Y-system and Deformed Thermodynamic Bethe Ansatz

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Abstract

We introduce a new tool, the Deformed TBA (Deformed Thermo-dynamic Bethe Ansatz), to analyze the monodromy problem of the cubic oscillator. The Deformed TBA is a system of five coupled nonlinear integral equations, which in a particular case reduces to the Zamolodchikov TBA equation for the 3-state Potts model. Our method generalizes the Dorey-Tateo analysis of the (monomial) cubic oscillator. We introduce a Y-system corresponding to the Deformed TBA and give it an elegant geometric interpretation.

1 Introduction

This is the first in a series of papers that we dedicate to studying the exact theory of the direct and inverse monodromy problem for the cubic anharmonic oscillator

$$\frac{d^2\psi(\lambda)}{d\lambda^2} = V(\lambda; a, b)\psi(\lambda) , \quad V(\lambda; a, b) = 4\lambda^3 - a\lambda - b . \quad (1)$$

The main purpose of the present paper is to introduce a novel instrument of analysis, that we call Deformed Thermodynamic Bethe Ansatz (Deformed TBA).

The monodromy problem of the cubic anharmonic oscillator is a fundamental and rather interesting problem in itself. Moreover, it is deeply interconnected with the study of singularities of solutions of the first Painlevé equations (see author's papers [Mas], [Mas10]).

Monodromy problems for anharmonic oscillators, and especially eigenvalue problems, has been intensively studied since early '70 years with a wealth of different methods: see for example the seminal papers [BW68], [Sim70], [Vor83], [BB98], [DT99], [BLZ01], [EG09].

Computations are often performed by means of the complex WKB methods (see for example [DT00], [BBM⁺01] and author's papers [Mas], [Mas10]) or refined asymptotic expansions (see [JSZJ09] and references therein). The first breakthrough towards an exact evaluation of the

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monodromy problem is the work of Dorey and Tateo [DT99]: they analyze anharmonic oscillators with a monomial potential $\lambda^n - E$ (n not necessarily 3) via the Thermodynamic Bethe Ansatz and other nonlinear integral equations (called sometimes Destri - de Vega equations). Subsequently Bazhanov, Lukyanov and Zamolodchikov generalized the Dorey-Tateo analysis to monomial potentials with a centrifugal term [BLZ01].

Here we generalize Dorey and Tateo approach to the general cubic potential. In Theorem 2 below, we show that the monodromy problem for the general cubic potential is encoded in a *nonlinear nonlocal Riemann-Hilbert problem*, which is equivalent (at least for small value of the parameter a in (1)) to the following system of nonlinear integral equations that we call Deformed Thermodynamic Bethe Ansatz:

$$\chi_l(\sigma) = \int_{-\infty}^{+\infty} \varphi_l(\sigma - \sigma') \Lambda_l(\sigma')\,, \sigma, \sigma' \in \mathbb{R}\,, l \in \mathbb{Z}_5 = \{-2, \ldots, 2\}\,. \quad (2)$$

Here

$$\Lambda_l(\vartheta) = \sum_{k \in \mathbb{Z}_5} e^{i\frac{2lk\pi}{5}} L_k(\sigma) \quad , \quad L_k(\sigma) = \ln\left(1 + e^{-\varepsilon_k(\sigma)}\right)\,,$$

$$\varepsilon_k(\sigma) = \frac{1}{5}\sum_{l \in \mathbb{Z}_5} e^{-i\frac{2lk\pi}{5}}\chi_l(\sigma) + \frac{\sqrt{\frac{\pi}{3}}\Gamma(1/3)}{2^{\frac{5}{3}}\Gamma(11/6)}e^\sigma + a\frac{\sqrt{3\pi}\Gamma(2/3)}{4^{\frac{2}{3}}\Gamma(1/6)}e^{\frac{\sigma}{5} - i\frac{2k\pi}{5}}\,,$$

$$\varphi_0(\sigma) = \frac{\sqrt{3}}{\pi}\frac{2\cosh(2\sigma)}{1 + 2\cosh(2\sigma)} \quad , \quad \varphi_1(\sigma) = -\frac{\sqrt{3}}{\pi}\frac{e^{-\frac{9}{5}\sigma}}{1 + 2\cosh(2\sigma)}$$

$$\varphi_2(\sigma) = -\frac{\sqrt{3}}{\pi}\frac{e^{-\frac{3}{5}\sigma}}{1 + 2\cosh(2\sigma)} \quad , \quad \varphi_{-1}(\sigma) = \varphi_1(-\sigma), \varphi_{-2}(\sigma) = \varphi_2(-\sigma)\,.$$

Here the pseudo-energy ε_k is related to the logarithm of the k-th Stokes multiplier (see Section 3 below).

