

On the classification of scalar evolutionary integrable equations in $2 + 1$ dimensions

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Abstract

We consider evolutionary equations of the form $u_t = F(u, w)$ where $w = D_x^{-1}D_y u$ is the nonlocality, and the right hand side F is polynomial in the derivatives of u and w . The recent paper [6] provides a complete list of integrable third order equations of this kind. Here we extend the classification to fifth order equations. Besides the known examples of Kadomtsev-Petviashvili (KP), Veselov-Novikov (VN) and Harry Dym (HD) equations, as well as fifth order analogues and modifications thereof, our list contains a number of equations which are apparently new. We conjecture that our examples exhaust the list of scalar polynomial integrable equations with the nonlocality w . The classification procedure consists of two steps. First, we classify quasilinear systems which may (potentially) occur as dispersionless limits of integrable scalar evolutionary equations. After that we reconstruct dispersive terms based on the requirement of the inheritance of hydrodynamic reductions of the dispersionless limit by the full dispersive equation.

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1 Introduction

The classification of integrable 1 + 1 dimensional scalar evolutionary equations,

$$u_t = F(u),$$

has been (and still is) a subject of active research within the soliton community. Here $u(x, t)$ is a scalar potential, and F denotes a differential expression which depends on x -derivatives of u up to some finite order. Although the general classification problem is still out of reach, quite a few important results were obtained under various additional assumptions on F (such as polynomiality, linearity in the highest derivative, etc). We refer to the review article [12] for a detailed discussion of the classification techniques involved, extensive lists of integrable equations within particularly interesting subclasses, and references.

In this paper we apply the novel perturbative approach outlined in [5, 6] to a similar problem in 2 + 1 dimensions, the area where very few classification results are currently available. The main challenge of higher dimensions is the non-locality of scalar evolutionary integrable equations: the corresponding right hand side F must contain nonlocal variables whose differential structure was clarified in [13]. Here we consider equations of the form

$$u_t = F(u, w) \tag{1}$$

where $u(x, y, t)$ is a scalar field and $w = D_x^{-1}D_y u$ is the simplest nonlocality (equivalently, w can be introduced via the relation $w_x = u_y$). We assume that the right hand side F is *polynomial* in the x - and y -derivatives of u and w , while the dependence on u and w themselves is allowed to be arbitrary. The paper [6] provides a complete list of integrable third order equations of the form (1),

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(\dots) + \epsilon^2(\dots), \tag{2}$$

where φ, ψ and η are functions of u and w , while the terms at ϵ and ϵ^2 are assumed to be homogeneous differential polynomials of the order two and three in the derivatives of u and w (one can show that all terms at ϵ have to vanish). We use the following weighting scheme: u and w are assumed to have order zero, their derivatives u_x, u_y, w_x, w_y are of order one, the expressions $u_{xx}, u_{xy}, u_{yy}, w_{yy}, u_x^2, u_x u_y, u_y^2, u_x w_y, u_y w_y, w_y^2$ are of order two, and so on. Assuming that the dispersionless limit of the equation (2),

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \quad w_x = u_y, \tag{3}$$

is linearly nondegenerate (the property to be clarified in Sect. 2.2), and satisfies the condition $\eta \neq 0$ (which is equivalent to the requirement that the dispersion relation of the system (3) defines an irreducible conic), we have the following result:

Theorem 1 [6] *Up to invertible transformations, the examples below provide a complete list of integrable third order equations (2) with $\eta \neq 0$ whose dispersionless limit is linearly nondegen-*

erate:

KP equation

$$u_t = uu_x + w_y + \epsilon^2 u_{xxx},$$

modified KP equation

$$u_t = (w - u^2/2)u_x + w_y + \epsilon^2 u_{xxx},$$

Gardner equation

$$u_t = (\beta w - \frac{\beta^2}{2}u^2 + \delta u)u_x + w_y + \epsilon^2 u_{xxx},$$

VN equation

$$u_t = (uw)_y + \epsilon^2 u_{yyy},$$

modified VN equation

$$u_t = (uw)_y + \epsilon^2 \left(u_{yy} - \frac{3}{4} \frac{u_y^2}{u} \right)_y,$$

HD equation

$$u_t = -2wu_y + uw_y - \frac{\epsilon^2}{u} \left(\frac{1}{u} \right)_{xxx},$$

deformed HD equation

$$u_t = \frac{\delta}{u^3} u_x - 2wu_y + uw_y - \frac{\epsilon^2}{u} \left(\frac{1}{u} \right)_{xxx},$$

Equation E₁

$$u_t = (\beta w + \beta^2 u^2)u_x - 3\beta u u_y + w_y + \epsilon^2 [B^3(u) - \beta u_x B^2(u)],$$

Equation E₂

$$u_t = \frac{4}{3} \beta^2 u^3 u_x + (w - 3\beta u^2)u_y + uw_y + \epsilon^2 [B^3(u) - \beta u_x B^2(u)],$$

here $B = \beta u D_x - D_y$, $\beta = \text{const}$.

