arXiv:1103.4048v2 [math-ph] 21 Sep 2011

A Class of Infinite-dimensional Frobenius Manifolds and Their Submanifolds

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Abstract

We construct a class of infinite-dimensional Frobenius manifolds on the space of pairs of certain even functions meromorphic inside or outside the unit circle. Via a bi-Hamiltonian recursion relation, the principal hierarchies associated to such Frobenius manifolds are found to be certain extensions of the dispersionless two-component BKP hierarchy. Moreover, we show that these manifolds contain finite-dimensional Frobenius submanifolds as defined on the orbit space of Coxeter groups of types B and D.

Key words: Frobenius manifold, principal hierarchy, two-component BKP hierarchy, bi-Hamiltonian structure

1 Introduction

The notion of Frobenius manifold was introduced by Dubrovin [10] to give a geometric description of the WDVV equation in the context of 2D topological field theory [17, 26]. This concept was discovered to be essential in characterizing a wide class of integrable systems that are related to branches of mathematical physics such as topological field theory, Gromov-Witten invariants and singularity theory, see [10, 13, 16, 18, 19] and references therein.

Recall that a *Frobenius algebra* $(A, e, <\ ,\ >)$ is a commutative associative algebra A with a unity e and a non-degenerate symmetric bilinear form (inner product) $<\ ,\ >$

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that is invariant with respect to the multiplication, i.e.,

$$< x \cdot y, z >=< x, y \cdot z >, \quad x, y, z \in A. \tag{1.1}$$

Definition 1.1 ([10]) *A manifold M is called a Frobenius manifold if on each tangent space T_tM a Frobenius algebra $(T_tM, e, < \ , \ >)$ is defined depending smoothly on $t \in M$, and the following axioms are fulfilled:*

(F1) *The inner product $< \ , \ >$ is a flat metric on M. Denote the Levi-Civita connection for this metric by ∇, then the unity vector field e satisfies $\nabla e = 0$.*

(F2) *Let c be the 3-tensor $c(x, y, z) :=< x \cdot y, z >$, then the 4-tensor $(\nabla_w c)(x, y, z)$ is symmetric in the vector fields x, y, z, w.*

(F3) *A so-called Euler vector field $E \in Vect(M)$ is fixed on M such that $\nabla\nabla E = 0$, and it satisfies*

$$[E, x \cdot y] - [E, x] \cdot y - x \cdot [E, y] = x \cdot y,$$
$$Lie_E < x, y > - < [E, x], y > - < x, [E, y] > = (2 - d) < x, y > .$$

Here d is a constant named as the charge of M.

On a Frobenius manifold M there exist *flat coordinates* $t^1, \ldots, t^n$ such that the unity vector field $e = \partial/\partial t^1$ and

$$(\eta_{\alpha\beta})_{n\times n} = \left(< \frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta} >\right)_{n\times n} \tag{1.2}$$

is a constant non-degenerate matrix. Denote the inverse of this matrix by $(\eta^{\alpha\beta})_{n\times n}$, then the product of the Frobenius algebra T_tM reads

$$\frac{\partial}{\partial t^\alpha} \cdot \frac{\partial}{\partial t^\beta} = c^\gamma_{\alpha\beta} \frac{\partial}{\partial t^\gamma}, \quad c^\gamma_{\alpha\beta} = \eta^{\gamma\epsilon} c_{\epsilon\alpha\beta} \tag{1.3}$$

with $c_{\alpha\beta\gamma} = c(\partial/\partial t^\alpha, \partial/\partial t^\beta, \partial/\partial t^\gamma)$. Here the Einstein convention of summation over repeated indices is used.

Note that the structure constants satisfy

$$c^\beta_{1\alpha} = \delta^\beta_\alpha, \quad c^\epsilon_{\alpha\beta} c^\sigma_{\epsilon\gamma} = c^\epsilon_{\alpha\gamma} c^\sigma_{\epsilon\beta}, \tag{1.4}$$

and there locally exists on M a smooth function F, called the *potential* of the Frobenius manifold, such that

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}, \quad Lie_E F = (3 - d)F + \text{ quadratic}. \tag{1.5}$$

The system (1.4), (1.5) is called the WDVV equation in topological field theory [17, 26], and its solution F is the free energy whose third derivatives $c_{\alpha\beta\gamma}$ are the 3-point correlator functions.

Conversely, suppose it is given a potential F satisfying (1.4) and (1.5) (which also implies a constant matrix $(\eta_{\alpha\beta})$, a unity and a Euler vector field), then one can recover the structure of a Frobenius manifold.

A Frobenius manifold M is said to be *semisimple* if the Frobenius algebras T_tM are semisimple at generic points $t \in M$. On a semisimple Frobenius manifold there exist so-called *canonical coordinates* that are given by the eigenvalues of the operator of multiplication by the Euler vector field [10, 12].

Due to the associativity property (1.4), there is a commutative associative algebra structure on every cotangent space T_t^*M of a Frobenius manifold M. More precisely, the structure constants for the basis $\{dt^\alpha\}$ are $c_\gamma^{\alpha\beta} = \eta^{\alpha\epsilon}c_{\epsilon\gamma}^\beta$. Hence on T_t^*M a symmetric bilinear form, named the *intersection form*, is defined by

$$(dt^\alpha, dt^\beta)^* := i_E(dt^\alpha \cdot dt^\beta). \tag{1.6}$$

A celebrated result is that this intersection form and the bilinear form $< dt^\alpha, dt^\beta >^* = \eta^{\alpha\beta}$ compose a flat pencil of metrics on T_t^*M. That is, every linear combination (with constant coefficients) of the two metrics is a flat metric, and the Christoffel symbols $\Gamma_\gamma^{\alpha\beta}$ of their Levi-Civita connections obey the same linear combination relation.

Every flat pencil of metrics defines a local bi-Hamiltonian structure of hydrodynamic type [14], which induces an integrable system of bi-Hamiltonian equations. Based on this fact, on the loop space $\{S^1 \to M\}$ of the Frobenius manifold M there is an integrable hierarchy carrying a bi-Hamiltonian structure of hydrodynamic type. Such a hierarchy is called the *principal hierarchy* [10, 16], in which the evolutionary equations of the lowest level, i.e., the *primary flows*, read

$$\frac{\partial t^\gamma}{\partial T^{\alpha,0}} = c_{\alpha\beta}^\gamma \frac{\partial t^\beta}{\partial x}, \quad \alpha, \gamma = 1, 2, \ldots, n \tag{1.7}$$

with x being the coordinate of S^1, and the flows of higher level can be determined via a bi-Hamiltonian recursion relation with Hamiltonian densities $\theta_{\alpha,p}(t)$ defined by

$$\theta_{\alpha,0} = \eta_{\alpha\beta}t^\beta, \quad \theta_{\alpha,1} = \frac{\partial F}{\partial t^\alpha}, \quad \frac{\partial^2 \theta_{\alpha,p}}{\partial t^\lambda \partial t^\mu} = c_{\lambda\mu}^\epsilon \frac{\partial \theta_{\alpha,p-1}}{\partial t^\varepsilon} \text{ for } p > 1. \tag{1.8}$$

Up to now various Frobenius manifolds of finite dimension have been constructed. A typical example is the construction of semisimple Frobenius manifolds on the orbit space of Coxeter groups [11, 10]. In Particular, for Frobenius manifolds on the orbit space of Weyl groups, the principal hierarchies are the dispersionless limit of Drinfeld-Sokolov hierarchies associated to untwisted affine Kac-Moody algebras with the zeroth vertex c_0 of the Dynkin diagram marked [9, 21, 10, 13], and they are closely related to singularity theory and Gromov-Witten invariants [16, 19]. Another important class of Frobenius manifolds are defined on the orbit space of the extended affine Weyl groups [15], for part of which the principal hierarchies are the dispersionless limit of the extended bigraded Toda hierarchies [2, 4].

The profound theory of finite-dimensional Frobenius manifold and integrable hierarchies has started recently being generalized to the case of infinite dimension. The first infinite-dimensional Frobenius manifold was constructed by Carlet, Dubrovin and Mertens [3] in consideration of a bi-Hamiltonian structure of the dispersionless 2D Toda hierarchy. The other published infinite-dimensional Frobenius manifold, proposed by Raimondo [23], underlies the dispersionless KP hierarchy.

It is well known that the KP hierarchy and the 2D Toda hierarchy can be reduced to the Gelfand-Dickey hierarchies and the extended bigraded Toda hierarchies (except the flows defined by the logarithm operator) respectively [8, 4, 2]. These reduced hierarchies are deformations of the principal hierarchy for the Frobenius manifolds on the orbit space of (extended) Coxeter groups of type A. Such reductions of integrable hierarchies as well as the bi-Hamiltonian structures are expected to be described by certain finite-dimensional Frobenius submanifolds of the infinite-dimensional ones. For instance, in [3] a submanifold of two dimensions was considered, whose principal hierarchy is the dispersionless extended Toda hierarchy [4], i.e., the extended bigraded Toda hierarchy [2] with the simplest parameter $(1, 1)$.

In this paper we will follow the approach of [3] and construct a series of infinite-dimensional Frobenius manifolds. For each of them, the principal hierarchy will be derived, which carries a bi-Hamiltonian structure of the dispersionless two-component BKP hierarchy (see [6, 25, 22] or Section 3 below for the definition). Furthermore, corresponding to the reductions of these bi-Hamiltonian structures studied in [27], we show that there are finite-dimensional Frobenius submanifolds coinciding with those defined on the orbit space of Coxeter groups of types B and D [1, 11, 10, 29]. In fact, it is the properties of such finite-dimensional Frobenius manifolds that provide much inspiration for us to come to a relatively neat description of the infinite-dimensional manifolds.

Let us state the main results. Given $0 < \epsilon < 1$, we introduce two sets of holomorphic even functions on disks of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ as follows:

$$\mathcal{H}^- = \left\{ f(z) = \sum_{i \geq 0} f_i \, z^{-2i} \mid f \text{ holomorphic on } |z| > 1 - \epsilon \right\},$$

$$\mathcal{H}^+ = \left\{ \hat{f}(z) = \sum_{i \geq 0} \hat{f}_i \, z^{2i} \mid \hat{f} \text{ holomorphic on } |z| < 1 + \epsilon \right\}.$$

For two arbitrary positive integers m and n we consider the following coset

$$\tilde{\mathcal{M}}_{m,n} = (z^{2m}, 0) + z^{2m-2}\mathcal{H}^- \times z^{-2n}\mathcal{H}^+. \tag{1.9}$$

An element of this coset reads $\boldsymbol{a}(z) = \big(a(z), \hat{a}(z)\big)$ where

$$a(z) = z^{2m} + \sum_{i \leq m} v_i z^{2i-2}, \quad \hat{a}(z) = \sum_{j \geq -n} \hat{v}_j z^{2j}. \tag{1.10}$$

With coordinates given by the coefficients v_i and $\hat{v}_j$ of such Laurent series, the coset $\tilde{\mathcal{M}}_{m,n}$ is viewed as an infinite-dimensional manifold. This manifold has tangent and cotangent spaces as follows:

$$T_{\boldsymbol{a}}\tilde{\mathcal{M}}_{m,n} = z^{2m-2}\mathcal{H}^- \times z^{-2n}\mathcal{H}^+, \quad T^*_{\boldsymbol{a}}\tilde{\mathcal{M}}_{m,n} = z^{-2m+1}\mathcal{H}^+ \times z^{2n-1}\mathcal{H}^-. \tag{1.11}$$

The pairing between a vector $\boldsymbol{\alpha} = (\alpha(z), \hat{\alpha}(z)) \in T_{\boldsymbol{a}}\tilde{\mathcal{M}}_{m,n}$ and a covector $\boldsymbol{\omega} = (\omega(z), \hat{\omega}(z)) \in T^*_{\boldsymbol{a}}\tilde{\mathcal{M}}_{m,n}$ is defined to be

$$\langle \boldsymbol{\alpha}, \boldsymbol{\omega} \rangle = \frac{1}{2\pi \mathbb{i}} \oint_{|z|=1} \big[\alpha(z)\omega(z) + \hat{\alpha}(z)\hat{\omega}(z)\big] dz. \tag{1.12}$$

For $(a(z), \hat{a}(z)) \in \tilde{\mathcal{M}}_{m,n}$ we let

$$\zeta(z) = a(z) - \hat{a}(z), \quad l(z) = a(z)_+ + \hat{a}(z)_-, \tag{1.13}$$

where the subscripts "$\pm$" mean the projections of a Laurent series to its nonnegative part and negative part respectively.