For $a = 0$, see equation (22) below, equations (2) reduce to the Thermodynamic Bethe Ansatz, introduced by Zamolodchikov [Zam90] to describe the thermodynamics of the 3-state Potts model and Lee-Yang model.

The paper is organized as follows. In Section 2 we give a geometric construction of the *space of monodromy data*. We realize the Stokes multipliers of the cubic oscillators as natural coordinates on the quotient $W_5/PSL(2,\mathbb{C})$, where W_5 is a dense open subset of $\left(\mathbb{P}^1\right)^5$ (the Cartesian product of five copies of $\mathbb{P}^1$). Section 3 is devoted to the construction of the deformed Y-system. In Section 4 we derive the Deformed Thermodynamic Bethe Ansatz. For convenience of the reader, we explain the basic theory of cubic oscillators (Stokes sectors, Stokes multipliers, subdominant solutions, etc ...) in the Appendix.

2 The space of monodromy data

Usually monodromy data of equation (1) are expressed in terms of Stokes multipliers, which are defined by means of a special set of solutions of the equation. In this section, following Nevanlinna [Nev70] and author's paper [Mas], we study the monodromy data from a geometric (hence invariant) viewpoint. Here and for the rest of the paper $\mathbb{Z}_5 = \{-2, \ldots, 2\}$.

Definition 1. *Let $\{\varphi, \chi\}$ be a basis of solution of (1).*
We call

$$w_k(\varphi, \chi) = \lim_{\substack{\lambda \to \infty \\ \left|\arg \lambda - \frac{2\pi k}{5}\right| < \frac{\pi}{5} - \varepsilon}} \frac{\varphi(\lambda)}{\chi(\lambda)} \in \mathbb{C} \cup \infty \,, \ k \in \mathbb{Z}_5 \,. \qquad (3)$$

the k-th asymptotic value.

We collect the main properties of the asymptotic values in the following

Lemma 1. *(i) Let $\varphi' = a\,\varphi + b\,\chi$ and $\chi' = c\,\varphi + d\,\chi'$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl(2, \mathbb{C})$. Then*

$$w_k(\varphi', \chi') = \frac{a\,w_k(\varphi, \chi) + b}{c\,w_k(\varphi, \chi) + d} \,. \qquad (4)$$

(ii) $w_{k-1}(\varphi, \chi) = w_{k+1}(\varphi, \chi)$ iff $\sigma_k(a, b) = 0$. Here $\sigma_k(a, b)$ is the k-th Stokes multipliers.

(iii) $w_{k+1}(\varphi, \chi) \neq w_k(\varphi, \chi)$

Proof. See [Mas]. □

Definition 2. *We define*

$$W_5 = \{(z_{-2}, z_{-1}, z_0, z_1, z_2), z_k \in \mathbb{C} \cup \infty, z_k \neq z_{k\pm 1}\} \,.$$

The group of automorphism of the Riemann sphere, called Möbius group or $PSL(2, \mathbb{C})$, has the following natural free action on W_5: let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C})$ then

$$T(z_{-2}, \ldots, z_2) = \left(\frac{az_{-2} + b}{cz_{-2} + d}, \ldots, \frac{az_2 + b}{cz_2 + d}\right) \,.$$

After formula (3) and Lemma 1(iii) every basis of solution of (1) determines a point in W_5. After the transformation law (4), the Schrödinger equation (1) determines an orbit of the $PSL(2, \mathbb{C})$ action. Hence, we define the *space of monodromy data* as follows.