The main result of this paper is a generalisation of the above classification to fifth order equations,

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(\dots) + \epsilon^2(\dots) + \epsilon^3(\dots) + \epsilon^4(\dots), \quad (4)$$

where the terms at ϵ^k are assumed to be homogeneous differential polynomials of the order $k+1$ in the derivatives of u and w , respectively. We also assume that the ϵ^4 term depends on at least one of the possible fifth derivatives $u_{xxxxx}, u_{xxxxy}, \dots$, i.e. the equation (4) is of order 5.

Theorem 2 *Up to invertible transformations, the examples below provide a complete list of integrable fifth order equations (4) with $\eta \neq 0$ whose dispersionless limit is linearly nondegenerate:*

BKP equation

$$u_t = 5(u^2 + w)u_x + 5uu_y - 5w_y + 5\epsilon^2(uu_{xxx} + u_{xxy} + u_x u_{xx}) + \epsilon^4 u_{xxxxx},$$

CKP equation

$$u_t = 5(u^2 + w)u_x + 5uu_y - 5w_y + 5\epsilon^2(uu_{xxx} + w_{xxx} + \frac{5}{2}u_x u_{xx}) + \epsilon^4 u_{xxxxx},$$

HD₅ equation

$$u_t = 15wu_y - 5uw_y + 5\epsilon^2 \left[\frac{u_{xxy}}{u^2} - \frac{3}{u} \left(\frac{u_x u_y}{u^2} \right)_x \right] - \frac{\epsilon^4}{2u^2} \left(\frac{1}{u^2} \right)_{xxxxx},$$

Equation E₃

$$u_t = 4\gamma^2 \frac{u_x}{u^5} + 5(3w - \frac{\gamma}{u^2})u_y - 5uw_y + 5\epsilon^2 \left[\frac{\gamma}{2u^2} \left(\frac{1}{u^2} \right)_{xxx} + \frac{u_{xxy}}{u^2} - \frac{3}{u} \left(\frac{u_x u_y}{u^2} \right)_x \right] - \frac{\epsilon^4}{2u^2} \left(\frac{1}{u^2} \right)_{xxxxx},$$

Equation E₄

$$u_t = 4\gamma^2 \frac{u_x}{u^5} + 5(3w - \frac{\gamma}{u^2})u_y - 5uw_y + 5\epsilon^2 \left[\frac{\gamma}{3u} \left(\frac{1}{u^3} \right)_{xxx} - \gamma \frac{u_x}{u^7} - \left(\frac{1}{u} \right)_{xxy} + \left(\frac{u_x u_y}{u^3} \right)_x - \frac{u_y}{4u^4} (2uu_{xx} - 3u_x^2) \right] - \epsilon^4 \left[\frac{1}{2u^2} \left(\frac{1}{u^2} \right)_{xxxxx} - \frac{15}{16} \left(\frac{(2uu_{xx} - 3u_x^2)^2}{u^8} \right)_x \right].$$

We point out that the last two examples from Theorem 2 are apparently new. The equation E_3 can be viewed as a deformation of the fifth order Harry Dym equation HD_5 : it reduces to HD_5 when $\gamma = 0$. Although each equation appearing in Theorems 1-2 gives rise to an integrable hierarchy, the corresponding higher flows will not belong to the class (1): they will necessarily have a more complicated nonlocality. Preliminary calculations suggest that there exist no seventh order equations of the form (1). This leads to the following

Conjecture *Up to invertible transformations, Theorems 1-2 provide a complete list of integrable evolutionary equations of the form (1) which are polynomial in the derivatives of u and w .*

Remark. The assumption of polynomiality is essential: there exist examples of integrable equations of the form (1) where the right hand side F is an infinite series in ϵ . As an illustration, let us consider integrable differential-difference equations of the Toda lattice,

$$v_t = v\Delta_-(w), \quad w_x = \Delta_+(v),$$

where

$$\Delta_-(w) = \frac{w(y) - w(y - \epsilon)}{\epsilon}, \quad \Delta_+(v) = \frac{v(y + \epsilon) - v(y)}{\epsilon}.$$

Introducing the variable u by the formula $\Delta_+(v) = u_y$, one can rewrite the equations of the Toda lattice in such a way that the nonlocality w will be of the required form,

$$u_t = D_y^{-1}\Delta_+(\Delta_+^{-1}(u_y)\Delta_-(w)), \quad w_x = u_y.$$

Expanding the first equation in powers of ϵ one obtains an infinite series,

$$u_t = uw_y + \frac{\epsilon^2}{12}(uw_{yy})_y + \dots, \quad w_x = u_y.$$

Examples of this type will be outside the scope of this paper.

The structure of the paper is as follows. Following [6], in Sect. 2.1 we review the classification of integrable quasilinear systems of the form (3). In Sect 2.2 we outline the general procedure which, starting with an integrable dispersionless system, allows one to systematically reconstruct dispersive corrections. This procedure is applied in Sect. 2.3 to the case of fifth order equations (4). For the reader's convenience, in Sect. 3 we present Lax pairs for all equations appearing in Theorems 1-2.