Let $\mathcal{M}_{m,n}$ be a submanifold of $\tilde{\mathcal{M}}_{m,n}$ that consists of points $(a(z), \hat{a}(z))$ satisfying the following conditions:

(M1) The function $\hat{a}(z)$ has a pole of order $2n$ at 0, namely, $\hat{v}_{-n} \neq 0$;

(M2) For $|z| = 1$,

$$a(z)\hat{a}'(z) - a'(z)\hat{a}(z) \neq 0, \quad \zeta'(z) \neq 0, \quad l'(z) \neq 0; \tag{1.14}$$

(M3) The winding number of the function $\zeta(z)$ around 0 is 2, such that $w(z) = \zeta(z)^{1/2}$ maps S^1 biholomorphicly to a simple smooth curve Γ around 0.

The manifold $\mathcal{M}_{m,n}$ is the one on which a Frobenius structure will be constructed, and our discussion is restricted to it below.

Introduce variables $\boldsymbol{t} = \{t^i\}_{i\in\mathbb{Z}}$, $\boldsymbol{h} = \{h^j\}_{j=1}^m$ and $\hat{\boldsymbol{h}} = \{\hat{h}^k\}_{k=1}^n$ as

$$t^i := \frac{2}{2i-1}\frac{1}{2\pi \mathbb{i}} \oint_{|z|=1} \zeta(z)^{\frac{-2i+1}{2}} dz, \quad i \in \mathbb{Z}; \tag{1.15}$$

$$h^j := -\frac{2m}{2j-1} \operatorname{res}_{z=\infty} l(z)^{\frac{2j-1}{2m}} dz, \quad j = 1, 2, \cdots, m; \tag{1.16}$$

$$\hat{h}^k := \frac{2n}{2k-1} \operatorname{res}_{z=0} l(z)^{\frac{2k-1}{2n}} dz, \quad k = 1, 2, \cdots, n. \tag{1.17}$$

One sees below that these variables give another system of coordinates on the manifold $\mathcal{M}_{m,n}$.

Theorem 1.2 *For any two positive integers m and n, the infinite dimensional manifold $\mathcal{M}_{m,n}$ is a semisimple Frobenius manifold of charge $d_m = 1 - \frac{1}{m}$ such that* *(i) The unity vector field* $\boldsymbol{e} = \partial/\partial h^m$*;*

(ii) The Euler vector field

$$\mathcal{E}_{m,n} = \sum_{i\in\mathbb{Z}} \frac{m(1-2i)+1}{2m} t^i \frac{\partial}{\partial t^i} + \sum_{j=1}^{m} \frac{j}{m} h^j \frac{\partial}{\partial h^j} + \sum_{k=1}^{n} \Big(\frac{2k-1}{2n} + \frac{1}{2m}\Big) \hat{h}^k \frac{\partial}{\partial \hat{h}^k}; \tag{1.18}$$

(iii) The potential

$$\mathcal{F}_{m,n} = \left(\frac{1}{2\pi \mathbb{i}}\right)^2 \oint\oint_{|z_1|<|z_2|} \left(\frac{1}{2}\zeta(z_1)\zeta(z_2) - \zeta(z_1) l(z_2) + l(z_1)\zeta(z_2)\right) \times \\ \times \log(z_2 - z_1)\, dz_1 dz_2 + F_{m,n}, \tag{1.19}$$

where $F_{m,n}$ *is a rational function of variables* $\boldsymbol{h} \cup \hat{\boldsymbol{h}}$ *determined by*

$$\frac{\partial^3 F_{m,n}}{\partial u\, \partial v\, \partial w} = -\big(\mathrm{res}_{z=\infty} + \mathrm{res}_{z=0}\big) \frac{\partial_u l(z) \cdot \partial_v l(z) \cdot \partial_w l(z)}{l'(z)} dz \tag{1.20}$$

for any $u, v, w \in \boldsymbol{h} \cup \hat{\boldsymbol{h}}$*.*

This theorem will be proved in Section 2, where a flat metric (with flat coordinates (1.15)–(1.17)), an intersection form and the canonical coordinates of this semisimple Frobenius manifold will also be given. We remark that this Frobenius structure is distinct from the one in [3] restricted to the space consisting of even Laurent series.

In section 3 we will show that the Frobenius manifold $\mathcal{M}_{m,n}$ is associated with a bi-Hamiltonian structure of hydrodynamic type, which is the one derived in [27] for the dispersionless two-component BKP hierarchy, see also [28]. Then via the bi-Hamiltonian recursion relation the principal hierarchy for $\mathcal{M}_{m,n}$ will be obtained.

Theorem 1.3 *Suppose the coefficients* v_i *and* $\hat{v}_j$ *in* (1.10) *are smooth functions of* $x \in S^1$*. Then the principal hierarchy associated to the Frobenius manifold* $\mathcal{M}_{m,n}$ *can be written in the following Lax form:*

$$\frac{\partial a(z)}{\partial T^{u,p}} = \{-(A_{u,p}(z))_-, a(z)\}, \quad \frac{\partial \hat{a}(z)}{\partial T^{u,p}} = \{(A_{u,p}(z))_+, \hat{a}(z)\}, \quad p \ge 0, \tag{1.21}$$

where $\{f, g\} = \partial f/\partial z \cdot \partial g/\partial x - \partial g/\partial z \cdot \partial f/\partial x$ *and*

$$A_{u,p}(z) = \begin{cases} \dfrac{1}{2i+1}\dfrac{1}{(2p)!!}\zeta(z)^{\frac{2i+1}{2}}\varphi(z)^p, & u = t^i \ (i\in\mathbb{Z}); \\[2ex] \dfrac{\Gamma\left(\frac{2m-2j+1}{2m}\right)}{2m\,\Gamma\left(p+1+\frac{2m-2j+1}{2m}\right)} a(z)^{p+\frac{2m-2j+1}{2m}}, & u = h^j \ (1\le j\le m); \\[2ex] \dfrac{\Gamma\left(\frac{2n-2k+1}{2n}\right)}{2n\,\Gamma\left(p+1+\frac{2n-2k+1}{2n}\right)} \hat{a}(z)^{p+\frac{2n-2k+1}{2n}}, & u = \hat{h}^k \ (1\le k\le n) \end{cases} \tag{1.22}$$

with $\varphi(z) = a(z) + \hat{a}(z)$*.*

Observe that the flows $\partial/\partial T^{h^j,p}$ and $\partial/\partial T^{\hat{h}^k,p}$ compose the dispersionless two-component BKP hierarchy. For this reason we call the system of equations (1.21) the *principal two-component BKP hierarchy.* Recall that this hierarchy depends on two parameters m and n.

In section 4 we will consider an $(m+n)$-dimensional submanifold $M_{m,n} \subset \mathcal{M}_{m,n}$ spanned by the flat coordinates $\boldsymbol{h} \cup \hat{\boldsymbol{h}}$. The following proposition is a corollary of the results of Sections 2 and 3.

Proposition 1.4 *The submanifold $M_{m,n}$ is a semisimple Frobenius submanifold of $\mathcal{M}_{m,n}$, i.e., $M_{m,n}$ carries a Frobenius structure projected from that of $\mathcal{M}_{m,n}$. The potential $F_{m,n}$ (given in* (1.19)*) of $M_{m,n}$ is a polynomial in $(h^1, \dots, h^m, \hat{h}^1, \dots, \hat{h}^n, 1/\hat{h}^1)$. It is a polynomial in the flat coordinates $(h^1, \dots, h^m, \hat{h}^1, \dots, \hat{h}^n)$ if and only if $n = 1$.*

The principal hierarchy associated to $M_{m,n}$ is the $(2m, 2n)$-reduction of the dispersionless two-component BKP hierarchy [7, 22], that is, the restriction of (1.21) *to $a(z) = \hat{a}(z) = l(z)$. Particularly when $n = 1$, the principal hierarchy is the dispersionless limit of the Drinfeld-Sokolov hierarchy of type $(D_{m+1}^{(1)}, c_0)$.*

One will see that $M_{m,n}$ coincides with the Frobenius manifold defined on the orbit space of the Coxeter group B_{m+n} by Bertola [1] with superpotential $l(z)$ given in (1.13), and constructed also by Zuo [29] in a different way. However, to our best knowledge, the principal hierarchy for the Frobenius manifold $M_{m,n}$ with general (m, n) has not been considered before.

The last section is devoted to the conclusion and outlook.

2 Construction of Frobenius manifolds

Let us begin to prove Theorem 1.2, i.e., to equip a semisimple Frobenius structure to the infinite-dimensional manifold

$$\mathcal{M}_{m,n} \subset (z^{2m}, 0) + z^{2m-2}\mathcal{H}^- \times z^{-2n}\mathcal{H}^+$$

constrained by conditions (M1) – (M3) above with any fixed positive integers m and n. To this end, we need to introduce a flat metric that is invariant with respect to a multiplication on each tangent space, and to write down the potential and the Euler vector field. Besides that, the canonical coordinates and the intersection form on this Frobenius manifold will be derived.

2.1 A flat metric

Recall that a system of coordinates of $\mathcal{M}_{m,n}$ is given by the coefficients of the series

$$\boldsymbol{a} = (a(z), \hat{a}(z)) = \left(z^{2m} + \sum_{i \le m} v_i z^{2i-2}, \sum_{j \ge -n} \hat{v}_j z^{2j} \right).$$

We identify the tangent space $T_{\boldsymbol{a}}\mathcal{M}_{m,n}$ with $z^{2m-2}\mathcal{H}^- \times z^{-2n}\mathcal{H}^+$ by

$$\partial = \ (\partial a(z), \partial \hat{a}(z)) \tag{2.1}$$

for any tangent vector ∂. For example, when $i \le m$ and $j \ge -n$ we have

$$\frac{\partial}{\partial v_i} = (z^{2i-2}, 0), \quad \frac{\partial}{\partial \hat{v}_j} = (0, z^{2j}). \tag{2.2}$$

According to the pairing (1.12), $T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$ is identified with $z^{-2m+1}\mathcal{H}^+ \times z^{2n-1}\mathcal{H}^-$, and its dual basis with respect to (2.2) reads

$$dv_i = (z^{-2i+1}, 0), \quad d\hat{v}_j = (0, z^{-2j-1}) \tag{2.3}$$

with $i \le m$ and $j \ge -n$.

Introduce two generating functions for covectors

$$da(p) := \sum_{i\le m} dv_i p^{2i-2} = \Big(\frac{p^{2m}}{z^{2m-1}(p^2-z^2)}, 0\Big), \quad |z| < |p|, \tag{2.4}$$

$$d\hat{a}(p) := \sum_{j\ge -n} d\hat{v}_j p^{2j} = \Big(0, \frac{z^{2n+1}}{p^{2n}(z^2-p^2)}\Big), \quad |z| > |p|. \tag{2.5}$$

The following lemma follows from the Cauchy integral formula.