Definition 3. *We call space of monodromy data the space of the orbits of the $PSL(2, \mathbb{C})$ action and denote it V_5:*

$$V_5 = W_5/PSL(2, \mathbb{C}) \ .$$

REMARK. *The author proved in [Mas] that the space of monodromy data V_5 is the moduli space of solutions of the first Painlevé equation. V_5 is a smooth manifold and $\mathcal{M}_{0,5} \subset V_5 \subset \overline{\mathcal{M}_{0,5}}$.*

In the rest of the section we construct natural coordinates of V_5 and interpret them in the spirit of cubic oscillator theory. Define

$$
\begin{aligned}
R_k : W_5 &\rightarrow \quad \mathbb{C} \ , \ k \in \mathbb{Z}_5 \ , \\
R_k(z_{-2}, \dots, z_2) &= \quad (z_{1+k}, z_{-2+k}; z_{-1+k}, z_{2+k}) \ , \quad (5)
\end{aligned}
$$

where $(a, b; c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$ is the cross ratio of four point on the sphere.

We collect the main properties of functions R_k in the following Lemma, whose easy proof is left to the reader.

Lemma 2. *(i)*

$$
\begin{aligned}
R_k(z_{-2}, \dots, z_2) &\neq \quad \infty \ , \ \forall (z_{-2}, \dots, z_2) \in W_5 \ , \\
R_k(z_{-2}, \dots, z_2) &= \quad 0 \ \textit{iff} \ z_{k-1} = z_{k+1} \ , \quad (6) \\
R_k(z_{-2}, \dots, z_2) &= \quad 0 \ \textit{iff} \ z_{k-1} = z_{k+2} \ \textit{or} \ z_{k+1} = z_{k-2} \ .
\end{aligned}
$$

(ii) The functions R_k are invariant under the $PSL(2, \mathbb{C})$ action. Hence they are well defined on V_5: with a small abuse of notation we denote R_k also the functions defined on V_5.

(iii) They satisfy the following set of quadratic relation [1]

$$R_{k-2} R_{k+2} = 1 - R_k \ , \ \forall k \in \mathbb{Z}_5 \ . \quad (7)$$

(iv) The pair R_k, R_{k+1} is a coordinate system of V_5 on the open subset $R_{k-2} \neq 0$. The pair of coordinate systems (R_k, R_{k+1}) and (R_{k+2}, R_{k-2}) form an atlas of V_5.

Definition 4. *We call any cubic polynomial of the form $V(\lambda; a, b) = 4\lambda^3 - a\lambda - b$ a cubic potential. The above formula identifies the space of cubic potentials with $\mathbb{C}_2 \ni (a, b)$. Through the asymptotic values (3), we define the monodromy map*

$$\mathcal{T} : \mathbb{C}^2 \rightarrow V_5 \ .$$

$\mathcal{T}(a, b)$ is the monodromy data of equation (1).

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[1] almost identical to the relations satisfied by the Stokes multipliers (24)

Nevanlinna [Nev70] proved that the monodromy map $\mathcal{T}$ is surjective.

With a small abuse of notation we define

$$R_k(a,b) = R_k \circ \mathcal{T}(a,b) . \tag{8}$$

The reader can verify that the Stokes multipliers (defined precisely in the Appendix) are (modulo multiplicative constant) the functions R_k:

$$\sigma_k(a,b) = iR_k(a,b) .$$

We have thus realized the Stokes multipliers geometrically as natural coordinates of the monodromy map from the space of cubic oscillators to the quotient $W_5/PSL(2,\mathbb{C})$.

REMARK. *The same construction presented here holds for anharmonic oscillators with polynomial potentials of any degree. If n is the degree, we denote V_{n+2} the space of monodromy data and $R_k^{(n+2)}$ the natural functions (defined by formula 5). For example V_3 is one point and V_4 is $\mathbb{C}^*$. The functions $R_k^{(n+2)}$ satisfy a system of algebraic relations similar to (8), which will be studied in a subsequent paper. The reader should notice that only in the cubic case the functions $R_k^{(n+2)}$ coincide with the Stokes multipliers.*

3 Y-system

Here we introduce the Y-system (12), which is a fundamental step in the derivation of the Deformed TBA.