2 Proof of Theorem 2

The proof consists of two steps. In Sect. 2.1 we review the classification of integrable quasilinear systems (3) which may (potentially) occur as dispersionless limits of integrable soliton equations. In Sect. 2.2 we discuss the general procedure of the reconstruction of dispersive corrections based on the requirement of the inheritance of hydrodynamic reductions. This procedure is applied to fifth order equations in Sect. 2.3, leading to the proof of Theorem 2.

2.1 Classification of integrable dispersionless limits

For a system of the form (3),

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \quad w_x = u_y,$$

the integrability conditions were obtained in [6] based on the results of [4]. They constitute an involutive system of second order PDEs for the coefficients φ, ψ and η ,

$$\begin{aligned}\varphi_{uu} &= -\frac{\varphi_w^2 + \psi_u \varphi_w - 2\psi_w \varphi_u}{\eta}, & \varphi_{uw} &= \frac{\eta_w \varphi_u}{\eta}, & \varphi_{ww} &= \frac{\eta_w \varphi_w}{\eta}, \\ \psi_{uu} &= \frac{-\varphi_w \psi_w + \psi_u \psi_w - 2\varphi_w \eta_u + 2\eta_w \varphi_u}{\eta}, & \psi_{uw} &= \frac{\eta_w \psi_u}{\eta}, & \psi_{ww} &= \frac{\eta_w \psi_w}{\eta}, \\ \eta_{uu} &= -\frac{\eta_w (\varphi_w - \psi_u)}{\eta}, & \eta_{uw} &= \frac{\eta_w \eta_u}{\eta}, & \eta_{ww} &= \frac{\eta_w^2}{\eta};\end{aligned}$$

we assume $\eta \neq 0$: this is equivalent to the requirement that the dispersion relation of the system (3) defines an irreducible conic. The integrability conditions are straightforward to solve. First of all, the equations for η imply that, modulo translations and rescalings, one can set $\eta = 1$, $\eta = u$ or $\eta = e^w h(u)$. We will consider all three possibilities case-by-case below. Notice that φ and ψ are defined up to additive constants which can always be set equal to zero via the Galilean transformations of the initial system (3). Moreover, the integrability conditions are form-invariant under transformations of the form

$$\tilde{\varphi} = \varphi - s\psi + s^2\eta, \quad \tilde{\psi} = \psi - 2s\eta, \quad \tilde{\eta} = \eta, \quad \tilde{u} = u, \quad \tilde{w} = w + su, \quad s = \text{const},$$

which correspond to the following transformations preserving the structure of system (3):

$$\tilde{x} = x - sy, \quad \tilde{y} = y, \quad \tilde{u} = u, \quad \tilde{w} = w + su.$$

All our classification results are formulated modulo this equivalence.

Case 1: $\eta = 1$. Then the remaining equations imply $\psi = \alpha w + f(u)$, $\varphi = \beta w + g(u)$, where f and g satisfy the linear ODEs

$$f'' = \alpha(f' - \beta), \quad g'' = 2\alpha g' - \beta f' - \beta^2.$$

The subcase $\alpha = 0$ leads to polynomial solutions of the form

$$\psi = \gamma u, \quad \varphi = \beta w - \frac{1}{2}\beta(\beta + \gamma)u^2 + \delta u. \quad (5)$$

Up to equivalence transformations, the case $\alpha \neq 0$ leads to exponential solutions,

$$\psi = w + \beta e^u, \quad \varphi = \alpha e^{2u}, \quad (6)$$

where α, β, γ are arbitrary constants.

Case 2: $\eta = u$. Then the remaining equations imply $\psi = \alpha w + f(u)$, $\varphi = \beta w + g(u)$, where f and g satisfy the linear ODEs

$$uf'' = \alpha(f' - \beta) - 2\beta, \quad ug'' = 2\alpha g' - \beta f' - \beta^2.$$

The case $\alpha \notin \{0, -1, -1/2\}$ leads to power-like solutions of the form

$$\psi = \alpha w + \gamma u^{\alpha+1}, \quad \varphi = \delta u^{2\alpha+1}. \quad (7)$$

The subcase $\alpha = 0$ leads to logarithmic solutions,

$$\psi = -2\beta u \ln u - \beta u, \quad \varphi = \beta w + \beta^2 u \ln^2 u + \delta u. \quad (8)$$

The subcase $\alpha = -1$ gives

$$\psi = -w + \gamma \ln u, \quad \varphi = \delta/u. \quad (9)$$

Finally, the subcase $\alpha = -1/2$ gives

$$\psi = -\frac{1}{2}w + \gamma\sqrt{u}, \quad \varphi = \delta \ln u. \quad (10)$$

Case 3: $\eta = e^w h(u)$. Then the remaining equations imply $\psi = e^w f(u)$, $\varphi = e^w g(u)$ where f , g and h satisfy the nonlinear system of ODEs,

$$h'' = f' - g, \quad g''h = 2fg' - gf' - g^2, \quad f''h = 2hg' - 2gh' + ff' - fg.$$

Although the structure of the general solution to this system is quite complicated, one can show that Case 3 cannot occur as the dispersionless limit of an integrable soliton equation.