Lemma 2.1 *The generating functions* (2.4) *and* (2.5) *have the following properties:*

(i) For any vector $\boldsymbol{\xi} = (\xi(z), \hat{\xi}(z)) \in T_{\boldsymbol{a}}\mathcal{M}_{m,n}$,

$$\langle da(p), \boldsymbol{\xi}\rangle = \xi(p), \quad \langle d\hat{a}(p), \boldsymbol{\xi}\rangle = \hat{\xi}(p); \tag{2.6}$$

(ii) For any covector $\boldsymbol{\omega} = (\omega(z), \hat{\omega}(z)) \in T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$,

$$\boldsymbol{\omega} = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \big[\omega(p)da(p) + \hat{\omega}(p)d\hat{a}(p)\big]dp. \tag{2.7}$$

Let us introduce a symmetric bilinear form on the cotangent space $T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$ as

$$< d\alpha(p), d\beta(q) >^* = \frac{q\beta'(q)}{q^2-p^2} + \frac{p\alpha'(p)}{p^2-q^2}, \tag{2.8}$$

where $\alpha'(p) = \partial\alpha(p)/\partial p$ and $\alpha, \beta \in \{a, \hat{a}\}$. Note the difference between the above formula and equation (1.16) in [3].

Since the pairing (1.12) is nondegenerate, the following linear map is well defined

$$\eta: \ T^*_{\boldsymbol{a}}\mathcal{M}_{m,n} \to T_{\boldsymbol{a}}\mathcal{M}_{m,n}, \quad \langle \boldsymbol{\omega}_1, \eta(\boldsymbol{\omega}_2)\rangle = < \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 >^* \tag{2.9}$$

for any $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$. More explicitly, given $\boldsymbol{\omega} = (\omega(z), \hat{\omega}(z)) \in T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$ we have

$$\begin{aligned}\eta(\boldsymbol{\omega})(z) =&\frac{1}{2\pi\mathbb{i}}\oint_{|z|=1}\big(\omega(p)\eta(da(p)) + \hat{\omega}(p)\eta(d\hat{a}(p))\big)dp\\ =&\big(a'(z)[\omega(z)+\hat{\omega}(z)]_- - [\omega(z)a'(z)+\hat{\omega}(z)\hat{a}'(z)]_-,\\ &-\hat{a}'(z)[\omega(z)+\hat{\omega}(z)]_+ + [\omega(z)a'(z)+\hat{\omega}(z)\hat{a}'(z)]_+\big),\end{aligned} \tag{2.10}$$

in which the following formulae have been used:

$$\eta(d\alpha(p)) = (< d\alpha(p), da(z) >^*, < d\alpha(p), d\hat{a}(z) >^*), \quad \alpha \in \{a, \hat{a}\}. \tag{2.11}$$

Lemma 2.2 *The linear map η defined in* (2.9) *is a bijection.*

Proof It follows from (2.10) that η is surjective, hence we only need to show that it is invertible.

Suppose $\boldsymbol{\xi} = \eta(\boldsymbol{\omega})$, where

$$\begin{aligned}\boldsymbol{\xi} =&\big(\xi(z), \hat{\xi}(z)\big) = \Big(\sum_{i\le m}\xi_i z^{2i-2}, \sum_{j\ge -n}\hat{\xi}_j z^{2j}\Big) \in T_{\boldsymbol{a}}\mathcal{M}_{m,n},\\ \boldsymbol{\omega} =&\big(\omega(z), \hat{\omega}(z)\big) = \Big(\sum_{i\le m}\omega_{-i}z^{-2i+1}, \sum_{j\ge -n+1}\hat{\omega}_{-j}z^{-2j+1}\Big) \in T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}.\end{aligned}$$

Recall $a'(z) - \hat{a}'(z) = \zeta'(z) \neq 0$ in (1.14), then from (2.10) we have

$$\omega(z)_+ = -\Big(\frac{\xi(z)-\hat{\xi}(z)}{a'(z)-\hat{a}'(z)}\Big)_+, \quad \hat{\omega}(z)_- = \Big(\frac{\xi(z)-\hat{\xi}(z)}{a'(z)-\hat{a}'(z)}\Big)_-. \tag{2.12}$$

On the other hand, consider

$$\xi(z)_+ = \big(a'(z)_+(\omega(z)+\hat{\omega}(z))_-\big)_+ \tag{2.13}$$

where

$$\begin{aligned}a'(z)_+ &= 2mz^{2m-1} + (2m-2)v_m z^{2m-3} + \cdots + 2v_2 z^1,\\ (\omega(z)+\hat{\omega}(z))_- &= \tilde{\omega}_{-1}z^{-1} + \tilde{\omega}_{-2}z^{-3} + \cdots + \tilde{\omega}_{-m}z^{-2m+1} + O(z^{-2m-1}), \quad z\to\infty\end{aligned}$$

with $\tilde{\omega}_j = \omega_j + \hat{\omega}_j$. Introduce the following invertible matrix

$$K_m = \begin{pmatrix} 2m & & & & \\ (2m-2)v_m & 2m & & & \\ (2m-4)v_{m-1} & (2m-2)v_m & 2m & & \\ \vdots & \ddots & \ddots & \ddots & \\ 2v_2 & \cdots & (2m-4)v_{m-1} & (2m-2)v_m & 2m \end{pmatrix}. \tag{2.14}$$

Equation (2.13) is rewritten to

$$\begin{pmatrix} \xi_m \\ \xi_{m-1} \\ \vdots \\ \xi_1 \end{pmatrix} = K_m \begin{pmatrix} \tilde{\omega}_{-1} \\ \tilde{\omega}_{-2} \\ \vdots \\ \tilde{\omega}_{-m} \end{pmatrix}. \tag{2.15}$$

Hence $\tilde{\omega}_{-1}, \dots, \tilde{\omega}_{-m}$ can be found from this linear equation. In combination with (2.12) one obtains $(\omega(z))_-$ and thus $\omega(z)$.

Similarly, from the second equation in (2.12) together with

$$\hat{\xi}(z)_- = -\big(\hat{a}'(z)_-(\omega(z) + \hat{\omega}(z))_+\big)_- \tag{2.16}$$

one gets $\hat{\omega}(z)$.

Therefore, $\boldsymbol{\omega}$ is uniquely determined by $\boldsymbol{\xi}$ via the relations (2.12), (2.13) and (2.16). The lemma is proved. □

With the help of the bijection η, we now define a symmetric bilinear form on the tangent space $T_{\boldsymbol{a}}\mathcal{M}_{m,n}$ as

$$< \partial_1, \partial_2 > := \langle \eta^{-1}(\partial_1), \partial_2 \rangle = < \eta^{-1}(\partial_1), \eta^{-1}(\partial_2) >^* . \tag{2.17}$$

Recall on $\mathcal{M}_{m,n}$ the functions

$$\zeta(z) = a(z) - \hat{a}(z), \quad l(z) = a(z)_+ + \hat{a}(z)_- \tag{2.18}$$

satisfy $\zeta'(z) \neq 0$ and $l'(z) \neq 0$ for $|z| = 1$.

Lemma 2.3 *For any vectors $\partial_1, \partial_2 \in T_{\boldsymbol{a}}\mathcal{M}_{m,n}$, the bilinear form* (2.17) *is represented as*

$$\begin{aligned} < \partial_1, \partial_2 > = &- \frac{1}{2\pi \mathbb{i}} \oint_{|z|=1} \frac{\partial_1 \zeta(z) \cdot \partial_2 \zeta(z)}{\zeta'(z)} dz \\ &- \operatorname{res}_{z=\infty} \frac{\partial_1 l(z) \cdot \partial_2 l(z)}{l'(z)} dz - \operatorname{res}_{z=0} \frac{\partial_1 l(z) \cdot \partial_2 l(z)}{l'(z)} dz. \end{aligned} \tag{2.19}$$

Proof Suppose $(\omega(z), \hat{\omega}(z)) = \eta^{-1}(\partial_1)$. We have

$$\begin{aligned} \langle \eta^{-1}(\partial_1), \partial_2 \rangle = &\frac{1}{2\pi \mathbb{i}} \oint_{|z|=1} \big(\omega(z)\partial_2 a(z) + \hat{\omega}(z)\partial_2 \hat{a}(z)\big) dz \\ = &\frac{1}{2\pi \mathbb{i}} \oint_{|z|=1} \big((\omega(z)_+ - \hat{\omega}(z)_-)(\partial_2 a(z) - \partial_2 \hat{a}(z))\big) dz \\ &+ \frac{1}{2\pi \mathbb{i}} \oint_{|z|=1} (\omega(z)_- + \hat{\omega}(z)_-)\partial_2 a(z) dz \end{aligned}$$

$$+\frac{1}{2\pi\mathbb{i}}\oint_{|z|=1}(\omega(z)_+ + \hat\omega(z)_+)\partial_2\hat a(z)dz. \tag{2.20}$$

Let I_1, I_2 and I_3 denote the three integrals on the right hand side respectively.

First, the formulae (2.12) now read

$$\omega(z)_+ = -\left(\frac{\partial_1\zeta(z)}{\zeta'(z)}\right)_+, \quad \hat\omega(z)_- = \left(\frac{\partial_1\zeta(z)}{\zeta'(z)}\right)_-,$$

hence

$$I_1 = -\frac{1}{2\pi\mathbb{i}}\oint_{|z|=1}\frac{\partial_1\zeta(z)\cdot\partial_2\zeta(z)}{\zeta'(z)}dz. \tag{2.21}$$

Second, by using the relation (2.15) we have

$$\begin{aligned} I_2 =& \tilde\omega_{-1}\partial_2 v_1 + \tilde\omega_{-2}\partial_2 v_2 + \cdots + \tilde\omega_{-m}\partial_2 v_m \\ =& (\partial_1 v_1, \partial_1 v_2, \ldots, \partial_1 v_m)K_m^{-1}\begin{pmatrix}\partial_2 v_m \\ \partial_2 v_{m-1} \\ \vdots \\ \partial_2 v_1\end{pmatrix}. \end{aligned} \tag{2.22}$$

Note that $l'(z)/z^{2m-1}$ is a polynomial in z^{-2}, and that

$$K_m = \left(\frac{l'(z)}{z^{2m-1}}\right)_{z^{-2}\to\Lambda}, \quad \Lambda = (\delta_{i,j+1})_{m\times m}.$$

If one writes $1/l'(z) = z^{-2m+1}(f_0 + f_{-1}z^{-2} + f_{-2}z^{-4} + \cdots)$, then

$$K_m^{-1} = \left(\frac{z^{2m-1}}{l'(z)}\right)_{z^{-2}\to\Lambda} = (f_{-i+j})_{m\times m}.$$

Thus

$$\begin{aligned} I_2 =& \sum_{i,j=1}^{m}\partial_1 v_i\cdot f_{-i+j}\cdot\partial_2 v_{m+1-j} \\ =& -\operatorname{res}_{z=\infty}\left(\sum_{i=1}^{m}\partial_1 v_i z^{2i-2}\sum_{j=1}^{m}\partial_2 v_j z^{2j-2}\cdot z^{-2m+1}\sum_{k\le 0}f_k z^{2k}\right)dz \\ =& -\operatorname{res}_{z=\infty}\frac{\partial_1 l(z)\cdot\partial_2 l(z)}{l'(z)}dz. \end{aligned} \tag{2.23}$$

In the same way one can check

$$I_3 = -\operatorname{res}_{z=0}\frac{\partial_1 l(z)\cdot\partial_2 l(z)}{l'(z)}dz. \tag{2.24}$$

Finally, we gather (2.21), (2.23) and (2.24) together and finish the proof of the lemma. □

Lemma 2.4 *The bilinear form* $<\ ,\ >$ *in* (2.19) *is a nondegenerate flat metric on* $\mathcal{M}_{m,n}$*, with flat coordinates given by* (1.15)–(1.17)*. More exactly,*

$$< \frac{\partial}{\partial t^{i_1}}, \frac{\partial}{\partial t^{i_2}} >= -\frac{1}{2}\,\delta_{i_1+i_2,0}, \quad i_1, i_2 \in \mathbb{Z}; \tag{2.25}$$

$$< \frac{\partial}{\partial h^{j_1}}, \frac{\partial}{\partial h^{j_2}} >= \frac{1}{2m}\delta_{j_1+j_2,m+1}, \quad j_1, j_2 \in \{1, \dots, m\}; \tag{2.26}$$

$$< \frac{\partial}{\partial \hat{h}^{k_1}}, \frac{\partial}{\partial \hat{h}^{k_2}} >= \frac{1}{2n}\delta_{k_1+k_2,n+1}, \quad k_1, k_2 \in \{1, \dots, n\} \tag{2.27}$$

and any other pairing between these vector fields vanishes.