We begin with an observation, probably due to Sibuya [Sib75]:

Lemma 3. *Let $\omega = e^{i\frac{2\pi}{5}}$ and R_k be defined as in (8). Then*

$$R_k(\omega^{-1}a, \omega b) = R_{k-2}(a,b) . \tag{9}$$

Proof. Denote $\varphi(\lambda; a, b)$ a solution of (1) whose Cauchy data do not depend on a, b. It is an entire function of three complex variables with some remarkable properties. For any $k \in \mathbb{Z}_5$ $\varphi(\omega^k \lambda; \omega^{2k}a, \omega^{3k}b)$ satisfies the same Schrödinger equation (1) as $\varphi(\lambda; a, b)$. Fix $\varphi(\lambda; a, b), \chi(\lambda; a, b)$ linearly independent solutions and define

$$\begin{aligned}
w_k(a,b) &= w_k(\varphi(\lambda; a, b), \chi(\lambda; a, b)) , \\
\bar{w}_k(a,b) &= w_l(\varphi(\omega^l \lambda; \omega^{2l}a, \omega^{3l}b), \chi(\omega^l \lambda; \omega^{2l}a, \omega^{3l}b)) .
\end{aligned}$$

Then $w_k(\omega^{2l}a, \omega^{3l}b) = \bar{w}_{k-l}(a,b)$. Choose $l = 2$ to obtain the thesis. $\qquad\square$

Due to equations (7) and relations (9), the holomorphic functions $R_k(a,b)$ satisfies the following system of functional equations, first studied by Sibuya [Sib75]

$$R_k(\omega^{-1}a, \omega b)R_k(\omega a, \omega^{-1}b) = 1 - R_k(a,b) , \ \forall k \in \mathbb{Z}_5. \tag{10}$$

We have collected all the elements to introduce the important *Y-functions* and *Y-system*.

We fix $a \in \mathbb{C}$ and define

$$Y_k(\vartheta) = -R_0(\omega^{-k}a, e^{\frac{6}{5}\vartheta}), \ k \in \mathbb{Z}_5 . \tag{11}$$

Sibuya's equation (10) is equivalent to the following system of functional equations, that we call *Deformed Y-system*:

$$Y_{k-1}(\vartheta - i\frac{\pi}{3})Y_{k+1}(\vartheta + i\frac{\pi}{3}) = 1 + Y_k(\vartheta) . \tag{12}$$

REMARK. *If $a = 0$, $Y_k = Y_0, \forall k$ and the system (12) reduces to just one equation, called Y-system, which was introduced by Zamolodchikov [Zam91] in relation with the Lee-Yang and 3-state Potts models. Dorey and Tateo [DT99] studied the Zamolodchikov Y-system in relation with the Schrödinger equation with potential $V(\lambda; 0, b) = 4\lambda^3 - b$.*

3.1 Analytic Properties of Y_k

In the following theorem we summarize the analytic properties of the Y-functions.

Theorem 1. *(i) For any $a \in \mathbb{C}$ and $k \in \mathbb{Z}_5$, Y_k is analytic and $i\frac{5\pi}{3}$ periodic. If a is real then $\overline{Y_k(\overline{\vartheta})} = Y_{-k}(\vartheta)$, where $\overline{}$ stands for complex conjugation.*

(ii) For any $a \in \mathbb{C}$ and $k \in \mathbb{Z}_5$,

$$\left| \frac{Y_k(\vartheta)}{\tilde{Y}_k(\vartheta)} - 1 \right| = O(e^{-Re\vartheta}), \ as \ Re\vartheta \to +\infty \ and \ |Im\vartheta| < \frac{\pi}{2},$$

$$\tilde{Y}_k(\vartheta) = \exp\left(Ae^{\vartheta} + Bae^{\frac{\vartheta}{5} - i\frac{2k\pi}{5}} \right) . \tag{13}$$

Here $A = \frac{\sqrt{\frac{\pi}{3}}\Gamma(1/3)}{2^{\frac{5}{3}}\Gamma(11/6)}$ and $B = \frac{\sqrt{3\pi}\Gamma(2/3)}{4^{\frac{2}{3}}\Gamma(1/6)}$.