2.2 Reconstruction of dispersive corrections

Given an integrable dispersionless system of the form (3), one has to reconstruct dispersive terms. This can be done by requiring that all hydrodynamic reductions of the dispersionless system are inherited by its dispersive counterpart [5, 6]. Following [6], we will illustrate this procedure with the example of the KP equation,

$$u_t = uu_x + w_y + \epsilon^2 u_{xxx}, \quad w_x = u_y.$$

The dispersionless KP (dKP) equation,

$$u_t = uu_x + w_y, \quad w_x = u_y,$$

possesses one-phase solutions of the form $u = R$, $w = w(R)$ where the phase $R(x, y, t)$ satisfies a pair of Hopf-type equations

$$R_y = \mu R_x, \quad R_t = (\mu^2 + R)R_x; \quad (11)$$

here $\mu(R)$ is an arbitrary function, and $w' = \mu$. Equivalently, one can say that Eqs. (11) constitute a one-component hydrodynamic reduction of the dKP equation. Although the dKP equation is known to possess infinitely many N -component reductions for arbitrary N [7, 8, 9, 10], one-component reductions will be sufficient for our purposes. The main observation of [5] is that *all* one-component reductions (11) can be deformed into reductions of the full KP equation by adding appropriate dispersive terms which are *polynomial* in the x -derivatives of R . Explicitly, one has the following formulae for the deformed one-phase solutions,

$$u = R, \quad w = w(R) + \epsilon^2 \left(\mu' R_{xx} + \frac{1}{2}(\mu'' - (\mu')^3) R_x^2 \right) + O(\epsilon^4), \quad (12)$$

notice that one can always assume that u remains undeformed modulo the Miura group [2]. The deformed equations (11) take the form

$$\begin{aligned} R_y &= \mu R_x \\ &+ \epsilon^2 \left(\mu' R_{xx} + \frac{1}{2} (\mu'' - (\mu')^3) R_x^2 \right) + O(\epsilon^4), \\ R_t &= (\mu^2 + R) R_x \\ &+ \epsilon^2 \left((2\mu\mu' + 1) R_{xx} + (\mu\mu'' - \mu(\mu')^3 + (\mu')^2/2) R_x^2 \right) + O(\epsilon^4), \end{aligned} \quad (13)$$

see [5]. In other words, the KP equation can be ‘decoupled’ into a pair of $(1+1)$ -dimensional equations (13) in infinitely many ways, indeed, $\mu(R)$ is an arbitrary function. The series in (12) and (13) contain even powers of ϵ only, and do not terminate in general.

Conversely, the requirement of the inheritance of all one-component reductions allows one to reconstruct dispersive terms: given the dKP equation, let us look for a third order dispersive extension in the form

$$u_t = uu_x + w_y + \epsilon(\dots) + \epsilon^2(\dots), \quad w_x = u_y, \quad (14)$$

where the terms at ϵ and ϵ^2 are homogeneous differential polynomials in the derivatives of u and w of the order two and three, respectively. We require that all one-component reductions (11) can be deformed accordingly, so that we have the following analogues of Eqs. (12) and (13),

$$u = R, \quad w = w(R) + \epsilon(\dots) + \epsilon^2(\dots) + O(\epsilon^3), \quad (15)$$

and

$$R_y = \mu R_x + \epsilon(\dots) + \epsilon^2(\dots) + O(\epsilon^3), \quad R_t = (\mu^2 + R) R_x + \epsilon(\dots) + \epsilon^2(\dots) + O(\epsilon^3), \quad (16)$$

respectively. In Eqs. (15) and (16), dots denote terms which are polynomial in the derivatives of R . Substituting Eqs. (15) into (14), and using (16) along with the consistency conditions $R_{ty} = R_{yt}$, one arrives at a complicated set of relations allowing one to uniquely reconstruct dispersive terms in (14): not surprisingly, we obtain that all terms at ϵ vanish, while the terms at ϵ^2 result in the familiar KP equation. Moreover, one only needs to perform calculations up to the order ϵ^4 to arrive at this result. It is important to emphasise that the above procedure is required to work for *arbitrary* μ : whenever one obtains a differential polynomial in μ which has to vanish due to the consistency conditions, all its coefficients have to be set equal to zero independently. Another observation is that the reconstruction procedure does not necessarily lead to a unique dispersive extension like in the dKP case: one and the same dispersionless system may possess essentially non-equivalent dispersive extensions. In particular, VN and modified VN equations from Theorem 1, as well as BKP and CKP equations from Theorem 2 have coinciding dispersionless limits.