Proof Recall on $\mathcal{M}_{m,n}$ we have assumed that $z \mapsto w(z) = \zeta(z)^{1/2}$ gives a biholomorphic map from S^1 to a simple curve Γ around $w = 0$. Since $\zeta(z)$ is an even function, then $w(-z) = -w(z)$. Hence the inverse holomorphic map $w \mapsto z(w)$ on a neighborhood of Γ satisfies $z(w) = -z(-w)$. We apply the Riemann-Hilbert decomposition of this function on the w-plane:

$$z(w) = f_+(w) + f_-(w), \quad w \in \Gamma_1, \tag{2.28}$$

where the functions $f_+(w)$ and $f_-(w)$ are holomorphic inside and outside the curve Γ respectively. From the definition (1.15) of t^i it follows that

$$t^i = \frac{2}{2\pi \mathrm{i}} \oint_\Gamma z(w) w^{-2i} dw,$$

hence

$$f_+(w) = \sum_{i \geq 1} \frac{t^i}{2} w^{2i-1}, \quad |w| \to 0; \tag{2.29}$$

$$f_-(w) = \sum_{i \leq 0} \frac{t^i}{2} w^{2i-1}, \quad |w| \to \infty. \tag{2.30}$$

On the other hand, we denote

$$\chi(z) := l(z)^{\frac{1}{2m}} \ \text{ near } \ \infty, \quad \hat{\chi}(z) := l(z)^{\frac{1}{2n}} \ \text{ near } \ 0. \tag{2.31}$$

Similarly as above, the inverse function $z(\chi)$ of $\chi(z)$ is expanded in the form

$$z(\chi) = \chi - \frac{h^1}{2m}\chi^{-1} - \frac{h^2}{2m}\chi^{-3} - \dots - \frac{h^m}{2m}\chi^{-2m+1} + O(\chi^{-2m-1}), \quad |\chi| \to \infty; \tag{2.32}$$

and the inverse function $z(\hat{\chi})$ of $\hat{\chi}(z)$ reads

$$z(\hat{\chi}) = \frac{\hat{h}^1}{2n}\hat{\chi}^{-1} + \frac{\hat{h}^2}{2n}\hat{\chi}^{-3} + \dots + \frac{\hat{h}^n}{2n}\hat{\chi}^{-2n+1} + O(\hat{\chi}^{-2n-3}), \quad |\hat{\chi}| \to \infty. \tag{2.33}$$

Observe that the functions $\zeta(z) = w(z)^2$ and $l(z) = \chi(z)^{2m} = \hat{\chi}(z)^{2n}$ are determined by the variables

$$\boldsymbol{t} \cup \boldsymbol{h} \cup \hat{\boldsymbol{h}} = \{t^i \mid i \in \mathbb{Z}\} \cup \{h^j \mid j = 1, \dots, m\} \cup \{\hat{h}^k \mid k = 1, \dots, n\}. \tag{2.34}$$

These two functions give the series $a(z)$ and $\hat{a}(z)$ by

$$a(z) = \zeta(z)_- + l(z), \quad \hat{a}(z) = -\zeta(z)_+ + l(z). \tag{2.35}$$

Thus (2.34) is indeed a system of coordinates on the manifold $\mathcal{M}_{m,n}$.

Let us compute the pairings (2.25)–(2.27). First, due to (2.28)–(2.30) we have

$$\frac{\partial z(w)}{\partial t^i} = \frac{1}{2} w^{2i-1},$$

hence

$$\frac{\partial w(z)}{\partial t^i} = -\frac{1}{2} w(z)^{2i-1} w'(z). \tag{2.36}$$

In the same way, for $\chi(z)$ and its inverse function one has

$$\begin{aligned} \frac{\partial z(\chi)}{\partial h^j} &= -\frac{1}{2m} \chi^{-2j+1} + O(\chi^{-2m-1}), \quad |\chi| \to \infty; \\ \frac{\partial \chi(z)}{\partial h^j} &= \frac{1}{2m} \chi(z)^{-2j+1} \chi'(z) + O(z^{-2m-1}), \quad |z| \to \infty. \end{aligned} \tag{2.37}$$

While for $\hat{\chi}(z)$ and its inverse function,

$$\begin{aligned} \frac{\partial z(\hat{\chi})}{\partial \hat{h}^k} &= \frac{1}{2n} \hat{\chi}^{-2k+1} + O(\hat{\chi}^{-2n-1}), \quad |\hat{\chi}| \to \infty; \\ \frac{\partial \hat{\chi}(z)}{\partial \hat{h}^k} &= -\frac{1}{2n} \hat{\chi}(z)^{-2k+1} \hat{\chi}'(z) + O(z^{2n+1}), \quad |z| \to 0. \end{aligned} \tag{2.38}$$

The formulae (2.36)–(2.38) lead to

$$\frac{\partial \zeta(z)}{\partial t^i} = -\, w(z)^{2i} w'(z) = -\frac{1}{2} w(z)^{2i-1} \zeta'(z), \tag{2.39}$$

$$\frac{\partial l(z)}{\partial h^j} = \left(\chi(z)^{2m-2j} \chi'(z)\right)_+, \quad \frac{\partial l(z)}{\partial \hat{h}^k} = -\left(\hat{\chi}(z)^{2n-2k} \hat{\chi}'(z)\right)_-, \tag{2.40}$$

$$\frac{\partial \zeta(z)}{\partial h^j} = \frac{\partial \zeta(z)}{\partial \hat{h}^k} = \frac{\partial l(z)}{\partial t^i} = \left(\frac{\partial l(z)}{\partial h^j}\right)_- = \left(\frac{\partial l(z)}{\partial \hat{h}^k}\right)_+ = 0. \tag{2.41}$$

Substituting them into (2.19) and by integration by parts, we have

$$\begin{aligned} < \frac{\partial}{\partial t^{i_1}}, \frac{\partial}{\partial t^{i_2}} > &= -\frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{w(z)^{2i_1+2i_2-1} w'(z)}{2} dz \\ &= -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{w^{2i_1+2i_2-1}}{2} dw = -\frac{1}{2} \delta_{i_1+i_2,0}, \end{aligned}$$

$$< \frac{\partial}{\partial h^{j_1}}, \frac{\partial}{\partial h^{j_2}} >= -\operatorname{res}_{z=\infty} \frac{\chi(z)^{2m-2j_1-2j_2+1}\chi'(z)}{2m} dz = \frac{1}{2m}\delta_{j_1+j_2,m+1},$$
$$< \frac{\partial}{\partial \hat h^{k_1}}, \frac{\partial}{\partial \hat h^{k_2}} >= -\operatorname{res}_{z=0} \frac{\hat\chi(z)^{2n+1-2k_1-2k_2}\hat\chi'(z)}{2n} dz = \frac{1}{2n}\delta_{k_1+k_2,n+1}.$$

Any other pairing between these vectors vanishes. The lemma is proved. □

Corollary 2.5 *For the flat metric* (2.19)*, the induced metric on the cotangent bundle of* $\mathcal{M}_{m,n}$ *is given by* (2.8)*.*

According to the identification (2.7), let us compute another basis $\{dt^i \mid i \in \mathbb{Z}\} \cup \{dh^j \mid 1 \le j \le m\} \cup \{d\hat h^k \mid 1 \le k \le n\}$ of $T^*\mathcal{M}_{m,n}$ that will be used later. By using the formulae (1.15)–(1.17) and (2.3), we have

$$\begin{aligned}
dt^i =& \sum_{r\le m} \frac{\partial t^i}{\partial v_r} dv_r + \sum_{r \ge -n} \frac{\partial t^i}{\partial \hat v_r} d\hat v_r \\
=& -\frac{1}{2\pi\mathrm{i}} \oint_{|p|=1} w(p)^{-2i-1} \sum_{r\le m} \frac{\partial \zeta(p)}{\partial v_r} dp \cdot (z^{-2r+1}, 0) \\
& -\frac{1}{2\pi\mathrm{i}} \oint_{|q|=1} w(q)^{-2i-1} \sum_{r\ge -n} \frac{\partial \zeta(q)}{\partial \hat v_r} dq \cdot (0, z^{-2r-1}) \\
=& \Big(-\big(w(z)^{-2i-1}\big)_{\ge -2m+1}, \big(w(z)^{-2i-1}\big)_{\le 2n-1} \Big), \qquad (2.42) \\
dh^j =& \sum_{r\le m} \frac{\partial h^j}{\partial v_r} dv_r + \sum_{r \ge -n} \frac{\partial h^j}{\partial \hat v_r} d\hat v_r \\
=& -\operatorname{res}_{p=\infty} \chi(p)^{2j-2m-1} \sum_{r\le m} \frac{\partial l(p)}{\partial v_r} dp \cdot (z^{-2r+1}, 0) \\
=& \Big(\big(\chi(z)^{2j-2m-1}\big)_{\ge -2m+1}, 0 \Big), \qquad (2.43) \\
d\hat h^k =& \sum_{r\le m} \frac{\partial \hat h^k}{\partial v_r} dv_r + \sum_{r \ge -n} \frac{\partial \hat h^k}{\partial \hat v_r} d\hat v_r \\
=& \operatorname{res}_{p=0} \hat\chi(p)^{2k-2n-1} \sum_{r\ge -n} \frac{\partial l(p)}{\partial v_r} z^{-2r-2} dp \cdot (0, z^{-2r-1}) \\
=& \Big(0, \big(\hat\chi(z)^{2k-2n-1}\big)_{\le 2n-1} \Big). \qquad (2.44)
\end{aligned}$$

Here and below we use notations $\left(\sum_r f_r z^r\right)_{\le s} = \sum_{r\le s} f_r z^r$ and $\left(\sum_r f_r z^r\right)_{\ge s} = \sum_{r\ge s} f_r z^r$.

2.2 Frobenius algebra

To construct a Frobenius algebra structure on the tangent spaces of $\mathcal{M}_{m,n}$, we start from a multiplication on the cotangent spaces (recall (2.4) and (2.5)):

$$d\alpha(p)\cdot d\beta(q) := \frac{q\beta'(q)}{q^2-p^2} d\alpha(p) + \frac{p\alpha'(p)}{p^2-q^2} d\beta(q), \quad \alpha,\beta \in \{a, \hat a\}. \qquad (2.45)$$

Lemma 2.6 *On $T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$ the following two statements hold true.*

(i) The multiplication (2.45) *is associative and commutative. More generally, for $\alpha_i \in \{a, \hat{a}\}$ one has*

$$d\alpha_1(p_1)\cdot d\alpha_2(p_2)\cdot\cdots\cdot d\alpha_k(p_k) = \sum_{i=1}^{k}\left(\prod_{j\neq i}\frac{p_j\alpha_j'(p_j)}{p_j^2-p_i^2}\right)d\alpha_i(p_i). \tag{2.46}$$

(ii) The bilinear form $<\ ,\ >^*$ defined in (2.8) *is invariant with respect to the multiplication* (2.45).