(iii) For any $a \in \mathbb{C}$ and any $K \in \mathbb{R}$, $Y_k(\vartheta)$ is bounded on $Re\vartheta \le K$. If $a = 0$, $\lim_{\vartheta \to -\infty} Y_k(\vartheta) = \frac{1+\sqrt{5}}{2}$.

(iv) If $e^{i\frac{2k\pi}{5}}a$ is real non negative then $Y_k(\vartheta) = 0$ implies $Im\vartheta = \pm\frac{5\pi}{6}$. If $a = 0$ then $Y_k(\vartheta) = -1$ implies $\vartheta = \pm i\frac{\pi}{2}$

(v) Fix $\varepsilon > 0$. If a is small enough, then for any $k \in \mathbb{Z}_5$, $Y_k(\vartheta) \neq 0, -1$ for any $\vartheta \in |Im\vartheta| \le \frac{\pi}{2} - \varepsilon$.

Proof. (i) Trivial.

(ii) These "WKB-like" estimates can be found in [Mas] Section 4 or in [Sib75].

(iii) The boundedness follows directly from the fact that $R_k(a, b)$ is analytic in $b = 0$. If $a = b = 0$, then for symmetry reasons one can choose φ, χ such that $w_k = e^{i\frac{2k\pi}{5}}$. This implies the thesis.

(iv) The statement is equivalent to Theorem 3 in the Appendix.

(v) Since Y_k depends analytically from the parameter a, it follows from (iv).

$\square$

4   Deformed TBA

This section is devoted to the derivation of the Deformed Thermodynamic Bethe Ansatz equations (2) [2].

In what follows we always make the following

ASSUMPTIONS 1. *We assume that there exists an $\varepsilon > 0$ such that*

(i) *every branch of $\ln Y_k$ is holomorphic on $|Im\vartheta| \leq \frac{\pi}{3} + \varepsilon$, and bounded for $\vartheta \to -\infty$. And*

(ii) *every branch of $\ln\left(1 + \frac{1}{Y_k}\right)$ is holomorphic on $|Im\vartheta| \leq +\varepsilon$, and bounded for $\vartheta \to -\infty$.*

From Theorem 1(iii, v) we know that the assumptions are valid if a is small enough.

We define the following bounded analytic functions on *the physical strip $Im\vartheta \leq \frac{\pi}{3}$*

$$\varepsilon_k(\vartheta) = \ln Y_k(\vartheta), \tag{14}$$

$$\delta_k(\vartheta) = \varepsilon_k(\vartheta) - \frac{\sqrt{\frac{\pi}{3}}\Gamma(1/3)}{2^{\frac{5}{3}}\Gamma(11/6)}e^{\vartheta} + a\frac{\sqrt{3\pi}\Gamma(2/3)}{4^{\frac{2}{3}}\Gamma(1/6)}e^{\frac{\vartheta}{5} - i\frac{2k\pi}{5}},$$

$$L_k(\vartheta) = \ln(1 + e^{-\varepsilon_k(\vartheta)}).$$

Here the branches of logarithms are fixed by requiring

$$\lim_{\sigma \to +\infty} \delta_k(\sigma + i\tau) = \lim_{\sigma \to +\infty} L_k(\sigma + i\tau) = 0, \ \forall |\tau| < \frac{\pi}{2}. \tag{15}$$

We remark that by Theorem 1(ii), this choice is always possible. We denote ε_k pseudo-energies in analogy with the undeformed TBA.

Due to the Y-system (12), the functions δ_k satisfy the following *nonlinear nonlocal Riemann-Hilbert problem*

$$\delta_{k-1}(\vartheta - i\frac{\pi}{3}) + \delta_{k+1}(\vartheta + i\frac{\pi}{3}) - \delta_k(\vartheta) = L_k(\vartheta), \ |Im\vartheta| \leq \varepsilon. \tag{16}$$

Here the boundary conditions are given by asymptotics (15).