Let us now turn to the general case of dispersionless equations of the form (3),

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \quad w_x = u_y.$$

The corresponding one-component reductions are of the form $u = R$, $w = w(R)$ where $R(x, y, t)$ satisfies a pair of Hopf-type equations

$$R_y = \mu R_x, \quad R_t = (\varphi + \psi\mu + \eta\mu^2) R_x;$$

here $\mu(R)$ is an arbitrary function, and $w' = \mu$. Let us seek a third order dispersive deformation of system (3) in the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(\dots) + \epsilon^2(\dots), \quad w_x = u_y,$$

and postulate that one-phase solutions can be deformed accordingly,

$$u = R, \quad w = w(R) + \epsilon(\dots) + \epsilon^2(\dots) + O(\epsilon^3),$$

where

$$R_y = \mu R_x + \epsilon(\dots) + \epsilon^2(\dots) + O(\epsilon^3), \quad R_t = (\varphi + \psi\mu + \eta\mu^2)R_x + \epsilon(\dots) + \epsilon^2(\dots) + O(\epsilon^3).$$

Proceeding as outlined above we reconstruct dispersive terms.

Remark. We point out that the formulae for dispersive deformations contain the expression

$$\eta_w \mu^3 + (\psi_w + \eta_u) \mu^2 + (\varphi_w + \psi_u) \mu + \varphi_u$$

in the denominator. Since μ is assumed to be arbitrary, this expression is nonzero unless φ, ψ, η satisfy the relations

$$\eta_w = 0, \quad \psi_w + \eta_u = 0, \quad \varphi_w + \psi_u = 0, \quad \varphi_u = 0. \quad (17)$$

These relations characterise the so-called *totally linearly degenerate systems*. Dispersive deformations of such systems do not inherit hydrodynamic reductions, and require a different approach which is beyond the scope of this paper.

2.3 Classification of fifth order equations

In this Section we summarize the classification results for integrable fifth order equations (4),

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(\dots) + \epsilon^2(\dots) + \epsilon^3(\dots) + \epsilon^4(\dots),$$

which are obtained by adding dispersive terms to integrable dispersionless candidates from Sect. 2.1 (one can show that all terms at ϵ and ϵ^3 have to vanish). Thus, we follow the classification of Sect. 2.1. We concentrate on the case when the ϵ^4 -terms contain at least one fifth order derivative of u or w , and skip all cases leading to third order equations which were already classified in [6].

Case 1: We verified that the exponential solutions (6) do not survive, so that all non-trivial examples come from the polynomial case (5),

$$\eta = 1, \quad \psi = \gamma u, \quad \varphi = \beta w - \frac{1}{2}\beta(\beta + \gamma)u^2 + \delta u.$$

A detailed analysis of dispersive deformations leads to the constraints $\gamma = \beta$, $\delta = 0$. Modulo rescalings, this gives BKP/CKP equations.

Case 2: One can prove that none of the logarithmic cases (8), (9) and (10) survive, so that all non-trivial examples come from the power case (7),

$$\eta = u, \quad \psi = \alpha w + \gamma u^{\alpha+1}, \quad \varphi = \delta u^{2\alpha+1}.$$

The further analysis leads to the only possibility $\alpha = -3$. Modulo rescalings, the case $\delta = \gamma = 0$ gives the HD_5 equation. The case of nonzero δ and γ leads to the new equations E_3 and E_4 .

Case 3: One can show that no examples from this class possess fifth order dispersive extensions.

3 Lax pairs

For the reader's convenience, in this section we bring together Lax pairs for all equations appearing in Theorems 1-2. We emphasise that our classification scheme does not assume the existence of a Lax pair: these come as the result of direct calculations once the classification is completed. We refer to [11, 17] for an alternative approach to the classification of integrable systems in 2 + 1 dimensions based on postulating the structure of a Lax pair.

3.1 Third order equations

Since both KP and modified KP equations are particular cases of the Gardner equation, we will skip the first two examples.

The **Gardner equation**,

$$u_t = (\beta w - \frac{\beta^2}{2}u^2 + \delta u)u_x + w_y + \epsilon^2 u_{xxx},$$

possesses the Lax pair [11]

$$\epsilon^2 \psi_{xx} + \frac{\epsilon}{\sqrt{3}}(\psi_y - \beta u \psi_x) + \frac{\delta}{6}u\psi = 0,$$

$$\epsilon \psi_t = 4\epsilon^3 \psi_{yyy} - \sqrt{3}\beta\epsilon^2(2\psi_{xx} + u_x\psi_x) + \epsilon(\beta w + \frac{\beta^2}{2}u^2 + \delta u)\psi_x + \epsilon\frac{\delta}{2}u_x - \frac{\beta\delta}{4\sqrt{3}}u^2 + \frac{\delta}{2\sqrt{3}}w.$$