Proof It is straightforward to check the first assertion. For the second one,

$$\begin{aligned}&< d\alpha(p)\cdot d\beta(q), d\gamma(r) >^*\\ =&\frac{q\beta'(q)}{q^2-p^2}\frac{r\gamma'(r)}{r^2-p^2}+\frac{p\alpha'(p)}{p^2-q^2}\frac{r\gamma'(r)}{r^2-q^2}+\frac{p\alpha'(p)}{p^2-r^2}\frac{q\beta'(q)}{q^2-r^2}\\ =& < d\alpha(p), d\beta(q)\cdot d\gamma(r) >^*,\end{aligned}$$

where $\alpha,\beta,\gamma\in\{a,\hat{a}\}$. The lemma is proved. □

The multiplication (2.45) can also be represented in Laurent series by using (2.7). In fact, for any $\boldsymbol{\omega}_i = (\omega_i(z), \hat{\omega}_i(z)) \in T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$ with $i = 1, 2$ one can verify

$$\begin{aligned}\boldsymbol{\omega}_1\cdot\boldsymbol{\omega}_2 =\Big(&\big[\omega_2(z)\big(\omega_1(z)a'(z)\big)_+ - \omega_2(z)\big(\hat{\omega}_1(z)\hat{a}'(z)\big)_-\\ &-\omega_1(z)\big(\omega_2(z)a'(z)\big)_- - \omega_1(z)\big(\hat{\omega}_2(z)\hat{a}'(z)\big)_-\big]_{\geq -2m+1}\\ &\big[\hat{\omega}_2(z)\big(\omega_1(z)a'(z)\big)_+ + \hat{\omega}_2(z)\big(\hat{\omega}_1(z)\hat{a}'(z)\big)_+\\ &+\hat{\omega}_1(z)\big(\omega_2(z)a'(z)\big)_+ - \hat{\omega}_1(z)\big(\hat{\omega}_2(z)\hat{a}'(z)\big)_-\big]_{\leq 2n-1}\Big).\end{aligned} \tag{2.47}$$

For this multiplication there exists a unity

$$\boldsymbol{e}^* := \left(\frac{z^{-2m+1}}{2m}, 0\right) = \frac{1}{2m}dh^1. \tag{2.48}$$

Proposition 2.7 *The cotangent space $T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$ is a Frobenius algebra with multiplication defined by* (2.45)*, unity $\boldsymbol{e}^*$, and non-degenerate invariant bilinear form* (2.8).

By virtue of the definitions of the bijection η in (2.9) and the flat metric (2.17), we have

Corollary 2.8 *The tangent space $T_{\boldsymbol{a}}\mathcal{M}_{m,n}$ is a Frobenius algebra such that*

(i) The multiplication between vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ is defined by

$$\boldsymbol{\xi}_1\cdot\boldsymbol{\xi}_2 := \eta\big(\eta^{-1}(\boldsymbol{\xi}_1)\cdot\eta^{-1}(\boldsymbol{\xi}_2)\big); \tag{2.49}$$

(ii) The unity vector is

$$\boldsymbol{e} := \eta\big(\boldsymbol{e}^*\big) = \big(1, 1\big) = \frac{\partial}{\partial h^m}; \tag{2.50}$$

(iii) The invariant inner product is given by (2.19).

2.3 The potential and the Euler vector field

We have introduced a flat metric $<\,,\,>$ and a Frobenius algebra structure on the tangent bundle of the manifold $\mathcal{M}_{m,n}$. Now let us compute the 3-tensor

$$c(\partial_1, \partial_2, \partial_3) = <\partial_1 \cdot \partial_2, \partial_3>, \quad \partial_1, \partial_2, \partial_3 \in T\mathcal{M}_{m,n}.$$

Lemma 2.9 *For $i_1, i_2, i_3 \in \mathbb{Z}$, $j_1, j_2, j_3 \in \{1, \ldots, m\}$ and $k_1, k_2, k_3 \in \{1, \ldots, n\}$ it holds that*

$$\begin{aligned}
<\frac{\partial}{\partial t^{i_1}} \cdot \frac{\partial}{\partial t^{i_2}}, \frac{\partial}{\partial t^{i_3}}> = &-\frac{1}{2\pi\mathbb{i}} \oint_{|z|=1} \frac{1}{4} w'(z) \Big[w(z)^{2i_1+2i_2-1} \Pi(w(z)^{2i_3} w'(z)) \\
&+ w(z)^{2i_1+2i_3-1} \Pi(w(z)^{2i_2} w'(z)) + w(z)^{2i_2+2i_3-1} \Pi(w(z)^{2i_1} w'(z)) \\
&- w(z)^{2(i_1+i_2+i_3-1)} \Pi(w(z) w'(z)) \Big] dz \\
&- \frac{1}{2\pi\mathbb{i}} \oint_{|z|=1} \frac{1}{4} l'(z) w(z)^{2(i_1+i_2+i_3-1)} w'(z) dz
\end{aligned} \tag{2.51}$$

with $\Pi(f(z)) = f(z)_+ - f(z)_-$, and

$$<\frac{\partial}{\partial t^{i_1}} \cdot \frac{\partial}{\partial t^{i_2}}, \frac{\partial}{\partial h^{j_3}}> = -\frac{1}{2\pi\mathbb{i}} \oint_{|z|=1} \frac{1}{2} \big(w(z)^{2i_1+2i_2-1} w'(z)\big)_- \big(\chi(z)^{2m-2j_3} \chi'(z)\big)_+ dz, \tag{2.52}$$

$$<\frac{\partial}{\partial t^{i_1}} \cdot \frac{\partial}{\partial t^{i_2}}, \frac{\partial}{\partial \hat{h}^{k_3}}> = \frac{1}{2\pi\mathbb{i}} \oint_{|z|=1} \frac{1}{2} \big(w(z)^{2i_1+2i_2-1} w'(z)\big)_+ \big(\hat{\chi}(z)^{2n-2k_3} \hat{\chi}'(z)\big)_- dz, \tag{2.53}$$

$$<\frac{\partial}{\partial h^{j_1}} \cdot \frac{\partial}{\partial h^{j_2}}, \frac{\partial}{\partial t^{i_3}}> = -\frac{1}{2\pi\mathbb{i}} \oint_{|z|=1} \frac{1}{2m} \big(\chi(z)^{2m-2j_1-2j_2+1} \chi'(z)\big)_+ \big(w(z)^{2i_3} w'(z)\big)_- dz, \tag{2.54}$$

$$<\frac{\partial}{\partial h^{j_1}} \cdot \frac{\partial}{\partial h^{j_2}}, \frac{\partial}{\partial \hat{h}^{k_3}}> = -\frac{1}{2\pi\mathbb{i}} \oint_{|z|=1} \frac{1}{2m} \big(\chi(z)^{2m-2j_1-2j_2+1} \chi'(z)\big)_+ \big(\hat{\chi}(z)^{2n-2k_3} \hat{\chi}'(z)\big)_- dz, \tag{2.55}$$

$$<\frac{\partial}{\partial \hat{h}^{k_1}} \cdot \frac{\partial}{\partial \hat{h}^{k_2}}, \frac{\partial}{\partial t^{i_3}}> = -\frac{1}{2\pi\mathbb{i}} \oint_{|z|=1} \frac{1}{2n} \big(\hat{\chi}(z)^{2n-2k_1-2k_2+1} \hat{\chi}'(z)\big)_- \big(w(z)^{2i_3} w'(z)\big)_+ dz, \tag{2.56}$$

$$<\frac{\partial}{\partial \hat{h}^{k_1}} \cdot \frac{\partial}{\partial \hat{h}^{k_2}}, \frac{\partial}{\partial h^{j_3}}> = -\frac{1}{2\pi\mathbb{i}} \oint_{|z|=1} \frac{1}{2n} \big(\hat{\chi}(z)^{2n-2k_1-2k_2+1} \hat{\chi}'(z)\big)_- \big(\chi(z)^{2m-2j_3} \chi'(z)\big)_+ dz, \tag{2.57}$$

$$\begin{aligned}
<\frac{\partial}{\partial h^{j_1}} \cdot \frac{\partial}{\partial h^{j_2}}, \frac{\partial}{\partial h^{j_3}}> = &-\frac{1}{2\pi\mathbb{i}} \oint_{|z|=1} \frac{1}{2m^2} \big(w(z) w'(z)\big)_- \big(\chi(z)^{2m-2j_1-2j_2-2j_3+2} \chi'(z)\big)_+ dz \\
&- \mathrm{res}_{z=\infty} \frac{\big(\chi(z)^{2m-2j_1} \chi'(z)\big)_+ \big(\chi(z)^{2m-2j_2} \chi'(z)\big)_+ \big(\chi(z)^{2m-2j_3} \chi'(z)\big)_+}{l'(z)} dz,
\end{aligned} \tag{2.58}$$

$$< \frac{\partial}{\partial \hat{h}^{k_1}} \cdot \frac{\partial}{\partial \hat{h}^{k_2}}, \frac{\partial}{\partial \hat{h}^{k_3}} >= -\frac{1}{2\pi \mathbb{i}} \oint_{|z|=1} \frac{1}{2n^2} \big(w(z)w'(z)\big)_+ \big(\hat{\chi}(z)^{2n-2k_1-2k_2-2k_3+2}\hat{\chi}'(z)\big)_- dz$$
$$+ \operatorname{res}_{z=0} \frac{\big(\hat{\chi}(z)^{2m-2k_1}\hat{\chi}'(z)\big)_- \big(\hat{\chi}(z)^{2m-2k_2}\hat{\chi}'(z)\big)_- \big(\hat{\chi}(z)^{2m-2k_3}\hat{\chi}'(z)\big)_-}{l'(z)} dz, \tag{2.59}$$

$$< \frac{\partial}{\partial t^{i_1}} \cdot \frac{\partial}{\partial h^{j_2}}, \frac{\partial}{\partial \hat{h}^{k_3}} >= 0. \tag{2.60}$$

Proof Denote $\eta_{uv} =< \partial_u, \partial_v >$ for $u, v \in \boldsymbol{t} \cup \boldsymbol{h} \cup \hat{\boldsymbol{h}}$, then we have

$$< \partial_u \cdot \partial_v, \partial_w >= \eta_{u\tilde{u}}\eta_{v\tilde{v}}\eta_{w\tilde{w}} < d\tilde{u} \cdot d\tilde{v}, d\tilde{w} >^* = \eta_{u\tilde{u}}\eta_{v\tilde{v}}\eta_{w\tilde{w}} \langle d\tilde{u} \cdot d\tilde{v}, \eta(d\tilde{w})\rangle. \tag{2.61}$$

Substitute the formulae (2.42)–(2.44) into the right hand side and use (2.47) and (2.10), then the lemma follows from a lengthy but straightforward computation. □

Lemma 2.10 *There exists a function $F_{m,n}$ depending rationally on the variables $\boldsymbol{h} \cup \hat{\boldsymbol{h}}$ such that*

$$\frac{\partial^3 F_{m,n}}{\partial u\, \partial v\, \partial w} = -\big(\operatorname{res}_{z=\infty} + \operatorname{res}_{z=0}\big) \frac{\partial_u l(z) \cdot \partial_v l(z) \cdot \partial_w l(z)}{l'(z)} dz \tag{2.62}$$

for any $u, v, w \in \boldsymbol{h} \cup \hat{\boldsymbol{h}}$.

This lemma will be proved in the Appendix. Now we have the following result.

Proposition 2.11 *On $\mathcal{M}_{m,n}$ the following function*

$$\mathcal{F}_{m,n} = \left(\frac{1}{2\pi \mathbb{i}}\right)^2 \oint \oint_{|z_1|<|z_2|} \left(\frac{1}{2}\zeta(z_1)\zeta(z_2) - \zeta(z_1)l(z_2) + l(z_1)\zeta(z_2)\right) \times$$
$$\times \log(z_2 - z_1) dz_1 dz_2 + F_{m,n} \tag{2.63}$$

satisfies

$$c(\partial_u, \partial_v, \partial_w) = \frac{\partial^3 \mathcal{F}_{m,n}}{\partial u\, \partial v\, \partial w}, \quad u, v, w \in \boldsymbol{t} \cup \boldsymbol{h} \cup \hat{\boldsymbol{h}}. \tag{2.64}$$

This implies that $\nabla_s c(\partial_u, \partial_v, \partial_w)$ is a symmetric 4-tensor, where ∇ stands for the Levi-Civita connection of the metric $< \,,\, >$.