The system (16) is $\mathbb{Z}_5$ invariant. Hence we diagonalize its linear part (the left hand side) by taking its *discrete Fourier transform* (also called *Wannier* transform):

$$\chi_l(\vartheta) = \sum_{k\in\mathbb{Z}_5} e^{i\frac{2kl\pi}{5}}\delta_k(\vartheta) \ , \quad \delta_k(\vartheta) = \frac{1}{5}\sum_{l\in\mathbb{Z}_5} e^{-i\frac{2kl\pi}{5}}\chi_l(\vartheta), \tag{17}$$

$$\Lambda_l(\vartheta) = \sum_{k\in\mathbb{Z}_5} e^{i\frac{2l\pi}{5}}L_k(\vartheta) \ , \quad L_k(\vartheta) = \frac{1}{5}\sum_{l\in\mathbb{Z}_5} e^{-i\frac{2kl\pi}{5}}\Lambda_l(\vartheta) \ .$$

The above defined functions satisfy the following system

$$e^{-i\frac{2l\pi}{5}}\chi_l(\vartheta + i\frac{\pi}{3}) + e^{i\frac{2l\pi}{5}}\chi_l(\vartheta - i\frac{\pi}{3}) - \chi_l(\vartheta) = \Lambda_l(\vartheta). \tag{18}$$

The system of functional equation (18), may be rewritten in the convenient form of a system of coupled integral equations (2).

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[2]The reader who wants to repeat all the computations below, should remember that $\sum_{l=0}^{n-1} e^{i\frac{2\pi l}{n}} = 0$.

Theorem 2. *If a is small enough, the functions χ_l satisfy the Deformed Thermodynamic Bethe Ansatz*

$$\chi_l(\sigma) = \int_{-\infty}^{+\infty} \varphi_l(\sigma - \sigma')\Lambda_l(\sigma')\,, \sigma, \sigma' \in \mathbb{R}\,. \qquad (2)$$

Here Λ_l are defined as in (14,17) and

$$
\begin{aligned}
\varphi_0(\sigma) &= \frac{\sqrt{3}}{\pi}\frac{2\cosh(\sigma)}{1+2\cosh(2\sigma)} \\
\varphi_1(\sigma) &= -\frac{\sqrt{3}}{\pi}\frac{e^{-\frac{9}{5}\sigma}}{1+2\cosh(2\sigma)} \\
\varphi_2(\sigma) &= -\frac{\sqrt{3}}{\pi}\frac{e^{-\frac{3}{5}\sigma}}{1+2\cosh(2\sigma)} \\
\varphi_{-1}(\sigma) &= -\frac{\sqrt{3}}{\pi}\frac{e^{\frac{9}{5}\sigma}}{1+2\cosh(2\sigma)} \\
\varphi_{-2}(\sigma) &= -\frac{\sqrt{3}}{\pi}\frac{e^{\frac{3}{5}\sigma}}{1+2\cosh(2\sigma)}\,.
\end{aligned}
\qquad (19)
$$

Proof. If a is small enough we know that the Assumptions 1 are valid. Hence the thesis follows from system (18) and the technical Lemma 4 below. $\qquad\square$

Lemma 4. *Let $f : |Im\vartheta| \leq \varepsilon \to \mathbb{C}$ be a bounded analytic function. Then for any $l \in \mathbb{Z}_5$ there exists a unique function F analytic and bounded on $|Im\vartheta| \leq \frac{\pi}{3} + \varepsilon$, such that*

$$e^{-i\frac{2\pi l}{5}}F(\vartheta + i\frac{\pi}{3}) + e^{i\frac{2\pi l}{5}}F(\vartheta - i\frac{\pi}{3}) - F(\vartheta) = f(\vartheta)\,, \forall\,|Im\vartheta| \leq \varepsilon\,.$$

Moreover, F is expressed through the following integral transform

$$F(\vartheta + i\tau) = \int_{-\infty}^{+\infty}\varphi_l(\vartheta + i\tau - \vartheta')f(\vartheta')d\sigma'\,, \ \forall\,|Im\vartheta| \leq \varepsilon, |\tau| \leq \frac{\pi}{3}\,, \ (20)$$

provided $|Im(\vartheta + i\tau - \vartheta')| < \frac{\pi}{3}$ and the integration path belongs to the strip $|Im\vartheta'| \leq \varepsilon$. Here φ_l is defined by formula (19).