The **VN equation**,

$$u_t = (uw)_y + \epsilon^2 u_{yyy},$$

possesses the Lax pair [16, 15]

$$\epsilon^2 \psi_{xy} + \frac{1}{3}u\psi = 0,$$

$$\psi_t = \epsilon^2 \psi_{yyy} + w\psi_y.$$

The **modified VN equation**,

$$u_t = (uw)_y + \epsilon^2 \left(u_{yy} - \frac{3}{4} \frac{u_y^2}{u} \right)_y,$$

possesses the Lax pair [1]

$$\epsilon^2 \psi_{xy} - \epsilon^2 \frac{u_y}{2u} \psi_x + \frac{1}{3}u\psi = 0,$$

$$\psi_t = \epsilon^2 \psi_{yyy} + w\psi_y + \frac{1}{2}w_y\psi.$$

The **HD equation**,

$$u_t = -2wu_y + uw_y - \frac{\epsilon^2}{u} \left(\frac{1}{u} \right)_{xxx},$$

possesses the Lax pair [11]

$$\frac{\epsilon}{u^2} \psi_{xx} + \frac{1}{\sqrt{3}} \psi_y = 0,$$

$$\psi_t = \frac{4\epsilon^2}{u^3} \psi_{xxx} + \left(\frac{2\sqrt{3}\epsilon w}{u^2} - \frac{6\epsilon^2 u_x}{u^4} \right) \psi_{xx}.$$

The **deformed HD equation**,

$$u_t = \frac{\delta}{u^3}u_x - 2wu_y + uw_y - \frac{\epsilon^2}{u} \left(\frac{1}{u} \right)_{xxx},$$

possesses the Lax pair [6]

$$\begin{aligned} \frac{\epsilon^2}{u^2}\psi_{xx} + \frac{\epsilon}{\sqrt{3}}\psi_y + \frac{\delta}{4u^2}\psi &= 0, \\ \psi_t &= \frac{4\epsilon^2}{u^3}\psi_{xxx} + \left(\frac{2\sqrt{3}\epsilon w}{u^2} - \frac{6\epsilon^2 u_x}{u^4} \right) \psi_{xx} + \frac{\delta}{u^3}\psi_x + \left(-\frac{3\delta u_x}{2u^4} + \frac{\sqrt{3}\delta w}{2\epsilon u^2} \right). \end{aligned}$$

The **Equation E₁**,

$$u_t = (\beta w + \beta^2 u^2)u_x - 3\beta w u_y + w_y + \epsilon^2[B^3(u) - \beta u_x B^2(u)],$$

possesses the Lax pair [6]

$$\begin{aligned} \epsilon^2\psi_{xy} &= \epsilon^2\beta w\psi_{xx} + \frac{1}{3}\psi, \\ \psi_t &= \epsilon^2\beta^3 u^3\psi_{xxx} - \epsilon^2\psi_{yyy} + 3\epsilon^2\beta^2 w u_y\psi_{xx} + \beta w\psi_x. \end{aligned}$$

The **Equation E₂**,

$$u_t = \frac{4}{3}\beta^2 u^3 u_x + (w - 3\beta u^2)u_y + uw_y + \epsilon^2[B^3(u) - \beta u_x B^2(u)],$$

possesses the Lax pair [6]

$$\begin{aligned} \epsilon^2\psi_{xy} &= \epsilon^2\beta w\psi_{xx} + \frac{1}{3}w\psi, \\ \psi_t &= \epsilon^2\beta^3 u^3\psi_{xxx} - \epsilon^2\psi_{yyy} + 3\epsilon^2\beta^2 w u_y\psi_{xx} + \frac{\beta^2}{3}u^3\psi_x + w\psi_y + \beta w u_y\psi. \end{aligned}$$

3.2 Fifth order equations

The **BKP equation**,

$$u_t = 5(u^2 + w)u_x + 5w u_y - 5w_y + 5\epsilon^2(w u_{xxx} + u_{xxy} + u_x u_{xx}) + \epsilon^4 u_{xxxxx},$$

possesses the Lax pair [11]

$$\begin{aligned} \psi_y + w\psi_x + \epsilon^2\psi_{xxx} &= 0, \\ \psi_t + 5(u^2 - w)\psi_x + \epsilon^2(15w\psi_{xxx} + 15u_x\psi_{xx} + 10u_{xx}\psi_x) + 9\epsilon^4\psi_{xxxxx} &= 0. \end{aligned}$$

The **CKP equation**,

$$u_t = 5(u^2 + w)u_x + 5w u_y - 5w_y + 5\epsilon^2(w u_{xxx} + u_{xxy} + \frac{5}{2}u_x u_{xx}) + \epsilon^4 u_{xxxxx},$$

possesses the Lax pair [11]

$$\psi_y + u\psi_x + \frac{1}{2}u_x\psi + \epsilon^2\psi_{xxx} = 0,$$

$$\psi_t + 5(u^2 - w)\psi_x + 5(uu_x - \frac{1}{2}u_y)\psi + \epsilon^2(15u\psi_{xxx} + \frac{45}{2}u_x\psi_{xx} + \frac{35}{2}u_{xx}\psi_x + 5u_{xxx}\psi) + 9\epsilon^4\psi_{xxxx} = 0.$$