Proof The proposition can be checked directly with the help of the formulae (2.39)–(2.41) and by integration by parts. For example,

$$\frac{\partial^3 \mathcal{F}_{m,n}}{\partial t^{i_1} \partial t^{i_2} \partial t^{i_3}}$$
$$= \left(\frac{1}{2\pi \mathbb{i}}\right)^2 \oint \oint_{|z_1|<|z_2|} \frac{\partial^2}{\partial t^{i_1} \partial t^{i_2}} \Big[-\frac{1}{2} w'(z_1) w(z_1)^{2i_3} \zeta(z_2) - \frac{1}{2} \zeta(z_1) w'(z_2) w(z_2)^{2i_3}$$

$$
\begin{aligned}
&+ w'(z_1)w(z_1)^{2i_3}l(z_2) - l(z_1)w'(z_2)w(z_2)^{2i_3}\Big] \log(z_2 - z_1)dz_1dz_2 \\
=&\Big(\frac{1}{2\pi\mathbb{i}}\Big)^2 \oint\oint_{|z_1|<|z_2|} \frac{\partial}{\partial t^{i_1}}\Big[\frac{w(z_1)^{2i_3+2i_2-1}w'(z_1)\zeta(z_2)}{4} + \frac{w(z_1)^{2i_3+1}w'(z_2)w(z_2)^{2i_2}}{2(2i_3+1)} \\
&- \frac{w'(z_1)w(z_1)^{2i_2}w(z_2)^{2i_3+1}}{2(2i_3+1)} - \frac{\zeta(z_1)w'(z_2)w(z_2)^{2i_3+2i_2-1}}{4} \\
&+ \frac{w'(z_1)w(z_1)^{2i_3+2i_2-1}l(z_2)}{2} - \frac{l(z_1)w(z_2)^{2i_3+2i_2-1}w'(z_2)}{2}\Big]\frac{dz_1dz_2}{z_2 - z_1} \\
=&\Big(\frac{1}{2\pi\mathbb{i}}\Big)^2 \oint\oint_{|z_1|<|z_2|} \frac{1}{4}\Big[w(z_1)^{2i_1+2i_2+2i_3-1}w'(z_1)w(z_2)w'(z_2) \\
&- w(z_1)^{2i_2+2i_3-1}w'(z_1)w(z_2)^{2i_1}w'(z_2) - w(z_1)^{2i_1+2i_3-1}w'(z_1)w(z_2)^{2i_2}w'(z_2) \\
&+ w(z_1)^{2i_3}w'(z_1)w(z_2)^{2i_1+2i_2-1}w'(z_2) - w(z_1)^{2i_1}w'(z_1)w(z_2)^{2i_2+2i_3-1}w'(z_2) \\
&+ w(z_1)^{2i_2}w'(z_1)w(z_2)^{2i_1+2i_3-1}w'(z_2) + w(z_1)^{2i_1}w'(z_1)w(z_2)^{2i_2+2i_3-1}w'(z_2) \\
&- w(z_1)w'(z_1)w(z_2)^{2i_1+2i_2+2i_3-1}w'(z_2) - w(z_1)^{2i_1+2i_2+2i_3-1}w'(z_1)l'(z_2) \\
&- l'(z_1)w(z_2)^{2i_1+2i_2+2i_3-1}w'(z_2)\Big]\frac{dz_1dz_2}{z_2 - z_1} \\
=&< \frac{\partial}{\partial t^{i_1}} \cdot \frac{\partial}{\partial t^{i_2}}, \frac{\partial}{\partial t^{i_3}} >, \qquad (2.65)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial^3\mathcal{F}_{m,n}}{\partial h^{j_1}\partial h^{j_2}\partial \hat{h}^{k_3}} \\
=&\Big(\frac{1}{2\pi\mathbb{i}}\Big)^2 \oint\oint_{|z_1|<|z_2|} \frac{\partial^2}{\partial h^{j_1}\partial h^{j_2}}\Big[\zeta(z_1)\big(\hat{\chi}(z_2)^{2n-2k_3}\hat{\chi}'(z_2)\big)_- \\
&- \big(\hat{\chi}(z_1)^{2n-2k_3}\hat{\chi}'(z_1)\big)_-\zeta(z_2)\Big]\log(z_2 - z_1)dz_1dz_2 \\
&- \big(\mathrm{res}_{z=\infty} + \mathrm{res}_{z=0}\big)\frac{\partial_{h^{j_1}}l(z)\cdot\partial_{h^{j_2}}l(z)\cdot\partial_{\hat{h}^{k_3}}l(z)}{l'(z)}dz \\
=&-\mathrm{res}_{z=\infty}\frac{\big(\chi(z)^{2n-2j_1-2j_2+1}\chi'(z)\big)_+\big(\hat{\chi}(z)^{2n-2k_3}\hat{\chi}'(z)\big)_-}{2m}dz \\
=&< \frac{\partial}{\partial h^{j_1}} \cdot \frac{\partial}{\partial h^{j_2}}, \frac{\partial}{\partial \hat{h}^{k_3}} > . \qquad (2.66)
\end{aligned}
$$

The other cases are similar. The proposition is proved. □

To show that $\mathcal{M}_{m,n}$ is a Frobenius manifold, we still need to fix a Euler vector field and show the quasi-homogeneity of $\mathcal{F}_{m,n}$.

We assign a degree to each of the variables as

$$
\deg\, t^i = \frac{m(1-2i)+1}{2m}, \quad \deg\, h^j = \frac{j}{m}, \quad \deg\, \hat{h}^k = \frac{2k-1}{2n} + \frac{1}{2m}, \qquad (2.67)
$$

and let

$$
\mathcal{E}_{m,n} = \sum_{i\in\mathbb{Z}} \frac{m(1-2i)+1}{2m}t^i\frac{\partial}{\partial t^i} + \sum_{j=1}^{m}\frac{j}{m}h^j\frac{\partial}{\partial h^j} + \sum_{k=1}^{n}\Big(\frac{2k-1}{2n} + \frac{1}{2m}\Big)\hat{h}^k\frac{\partial}{\partial \hat{h}^k}. \qquad (2.68)
$$

Lemma 2.12 *The function $\mathcal{F}_{m,n}$ is quasi-homogeneous with respect to the degrees* (2.67)*. More precisely, we have (cf.* (1.5)*)*

$$Lie_{\mathcal{E}_{m,n}}\mathcal{F}_{m,n} = (3-d_m)\mathcal{F}_{m,n}, \quad d_m = 1-\frac{1}{m}. \tag{2.69}$$

Proof In addition to (2.67) we assume deg $z = 1/2m$. Note that both functions $\zeta(z)$ and $l(z)$ are homogeneous of degree 1, hence $\mathcal{F}_{m,n}$ has degree $2+1/m$. The lemma is proved. □

Up to now, we have shown that $\mathcal{M}_{m,n}$ is a Frobenius manifold, on which the multiplication is defined by (2.49) with unity $\boldsymbol{e} = \partial/\partial h^m$ and invariant flat metric (2.19), the Euler vector field is $\mathcal{E}_{m,n}$, and the potential is $\mathcal{F}_{m,n}$.

2.4 Semisimplicity

The final step in proving Theorem 1.2 is to show the semisimplicity of $\mathcal{M}_{m,n}$.

Let

$$d\mu(z) = \frac{da(z)}{a'(z)} - \frac{d\hat{a}(z)}{\hat{a}'(z)}, \quad z\in S^1. \tag{2.70}$$

This is a generating function for a basis of the cotangent space $T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$ by taking the Riemann-Hilbert decomposition with respect to S^1. According to (2.8) and (2.45), it is easy to show the following lemma.

Lemma 2.13 *The following formulae hold true:*

$$< d\mu(p), d\mu(q) >^* = -\frac{\zeta'(p)}{a'(p)\hat{a}'(p)}\delta_0(p-q), \tag{2.71}$$

$$d\mu(p)\cdot d\mu(q) = \delta_0(p-q)\, d\mu(p), \tag{2.72}$$

where $\delta_0(p-q) = \sum_{k\in\mathbb{Z}}\left(p^{2k}/q^{2k+1}\right)$ satisfies

$$\frac{1}{2\pi\mathbb{i}}\oint_{|q|=1} f(q)\delta_0(p-q)\, dq = \begin{cases} f(p), & f(-q)=f(q); \\ 0, & f(-q)=-f(q). \end{cases} \tag{2.73}$$

The formula (2.72) implies that no nilpotent elements exist in the Frobenius algebra $T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$, hence the Frobenius algebra $T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$ is semisimple and so is $T_{\boldsymbol{a}}\mathcal{M}_{m,n}$. Thus we arrive at the following result.

Proposition 2.14 *The Frobenius manifold $\mathcal{M}_{m,n}$ is semisimple.*

Proof of Theorem 1.2 The theorem follows from a combination of Lemmas 2.4 and 2.12, Corollary 2.8, Propositions 2.11 and 2.14. □

We want to find canonical coordinates on a subset $\mathcal{M}^s_{m,n} \subset \mathcal{M}_{m,n}$ consisting of generic points $(a(z), \hat{a}(z))$ that satisfy the following conditions

(S1) For $z \in S^1$,

$$a'(z)\hat{a}''(z) - \hat{a}'(z)a''(z) \neq 0; \tag{2.74}$$

(S2) The function $\sigma(z) = \big(a'(z)/\zeta'(z)\big)^{1/2}$ is holomorphic and injective on S^1 such that a smooth simple closed curve is defined by

$$\Sigma := \left\{ \sigma = \left(\frac{a'(z)}{\zeta'(z)}\right)^{\frac{1}{2}} \mid z \in S^1 \right\}. \tag{2.75}$$

Denote the inverse function of $\sigma(z)$ by $z = z(\sigma) : \Sigma \to S^1$, then

$$\big[\sigma^2 \hat{a}'(z) + (1-\sigma^2)a'(z)\big]_{z=z(\sigma)} = 0, \quad \sigma \in \Sigma. \tag{2.76}$$

Let

$$u_\sigma := \big[\sigma^2 \hat{a}(z) + (1-\sigma^2)a(z)\big]_{z=z(\sigma)}, \quad \sigma \in \Sigma. \tag{2.77}$$

One sees that the exact 1-form du_σ is just a normalization of the generating function $d\mu(z)$, more exactly,

$$du_\sigma = \left(\frac{a'(z)}{\zeta'(z)} d\hat{a}(z) - \frac{\hat{a}'(z)}{\zeta'(z)} da(z)\right)_{z=z(\sigma)} = \left(-\frac{a'(z)\hat{a}'(z)}{\zeta'(z)} d\mu(z)\right)_{z=z(\sigma)}. \tag{2.78}$$

Thus from Lemma 2.13 it follows that u_σ is a canonical coordinate on $\mathcal{M}^s_{m,n}$ (in the sense of [3]). In fact, it can be checked that $\mathcal{E}_{m,n}(u_\sigma) = u_\sigma$ by using the formulae (2.82) below.

2.5 The intersection form

Define the intersection form on the cotangent space $T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$ as (cf. (1.6))

$$\big(d\alpha(p), d\beta(q)\big)^* := i_{\mathcal{E}_{m,n}}\big(d\alpha(p)\cdot d\beta(q)\big), \quad \alpha, \beta \in \big\{a, \hat{a}\big\}. \tag{2.79}$$

Lemma 2.15 *For $\alpha, \beta \in \big\{a, \hat{a}\big\}$ it holds that*

$$\big(d\alpha(p), d\beta(q)\big)^* = \frac{q\beta'(q)}{q^2 - p^2}\alpha(p) + \frac{p\alpha'(p)}{p^2 - q^2}\beta(q). \tag{2.80}$$

Proof In consideration of the degree of coordinates in (2.67), one can express the Euler vector field (2.68) in the coordinates $\{v_i\}_{i=m}^{-\infty} \cup \{\hat{v}_j\}_{j=-n}^{+\infty}$ as follows:

$$\mathcal{E}_{m,n} = \sum_{i\le m} \frac{m+1-i}{m} v_i \frac{\partial}{\partial v_i} + \sum_{j \ge -n} \frac{m-j}{m} \hat{v}_j \frac{\partial}{\partial \hat{v}_j}. \tag{2.81}$$

Actually this formula can also be checked by letting (2.81) act on the flat coordinates (1.15)–(1.17).