Proof. Uniqueness: let F_1, F_2 be bounded solution of the functional equation

$$e^{-i\frac{2\pi l}{5}}F_j(\vartheta + i\frac{\pi}{3}) + e^{i\frac{2\pi l}{5}}F_j(\vartheta - i\frac{\pi}{3}) - F_j(\vartheta) = f(\sigma)\,, \ j = 1, 2, |Im\vartheta| \leq \varepsilon\,.$$

Their difference $G = F_1 - F_2$ satisfies

$$e^{-i\frac{2\pi l}{5}}G(\vartheta + i\frac{\pi}{3}) + e^{i\frac{2\pi l}{5}}G(\vartheta - i\frac{\pi}{3}) - G(\vartheta) = 0\,.$$

Hence G extends to an entire, $2i\pi$ periodic, bounded function. Therefore G is a constant and the only constant satisfying the functional relation is zero.

Existence: one notices that if $\theta \neq \pm in\frac{\pi}{3}$, $n \in \mathbb{Z}$ then

$$e^{-i\frac{2\pi l}{5}}\varphi_l(\theta + i\frac{\pi}{3}) + e^{i\frac{2\pi l}{5}}\varphi_l(\theta - i\frac{\pi}{3}) - \varphi_l(\theta) = 0 \, , \ \forall l \in \mathbb{Z}_5 \, .$$

Then a rather standard computation shows that the function F defined through formula (20) satisfies all the desired properties. $\square$

REMARK. *Once the system of integral equations (2) is solved for $\sigma \in \mathbb{R}$, one can use the same set of integral equations as explicit formulas to extend the functions $\chi_l(\theta)$ on $|Im\theta| \leq \frac{\pi}{3}$. Then one can use the Y-system (12) to extend the Y functions on the entire fundamental strip $|Im\vartheta| \leq \frac{5\pi}{6}$.*

REMARK. *While the Y-system equations do not depend on the parameter a (the coefficient of the linear term of the potential $4\lambda^3 - a\lambda - b$), on the contrary the Deformed TBA equations depend on it since it enters explicitely into the definition of functions Λ_l.*

4.1   The case $a = 0$

If $a = 0$ then $\delta_k = \delta_0$, $L_k = L_0 \forall k$. Therefore, δ_0 satisfy the single functional equation

$$\delta_0(\vartheta - i\frac{\pi}{3}) + \delta_0(\vartheta + i\frac{\pi}{3}) - \delta_0(\vartheta) = L_0(\vartheta) \, , \ |Im\vartheta| \leq \varepsilon \, . \qquad (21)$$

Similar reasoning as in Theorem 2 shows that δ_0 satisfies the following nonlinear integral equation (as it was firstly discovered by Dorey and Tateo [DT99])

$$\delta_0(\sigma) = \int_{-\infty}^{+\infty} \varphi_0(\sigma - \sigma') \ln\left(1 + \exp\left(-(\delta_0(\sigma') + Ae^{\sigma'})\right)\right), \sigma, \sigma' \in \mathbb{R} \, . \tag{22}$$

Here φ_0 is defined as in formula (19) and $A = \frac{\sqrt{\frac{\pi}{3}}\Gamma(1/3)}{2^{\frac{5}{3}}\Gamma(11/6)}$.

Equation (22) is called Thermodynamic Bethe Ansatz and was introduced by Zamolodchikov [Zam90] to describe the Thermodynamics of the 3-state Potts and Lee-Yang models.

5   Concluding Remarks

We have given a geometric construction of the *space of monodromy data* of cubic oscillators and we have used such construction to analyze the direct monodromy problem.

We have shown that for small value of the deformation parameter (the coefficient of the linear term of the potential) the direct monodromy problem defines a nonlinear nonlocal Riemann-Hilbert problem which is equivalent to a system of nonlinear equations, that we call Deformed Thermodynamic Bethe Ansatz (Deformed TBA). Our approach can be easily generalized to anharmonic oscillators of any

order. We will give the details in a subsequent publication (see also the Remark at the end of Section 2).

As it was said in the introduction, this is just the first of a series of paper that we dedicate to the Deformed TBA. Our rather ambitious purpose is to use the Deformed TBA to construct analytical and numerical tools to effectively solve the monodromy problem of the general cubic oscillator. We plan to pursue our study of the Deformed TBA equations in different directions to achieve such a goal.