The **HD equation**,

$$u_t = 15wu_y - 5uw_y + 5\epsilon^2 \left[\frac{u_{xy}}{u^2} - \frac{3}{u} \left(\frac{u_x u_y}{u^2} \right)_x \right] - \frac{\epsilon^4}{2u^2} \left(\frac{1}{u^2} \right)_{xxxx},$$

possesses the Lax pair [11]

$$\psi_y + \frac{\epsilon^2}{u^3}\psi_{xxx} = 0,$$

$$\psi_t + 15\epsilon^2 \frac{w}{u^3}\psi_{xxx} + \epsilon^4 \left[\frac{9}{u^5}\psi_{xxxx} - 45\frac{u_x}{u^6}\psi_{xxx} + \frac{15}{u^3} \left(\frac{1}{u^2} \right)_{xx} \psi_{xxx} \right] = 0.$$

The **Equation E₃**,

$$u_t = 4\gamma^2 \frac{u_x}{u^5} + 5(3w - \frac{\gamma}{u^2})u_y - 5uw_y + 5\epsilon^2 \left[\frac{\gamma}{2u^2} \left(\frac{1}{u^2} \right)_{xxx} + \frac{u_{xy}}{u^2} - \frac{3}{u} \left(\frac{u_x u_y}{u^2} \right)_x \right] - \frac{\epsilon^4}{2u^2} \left(\frac{1}{u^2} \right)_{xxxx},$$

possesses the Lax pair

$$\psi_y - \frac{\gamma}{u^3}\psi_x + \frac{\epsilon^2}{u^3}\psi_{xxx} = 0,$$

$$\psi_t + \left(\frac{6\gamma^2}{u^5} - \frac{15\gamma w}{u^3} \right) \psi_x + 15\epsilon^2 \left[\left(\frac{w}{u^3} - \frac{\gamma}{u^5} \right) \psi_{xxx} + \frac{3\gamma u_x}{u^6} \psi_{xx} + \frac{2\gamma}{u^3} \left(\frac{u_x}{u} \right)_x \psi_x \right] + \epsilon^4 \left[\frac{9}{u^5}\psi_{xxxx} - 45\frac{u_x}{u^6}\psi_{xxx} + \frac{15}{u^3} \left(\frac{1}{u^2} \right)_{xx} \psi_{xxx} \right].$$

The **Equation E₄**,

$$u_t = 4\gamma^2 \frac{u_x}{u^5} + 5(3w - \frac{\gamma}{u^2})u_y - 5uw_y + 5\epsilon^2 \left[\frac{\gamma}{3u} \left(\frac{1}{u^3} \right)_{xxx} - \gamma \frac{u_x^3}{u^7} - \left(\frac{1}{u} \right)_{xxy} + \left(\frac{u_x u_y}{u^3} \right)_x - \frac{u_y}{4u^4} (2uu_{xx} - 3u_x^2) \right] - \epsilon^4 \left[\frac{1}{2u^2} \left(\frac{1}{u^2} \right)_{xxxx} - \frac{15}{16} \left(\frac{(2uu_{xx} - 3u_x^2)^2}{u^8} \right)_x \right],$$

possesses the Lax pair

$$\psi_y + \left(\frac{\gamma u_x}{u^4} + \frac{u_y}{2u} \right) \psi - \frac{\gamma}{u^3}\psi_x + \epsilon^2 \left[\frac{1}{u^3}\psi_{xxx} + \left(\frac{3}{2} \frac{u_{xx}}{u^4} - \frac{9}{4} \frac{u_x^2}{u^5} \right) \psi_x + \left(\frac{1}{2u} \left(\frac{u_{xx}}{u^3} \right)_x + \frac{3}{4} \frac{u_x^3}{u^6} \right) \psi \right] = 0,$$