It is easy to see

$$\mathcal{E}_{m,n}(\alpha(z)) = \alpha(z) - \frac{z}{2m}\alpha'(z), \quad \alpha, \beta \in \{a, \hat{a}\}. \tag{2.82}$$

Substituting (2.45) into (2.79) we have

$$\begin{aligned}
\big(d\alpha(p), d\beta(q)\big)^* =& \left\langle \mathcal{E}_{m,n}, \frac{q\beta'(q)}{q^2-p^2}da(p) + \frac{pa'(p)}{p^2-q^2}d\beta(q) \right\rangle \\
=& \frac{q\beta'(q))}{q^2-p^2}\Big(\alpha(p) - \frac{p}{2m}\alpha'(p)\Big) + \frac{p\alpha'(p)}{p^2-q^2}\Big(\beta(q) - \frac{q}{2m}\beta'(q)\Big) \\
=& \frac{q\beta'(q)}{q^2-p^2}\alpha(p) + \frac{pa'(p)}{p^2-q^2}\beta(q).
\end{aligned}$$

The lemma is proved. □

Similar to the case of the flat metric (2.8), the intersection form (2.79) induces another metric on the tangent space $T_{\boldsymbol{a}}\mathcal{M}_{m,n}$.

In fact, there is a linear map

$$g: \ T^*_{\boldsymbol{a}}\mathcal{M}_{m,n} \to T_{\boldsymbol{a}}\mathcal{M}_{m,n} \tag{2.83}$$

such that

$$\langle \boldsymbol{\omega}_1, g(\boldsymbol{\omega}_2)\rangle = (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)^*$$

for any $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$. Given any $\boldsymbol{\omega} = \big(\omega(z), \hat{\omega}(z)\big) \in T^*_{\boldsymbol{a}}\mathcal{M}_{m,n}$, the formulae (cf. (2.11) and (2.10))

$$g(d\alpha(p)) = \Big(\big(d\alpha(p), da(z)\big)^*, \big(d\alpha(p), d\hat{a}(z)\big)^*\Big), \quad \alpha \in \{a, \hat{a}\} \tag{2.84}$$

lead to

$$\begin{aligned}
g(\boldsymbol{\omega})(z) =& \Big(a'(z)\big(a(z)\omega(z) + \hat{a}(z)\hat{\omega}(z)\big)_- - a(z)\big(a'(z)\omega(z) + \hat{a}'(z)\hat{\omega}(z)\big)_-, \\
& - \hat{a}'(z)\big(a(z)\omega(z) + \hat{a}(z)\hat{\omega}(z)\big)_+ + \hat{a}(z)\big(a'(z)\omega(z) + \hat{a}'(z)\hat{\omega}(z)\big)_+\Big).
\end{aligned} \tag{2.85}$$

This shows that the map g is surjective. On the other hand, suppose $g(\boldsymbol{\omega}) = \big(\xi(z), \hat{\xi}(z)\big) \in T_{\boldsymbol{a}}\mathcal{M}_{m,n}$, then

$$\omega(z) = \Big[\frac{1}{a(z)}\Big(\frac{\hat{a}(z)\xi(z) - a(z)\hat{\xi}(z)}{a(z)\hat{a}'(z) - a'(z)\hat{a}(z)}\Big)_+\Big]_{\geq -2m+1}, \tag{2.86}$$

$$\hat{\omega}(z) = -\Big[\frac{1}{\hat{a}(z)}\Big(\frac{\hat{a}(z)\xi(z) - a(z)\hat{\xi}(z)}{a(z)\hat{a}'(z) - a'(z)\hat{a}(z)}\Big)_-\Big]_{\leq 2n-1}. \tag{2.87}$$

Thus g is injective. Therefore, the linear map g defined in (2.83) is a bijection.

Now we define a bilinear form on $T_{\boldsymbol{a}}\mathcal{M}_{m,n}$ as

$$\big(\partial_1, \partial_2\big) := \langle g^{-1}(\partial_1), \partial_2\rangle = \big(g^{-1}(\partial_1), g^{-1}(\partial_2)\big)^* \tag{2.88}$$

Proposition 2.16 *The Frobenius manifold $\mathcal{M}_{m,n}$ has an intersection form* (2.79)*, which induces a flat metric as*

$$\big(\partial_1, \partial_2\big) = -\frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{\partial_1 \log\big(a(z)/\hat{a}(z)\big) \cdot \partial_2 \log\big(a(z)/\hat{a}(z)\big)}{\partial_z \log\big(a(z)/\hat{a}(z)\big)} dz \tag{2.89}$$

with arbitrary vectors $\partial_1, \partial_2 \in T_{\boldsymbol{a}}\mathcal{M}_{m,n}$.

Proof The equality (2.89) follows from the formulae (2.86)–(2.88). The flatness of this metric can be verified with the same method as used in Lemma 2.4. □

3 Principal two-component BKP hierarchy

We are to write down the principal hierarchy for the Frobenius manifold $\mathcal{M}_{m,n}$. First let us consider the bi-Hamiltonian structure associated to this Frobenius manifold, i.e., two compatible Hamiltonian structures corresponding to the flat metrics (2.8) and (2.79).

Let $\mathcal{L}\mathcal{M}_{m,n}$ be the loop space of smooth maps from S^1 to the manifold $\mathcal{M}_{m,n}$. In this space a point is expressed as $\boldsymbol{a} = \big(a(z,x), \hat{a}(z,x)\big)$ in (1.10) with the coefficients being smooth functions of $x \in S^1$. The tangent space and cotangent space at any point of $\mathcal{L}\mathcal{M}_{m,n}$ are composed of smooth maps from S^1 to $z^{2m-2}\mathcal{H}^- \times z^{-2n}\mathcal{H}^+$ and to $z^{-2m+1}\mathcal{H}^+ \times z^{2n-1}\mathcal{H}^-$ respectively. The pairing between a cotangent vector $\boldsymbol{\omega} = \big(\omega(z,x), \hat{\omega}(z,x)\big)$ and a tangent vector $\boldsymbol{\xi} = \big(\xi(z,x), \hat{\xi}(z,x)\big)$ is (cf. (1.12))

$$\langle \boldsymbol{\omega}, \boldsymbol{\xi} \rangle = \frac{1}{2\pi \mathrm{i}} \oint_{S^1} \oint_{|z|=1} \big[\omega(z,x)\xi(z,x) + \hat{\omega}(z,x)\hat{\xi}(z,x)\big] dz\, dx. \tag{3.1}$$

Without confusion we will simply write $(\alpha(z,x), \hat{\alpha}(z,x))$ as $(\alpha(z), \hat{\alpha}(z))$ below.

There uniquely exist

$$\lambda(z) = z + \cdots = a(z)^{\frac{1}{2m}} \text{ near } \infty, \quad \hat{\lambda}(z) = \frac{\hat{h}^1}{2n} z^{-1} + \cdots = \hat{a}(z)^{\frac{1}{2n}} \text{ near } 0. \tag{3.2}$$

On the loop space $\mathcal{L}\mathcal{M}_{m,n}$ a hierarchy of evolutionary equations is defined as

$$\frac{\partial \lambda(z)}{\partial s_k} = \{(\lambda(z)^k)_+, \lambda(z)\}, \quad \frac{\partial \hat{\lambda}(z)}{\partial s_k} = \{(\lambda(z)^k)_+, \hat{\lambda}(z)\}, \tag{3.3}$$

$$\frac{\partial \lambda(z)}{\partial \hat{s}_k} = \{-(\hat{\lambda}(z)^k)_-, \lambda(z)\}, \quad \frac{\partial \hat{\lambda}(z)}{\partial \hat{s}_k} = \{-(\hat{\lambda}(z)^k)_-, \hat{\lambda}(z)\}, \tag{3.4}$$

where $k \in \mathbb{Z}_+^{odd}$ and the Lie bracket $\big\{f, g\big\} := \partial f/\partial z \cdot \partial g/\partial x - \partial g/\partial z \cdot \partial f/\partial x$. This hierarchy is called the dispersionless two-component BKP hierarchy.

Remark 3.1 *The system of Lax equations* (3.3), (3.4) *was first written down by Takasaki [25] as the universal hierarchy underlying the D-type topological Landau-Ginzburg models. The name presently used is after the bilinear equations (whose dispersionless limit is equivalent to* (3.3), (3.4)*) introduced by Date, Jimbo, Kashiwarw and Miwa [6] considering the neutral free-fermion realization of the Lie algebra with infinite Dynkin diagram D_∞, see also [20, 21, 22, 24].*

A map $\mathcal{P}$ from the cotangent space to the tangent space of $\mathcal{LM}_{m,n}$ is a Poisson structure if it defines a Poisson bracket between local functionals on $\mathcal{LM}_{m,n}$ as

$$\big\{F, H\big\}_{\mathcal{P}} = \langle dF, \mathcal{P}(dH)\rangle. \tag{3.5}$$

Here dF denotes the gradient of the functional F, namely, the covector on $\mathcal{LM}_{m,n}$ determined by $\delta F = \langle \delta \boldsymbol{a}, dF\rangle$; and dH means the same.

Proposition 3.2 ([27]) *The space $\mathcal{LM}_{m,n}$ carries the following two compatible Poisson structures:*

$$\begin{aligned}
&\mathcal{P}_1(\omega(z), \hat\omega(z)) \\
&=\Big(-(\{a(z), \omega(z)\} + \{\hat a(z), \hat\omega(z)\})_- + \{a(z), (\omega(z) + \hat\omega(z))_-\}, \\
&\qquad (\{a(z), \omega(z)\} + \{\hat a(z), \hat\omega(z)\})_+ - \{\hat a(z), (\omega(z) + \hat\omega(z))_+\}\Big),
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\mathcal{P}_2(\omega(z), \hat\omega(z)) \\
&=\Big(-a(z)(\{a(z), \omega(z)\} + \{\hat a(z), \hat\omega(z)\})_- + \{a(z), (a(z)\omega(z) + \hat a(z)\hat\omega(z))_-\}, \\
&\qquad \hat a(z)(\{a(z), \omega(z)\} + \{\hat a(z), \hat\omega(z)\})_+ - \{\hat a(z), (a(z)\omega(z) + \hat a(z)\hat\omega(z))_+\}\Big).
\end{aligned} \tag{3.7}$$

They give Poisson brackets $\{\ ,\ \}_{1,2}$ that represent the dispersionless two-component BKP hierarchy (3.3), (3.4) *to a bi-Hamiltonian form as*

$$\frac{\partial F}{\partial s_k} = \{F, H_{k+2m}\}_1 = \{F, H_k\}_2, \quad \frac{\partial F}{\partial \hat s_k} = \{F, \hat H_{k+2n}\}_1 = \{F, \hat H_k\}_2 \tag{3.8}$$

with $k \in \mathbb{Z}_+^{\text{odd}}$ and

$$H_k = \frac{2m}{k}\frac{1}{2\pi\mathrm{i}}\oint_{S^1}\oint_{|z|=1} \lambda(z)^k dz\, dx,\ \ \hat H_k = \frac{2n}{k}\frac{1}{2\pi\mathrm{i}}\oint_{S^1}\oint_{|z|=1} \hat\lambda(z)^k dz\, dx. \tag{3.9}$$

Introduce two generating functions for local functionals as

$$a(p, y) = p^{2m} + \sum_{i \le m} v_i(y) p^{2i-2}, \quad \hat a(p, y) = \sum_{j \ge -n} \hat v_j(y) p^{2j}. \tag{3.10}$$

Their gradients are (cf. (2.4), (2.5))

$$da(p, y) = \Big(\frac{p^{2m}}{z^{2m-1}(p^2 - z^2)}\delta(x - y), 0\Big), \quad |z| < |p|; \tag{3.11}$$

$$d\hat{a}(p,y) = \Big(0, \frac{z^{2n+1}}{p^{2n}(z^2-p^2)}\delta(x-y)\Big), \quad |z|>|p|. \tag{3.12}$$

With the help of these notations, the Poisson brackets (3.6) and (3.7) can be expressed in the form

$$\begin{aligned}
\{\alpha(p,x),\beta(q,y)\}_1 =& \Big[\frac{q\partial_q\beta(q,x)}{q^2-p^2} + \frac{p\partial_p\alpha(p,x)}{p^2-q^2}\Big]\delta'(x-y) \\
&+ \Big[\frac{p^2+q^2}{(p^2-q^2)^2}\big(\partial_x\alpha(p,x) - \partial_x\beta(q,x)\big)\Big]\delta(x-y), \qquad (3.13)\\
\{\alpha(p,x),\beta(q,y)\}_2 =& \Big[\frac{q\partial_q\beta(q,x)}{q^2-p^2}\alpha(p,x) + \frac{p\partial_p\alpha(p,x)}{p^2-q^2}\beta(q,x)\Big]\delta'(x-y) \\
&+ \Big[\frac{p^2+q^2}{(p^2-q^2)^2}\big(\partial_x\alpha(p,x)\cdot\beta(q,x) - \alpha(p,x)\partial_x\beta(q,x)\big) \\
&+ \frac{p\partial_p\alpha(p,x)\cdot\partial_x\beta(q,x) - q\alpha(p,x)\partial_q\partial_x\beta(q,x)}{p^2-q^2}\Big]\delta(x-y), \qquad (3.14)
\end{aligned}$$

where $\alpha, \beta \in \{a, \hat{a}\}$. These Poisson brackets are of hydrodynamic type.