In particular, together with A. Moro and R. Tateo we are implementing an algorithm to solve numerically the equations and to understand for which values of the parameter a the Assumptions 1 fail. Consequently, we will modify the Riemann-Hilbert problem for taking into account the multivaluedness of the functions Λ_l, χ_l.

At the same time, we are going to study analytically the convergence of successive approximation schemes to solve the Deformed TBA.

As it is known from author previous works [Mas],[Mas10] (see also [KT05]), cubic oscillators are deeply interconnected with the first Painlevé equation and in particular with the distribution of poles of its solutions. As an application of the theory developed here, we are going to investigate the poles of special solution of first Painlevé equation. In particular we will study the tritronquée solution, since it is relevant to describe the singular limit of the focusing Nonlinear Schrödinger equation [DGK09], [BT10].

6 Appendix

The reader expert in anharmonic oscillators theory will skip this Appendix; for her, it will be enough to know that we denote $\sigma_k(a, b)$ the $k-th$ Stokes multipliers of equation (1). Here we review briefly the standard way, i.e. by means of Stokes multipliers, of introducing the monodromy problem for equation (1). All the statements of this section are proved in Appendix A of author's paper [Mas] and in Sibuya's book [Sib75] .

Lemma 5. *Fix* $k \in \mathbb{Z}_5 = \{-2, \dots, 2\}$, *define the branch of* $\lambda^{\frac{1}{2}}$ *by requiring*

$$\lim_{\substack{\lambda \to \infty \\ \arg \lambda = \frac{2\pi k}{5}}} \mathrm{Re}\lambda^{\frac{5}{2}} = +\infty$$

while choose the branch of $\lambda^{\frac{1}{4}}$ *globally on the complex plane minus the negative real axis such that it is positive on the positive real axis. Then there exists a unique solution* $y_k(\lambda; a, b)$ *of equation (1) such that*

$$\lim_{\substack{\lambda \to \infty \\ \left|\arg \lambda - \frac{2\pi k}{5}\right| < \frac{3\pi}{5} - \varepsilon}} \frac{y_k(\lambda; a, b)}{\lambda^{-\frac{3}{4}} e^{-\frac{4}{5}\lambda^{\frac{5}{2}} + \frac{a}{2}\lambda^{\frac{1}{2}}}} \to 1, \ \forall \varepsilon > 0 \, . \qquad (23)$$

Definition. *We denote* $\left\{\lambda \in \mathbb{C}, \left|\arg \lambda - \frac{2k\pi}{5}\right| < \frac{\pi}{5}\right\}$ *the k-th Stokes sector. We denote* y_k, *defined in Lemma above, the k-th subdominant solution or the solution subdominant in the k-th sector.*

From the asymptotics (23), it follows that y_k and y_{k+1} are linearly independent and the following equations hold true:

$$
\begin{aligned}
y_{k-1}(\lambda; a, b) &= y_{k+1}(\lambda; a, b) + \sigma_k(a, b) y_k(\lambda; a, b), \\
-i\sigma_{k+3} &= 1 + \sigma_k(a, b)\sigma_{k+1}(a, b), \ \forall k \in \mathbb{Z}_5.
\end{aligned}
\tag{24}
$$

Definition. *The entire functions $\sigma_k(a, b)$ are called Stokes multipliers. The quintuplet of Stokes multipliers $\sigma_k(a, b), k \in \mathbb{Z}_5$ is called the monodromy data of equation (1).*

6.1 PT-symmetric anharmonic oscillator

The eigenvalues problem $\sigma_k = 0, k \in \mathbb{Z}_5$ is PT symmetric if $\omega^k a \in \mathbb{R}$ (the study of PT symmetric oscillators began in the seminal paper [BB98]). Dorey, Dunning, Tateo [DDT01] and Shin [Shi02] proved the following result about PT symmetric cubic oscillator.

Theorem 3. *Fix $k \in \mathbb{Z}_5$. Suppose $\sigma_k(a, b) = 0$ and $\omega^{2k}a$ is real and positive. Then $\omega^{3k}b$ is real and negative.*

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