$$\begin{aligned}
& \psi_t + \left(\frac{15(u^3 u_y + 2\gamma u_x)w}{2u^4} - \frac{10\gamma u_y}{u^3} - \frac{10\gamma^2 u_x}{u^6} - \frac{5}{2}w_y \right) \psi + \left(6\frac{\gamma^2}{u^5} - 15\frac{\gamma w}{u^3} \right) \psi_x + \\
& + \epsilon^2 \left[\frac{15(wu^2 - \gamma)}{u^5} \psi_{xxx} + \left(\frac{60\gamma u_x}{u^6} + \frac{15u_y}{2u^3} \right) \psi_{xx} + \right. \\
& + \left(\frac{30\gamma}{u^2} \left(\frac{u_x}{u^4} \right)_x + \frac{15u_x u_y}{2u^4} + \frac{(90uu_{xx} - 135u_x^2)w}{4u^5} \right) \psi_x + \\
& + \left. \left(5\gamma \left(\frac{u_{xx}}{u^6} \right)_x - \frac{15\gamma}{u} \left(\frac{u_x^2}{u^6} \right)_x + \frac{5}{2} \left(\frac{u_{xy}}{u^3} \right)_x + \frac{15}{2u} w \left(\frac{u_{xx}}{u^3} \right)_x + \frac{45u_x^3}{4u^6} w + \frac{15u_x^2 u_y}{2u^5} \right) \psi \right] + \\
& + \epsilon^4 \left[\frac{9}{u^5} \psi_{xxxxx} - \frac{45u_x}{u^6} \psi_{xxxx} + \left(-\frac{15u_{xx}}{2u^6} + \frac{225u_x^2}{4u^7} \right) \psi_{xxx} + \left(30 \left(\frac{u_{xx}}{u^6} \right)_x + 180 \frac{u_x^3}{u^8} \right) \psi_{xx} + \right. \\
& + \left(\frac{45}{2} \left(\frac{u_{xxx}}{u^6} \right)_x - 60 \left(\frac{u_x u_{xx}}{u^7} \right)_x - \frac{105u_{xx}^2}{2u^7} + \frac{825}{6u^4} \left(\frac{u_x^3}{u^4} \right)_x - \frac{235u_x^4}{8u^9} \right) \psi_x \\
& + \left(-\left(\frac{1}{u^5} \right)_{xxxxx} + \frac{195}{2} \left(\frac{u_x u_{xxx}}{u^7} \right)_x + \frac{135}{2} \left(\frac{u_{xx}^2}{u^7} \right)_x - \frac{4605}{4} \left(\frac{u_x^2 u_{xx}}{u^8} \right)_x - \frac{165u_x u_{xx}^2}{2u^8} + \right. \\
& + \left. \frac{3375}{2} \left(\frac{u_x^4}{u^9} \right)_x + \frac{3645u_x^5}{8u^{10}} \right) \psi \Big].
\end{aligned}$$

We do not exclude a possibility that simpler Lax pair can be found in this case.

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References

- [1] L.V. Bogdanov, Veselov-Novikov equation as a natural two-dimensional generalization of the Korteweg-de Vries equation, *Theor. and Math. Phys.* **70** (1987) 309–314.
- [2] B.A. Dubrovin and Youjin Zhang, Bi-Hamiltonian hierarchies in 2D topological field theory at one-loop approximation, *Comm. Math. Phys.* **198** (1998) no. 2, 311–361.
- [3] E.V. Ferapontov and K.R. Khusnutdinova, On integrability of (2+1)-dimensional quasilinear systems, *Comm. Math. Phys.* **248** (2004) 187–206.
- [4] E.V. Ferapontov and K.R. Khusnutdinova, The characterization of 2-component (2+1)-dimensional integrable systems of hydrodynamic type, *J. Phys. A: Math. Gen.* **37**, no. 8 (2004) 2949–2963.
- [5] E.V. Ferapontov and A. Moro, Dispersive deformations of hydrodynamic reductions of 2D dispersionless integrable systems, *J. Phys. A: Math. Theor.* **42** (2009) 035211, 15pp.
- [6] E.V. Ferapontov, A. Moro and V.S. Novikov, Integrable equations in 2 + 1 dimensions: deformations of dispersionless limits, *J. Phys. A: Math. Theor.* **42** (2009) (18pp).
- [7] J. Gibbons and Y. Kodama, A method for solving the dispersionless KP hierarchy and its exact solutions. II, *Phys. Lett. A* **135** (1989) 167–170.

- [8] J. Gibbons and S.P. Tsarev, Reductions of the Benney equations, *Phys. Lett. A* **211** (1996) 19–24.
- [9] J. Gibbons and S.P. Tsarev, Conformal maps and reductions of the Benney equations, *Phys. Lett. A* **258** (1999) 263–271.
- [10] Yu. Kodama, A method for solving the dispersionless KP equation and its exact solutions, *Phys. Lett. A* **129**, no. 4 (1988) 223–226.
- [11] B.G. Konopelchenko and V.G. Dubrovsky, Some new integrable nonlinear evolution equations in 2+1 dimensions, *Phys. Letters A* **102**, N 1, 2 (1984) 15–17.
- [12] A.V. Mikhailov, A. B Shabat and V.V. Sokolov, The symmetry approach to classification of integrable equations. What is integrability?, 115–184, Springer Ser. Nonlinear Dynam., Springer, Berlin, 1991.
- [13] A.V. Mikhailov and R.I. Yamilov, Towards classification of $(2 + 1)$ -dimensional integrable equations. Integrability conditions, *I. J. Phys. A* **31**, no. 31 (1998) 6707–6715.
- [14] A.V. Mikhailov and V.S. Novikov, Perturbative symmetry approach, *J. Phys. A* **35**, no. 22 (2002) 4775–4790.
- [15] L.P. Nizhnik, Integration of multidimensional nonlinear equations by the method of inverse problem, *DAN SSSR*, **254** (1980) 332.
- [16] A.P. Veselov and S.P. Novikov, Finite-gap two-dimensional potential Schrödinger operators. Explicit formulae and evolution equations, *DAN SSSR*, **279** (1984) 20.
- [17] J.-P. Wang, On the structure of $(2 + 1)$ -dimensional commutative and noncommutative integrable equations. *J. Math. Phys.* **47**, no. 11 (2006) 113508, 19 pp.