blaszak, Dispersionful analogue of the Whitham hierarchy, arXiv:0707.1082.

[30] S.P. Tsarev, Geometry of Hamiltonian systems of hydrodynamic type. Generalized hodograph method, Izvestija AN USSR Math. **54** (1990) 1048–1068.

[31] N.V. Ustinov, Darboux transformations, infinitesimal symmetries and conservation laws for the nonlocal two-dimensional Toda lattice, J. Phys. A: Math. Gen. **35** (2002) 6963?6972.

[32] A.P. Veselov and S.P. Novikov, Finite-gap two-dimensional potential Schrödinger operators. Explicit formulae and evolution equations, DAN SSSR, **279** (1984) 20.

[33] Wang, Jing Ping, On the structure of (2+1)-dimensional commutative and noncommutative integrable equations. J. Math. Phys. **47**, no. 11 (2006) 113508, 19 pp.

[34] A. Fujimoto and Y. Watanabe, Polynomial evolution equations of not normal type admitting nontrivial symmetries. Phys. Lett. A **136**, no. 6 (1989) 294–299.

[35] A.V. Mikhailov, V.V. Sokolov and A.B. Shabat, The symmetry approach to classification of integrable equations, in What is integrability? (V.E. Zakharov, Ed.) pp. 115-184, Springer series in Nonlinear Dynamics, 1991.

[36] N.H. Ibragimov, Transformation groups applied to mathematical physics (Dordrecht:Reidel), 1985.

[37] E.A. Zabolotskaya and R.V. Khokhlov, Quasi-plane waves in the nonlinear acoustics of confined beams, Sov. Phys. Acoust. **15** (1969) 35–40.

[38] V.E. Zakharov, Dispersionless limit of integrable systems in $2+1$ dimensions, in Singular Limits of Dispersive Waves, Ed. N.M. Ercolani et al., Plenum Press, NY (1994) 165–174.

[39] V.E. Zakharov and E.I. Schulman, Integrability of nonlinear systems and perturbation theory, in: What is integrability?, 185–250, Springer Ser. Nonlinear Dynam., Springer, Berlin, 1991.

[40] A.I. Zenchuk, The spectral problem and particular solutions to the $(2+1)$-dimensional integrable generalization of the Camassa-Holm equation, Physica D **152-153** (2001) 178–188.