Comparing the coefficients of $\delta'(x-y)$ in (3.13), (3.14) with the metrics (2.8) and (2.80) on the Frobenius manifold $\mathcal{M}_{m,n}$, we have the following result.

Proposition 3.3 *The bi-Hamiltonian structure associated to the Frobenius manifold $\mathcal{M}_{m,n}$ is given by* (3.13) *and* (3.14) *for the dispersionless two-component BKP hierarchy.*

To obtain the principal hierarchy associated to $\mathcal{M}_{m,n}$, let us introduce the Hamiltonians

$$\mathcal{H}_{u,p-1} = \int \theta_{u,p}\, dx, \quad p \geq 0 \tag{3.15}$$

with

$$\theta_{u,p} = \begin{cases} \dfrac{1}{2i+1}\dfrac{1}{(2p)!!}\dfrac{1}{2\pi\mathbb{i}}\displaystyle\oint_{|z|=1}\zeta(z)^{\frac{2i+1}{2}}\varphi(z)^p\, dz, & u = t^i\ (i\in\mathbb{Z}); \\[2ex] -\dfrac{\Gamma\left(\frac{2m-2j+1}{2m}\right)}{2m\,\Gamma\left(p+1+\frac{2m-2j+1}{2m}\right)}\mathrm{res}_{z=\infty}a(z)^{p+\frac{2m-2j+1}{2m}}\, dz, & u = h^j\ (1\le j\le m); \\[2ex] \dfrac{\Gamma\left(\frac{2n-2k+1}{2n}\right)}{2n\,\Gamma\left(p+1+\frac{2n-2k+1}{2n}\right)}\mathrm{res}_{z=0}\hat{a}(z)^{p+\frac{2n-2k+1}{2n}}\, dz, & u = \hat{h}^k\ (1\le k\le n). \end{cases} \tag{3.16}$$

Here $\zeta(z) = a(z) - \hat{a}(z)$ and $\varphi(z) = a(z) + \hat{a}(z)$.

Theorem 3.4 *The principal hierarchy associated to the Frobenius manifold $\mathcal{M}_{m,n}$ consists of the following bi-Hamiltonian flows*

$$\frac{\partial F}{\partial T^{u,p}} = \left\{F, \mathcal{H}_{u,p}\right\}_1 = \left(p + \frac{1}{2} + \mu_u\right)^{-1} \left\{F, \mathcal{H}_{u,p-1}\right\}_2, \quad p \geq 0, \tag{3.17}$$

where

$$\mu_u = \begin{cases} i, & u = t^i \ (i \in \mathbb{Z}); \\ \dfrac{m - 2j + 1}{2m}, & u = h^j \ (1 \leq j \leq m); \\ \dfrac{n - 2k + 1}{2n}, & u = \hat{h}^k \ (1 \leq k \leq n). \end{cases} \tag{3.18}$$

Proof In comparison with (1.15)–(1.17), we have

$$\theta_{t^i,0} = -\frac{1}{2} t^{-i}, \quad \theta_{h^j,0} = \frac{1}{2m} h^{m+1-j}, \quad \theta_{\hat{h}^k,0} = \frac{1}{2n} \hat{h}^{n+1-k},$$

which are densities of Casimirs for the first Poisson bracket (cf. (1.8)). According to the theory of Frobenius manifold and principal hierarchy [10, 16], we only need to check the recursion relation (3.17).

Take $u = t^i$ for example. It is easy to compute the gradients at point $(a(z), \hat{a}(z))$:

$$d\mathcal{H}_{t^i,p-1} = \frac{1}{(2i+1)\,(2p)!!} \left(\left(\frac{2i+1}{2} \zeta(z)^{\frac{2i-1}{2}} \varphi(z)^p + p\, \zeta(z)^{\frac{2i+1}{2}} \varphi(z)^{p-1} \right)_{\geq -2m+1}, \right.$$
$$\left. \left(-\frac{2i+1}{2} \zeta(z)^{\frac{2i-1}{2}} \varphi(z)^p + p\, \zeta(z)^{\frac{2i+1}{2}} \varphi(z)^{p-1} \right)_{\leq 2n-1} \right).$$

Substituting them into (3.6) and (3.7), we have

$$\mathcal{P}_1(d\mathcal{H}_{t^i,p}) = \left(\left\{a(z), (A_{t^i,p}(z))_-\right\}, \left\{(A_{t^i,p}(z))_+, \hat{a}(z)\right\} \right), \tag{3.19}$$

$$\mathcal{P}_2(d\mathcal{H}_{t^i,p-1}) = \left(p + i + \frac{1}{2}\right) \left(\left\{a(z), (A_{t^i,p}(z))_-\right\}, \left\{(A_{t^i,p}(z))_+, \hat{a}(z)\right\} \right) \tag{3.20}$$

with

$$A_{t^i,p}(z) = \frac{1}{2i+1} \frac{1}{(2p)!!} \zeta(z)^{\frac{2i+1}{2}} \varphi(z)^p.$$

The other cases are similar. The theorem is proved. □

Proof of Theorem 1.3 This is a corollary of Theorem 3.4. □

Remark 3.5 *To obtain the primary flows $\partial/\partial T^{u,0}$ of the principal hierarchy, one can also use the formula* (1.7) *and the* 3*-point functions computed in Section 2.3. The result coincides with that in theorem 1.3.*

Observe that the principal hierarchies with different m and n have their common part as the dispersionless two-component BKP hierarchy. In fact, comparing (1.21) with (3.3) and (3.4) one has

$$T^{h^j,p} = \frac{2m\,\Gamma\left(p+1+\frac{2m-2j+1}{2m}\right)}{\Gamma\left(\frac{2m-2j+1}{2m}\right)} s_{2mp+2m-2j+1}, \quad 1 \le j \le m;$$
$$T^{\hat{h}^k,p} = \frac{2n\,\Gamma\left(p+1+\frac{2n-2k+1}{2n}\right)}{\Gamma\left(\frac{2n-2k+1}{2n}\right)} \hat{s}_{2np+2n-2k+1}, \quad 1 \le k \le n.$$

Definition 3.6 *The hierarchy* (1.21) *(or equivalently* (3.17)*) is called the* $(2m, 2n)$*-principal two-component BKP hierarchy.*

We end this section with two remarks on the principal two-component BKP hierarchy.

Remark 3.7 *Recall the canonical coordinates u_σ in Section 2.4. The principal hierarchy* (1.21) *can also be represented in the following linear form:*

$$\frac{\partial u_\sigma}{\partial T^{u,p}} = \mathcal{A}_{u,p}(z)\big|_{z=z(\sigma):\,\Sigma\to S^1} \frac{\partial u_\sigma}{\partial x}, \tag{3.21}$$

where

$$\mathcal{A}_{u,p}(z) = \begin{cases} \left(A_{u,p}(z)\left(\frac{2i+1}{2\zeta(z)} + \frac{p}{\varphi(z)}\right)\frac{\partial a(z)}{\partial z}\right)_+ \\ \quad + \left(A_{u,p}(z)\left(\frac{2i+1}{2\zeta(z)} - \frac{p}{\varphi(z)}\right)\frac{\partial \hat{a}(z)}{\partial z}\right)_-, & u = t^i \ (i \in \mathbb{Z}); \\ \left(\frac{\partial A_{u,p}(z)}{\partial z}\right)_+, & u = h^j \ (1 \le j \le m); \\ -\left(\frac{\partial A_{u,p}(z)}{\partial z}\right)_-, & u = \hat{h}^k \ (1 \le k \le n) \end{cases}$$

with $A_{u,p}$ given in (1.22).

Remark 3.8 *In [5] Chen and Tu rewrote the dispersionless two-component BKP hierarchy to a system of dispersionless Hirota equations of a free energy $\mathcal{F}$. Given any solution $\mathcal{F}$ of such Hirota equations, one introduces*

$$\mathcal{F}_{i,j,k} = \partial_{s_i}\partial_{s_j}\partial_{s_k}\mathcal{F}, \quad i,j,k \in \mathbb{Z}^{odd}$$

with $s_{-k} = \hat{s}_k$ for $k \in \mathbb{Z}^{odd}_+$, and defines structure constants $C^l_{i,j}$ such that

$$\mathcal{F}_{i,j,k} = \sum_{l\in\mathbb{Z}^{odd}} C^l_{i,j}\mathcal{F}_{k,l,1}, \quad i,j,k \in \mathbb{Z}^{odd}.$$

Then $\mathcal{F}$ satisfies the following WDVV equations [5]

$$\sum_{l\in\mathbb{Z}^{odd}} C_{i,j}^{l}\mathcal{F}_{l,k,h} = \sum_{l\in\mathbb{Z}^{odd}} C_{i,k}^{l}\mathcal{F}_{l,j,h}, \quad i,j,k,h\in\mathbb{Z}^{odd}. \tag{3.22}$$

Observe that the associativity equations (3.22) *are different from those for the Frobenius manifolds $\mathcal{M}_{m,n}$. However, if we restrict the potential $\mathcal{F}_{m,n}$ of $\mathcal{M}_{m,n}$ to*

$$\tilde{\mathcal{F}}_{m,n} = \mathcal{F}_{m,n}|_{T^{t^i,p}=0,\ T^{h^j,p}=\delta_{p,0}h^j,\ T^{\hat{h}^k,p}=\delta_{p,0}\hat{h}^k} \tag{3.23}$$

and let $m,n\to\infty$, then the limit of $\tilde{\mathcal{F}}_{m,n}$ coincides with $\mathcal{F}$ up to a linear transformation of the time variables.

4 Finite-dimensional Frobenius submanifolds

The manifold $\mathcal{M}_{m,n}$ can be represented as $\mathcal{M}_{m,n}^0\times M_{m,n}$. Here $\mathcal{M}_{m,n}^0$ and $M_{m,n}$ are submanifolds spanned by the flat coordinates $\{t^i \mid i\in\mathbb{Z}\}$ and $\{h^j\}_{j=1}^m\cup\{\hat{h}^k\}_{k=1}^n$ respectively. In other words, they are characterized by the functions $\zeta(z)$ and $l(z)$ given in (1.13) respectively.

Let us consider the $(m+n)$-dimensional submanifold $M_{m,n}$. For convenience its coordinates are redenoted as (recall (1.16) and (1.17))

$$w^\alpha = \begin{cases} h^{m+1-\alpha}, & 1\le\alpha\le m; \\ \hat{h}^{m+n+1-\alpha}, & m+1\le\alpha\le m+n. \end{cases} \tag{4.1}$$

The projection $\mathcal{M}_{m,n}\to M_{m,n}$ induces a Frobenius structure on $M_{m,n}$. After some straightforward calculation the following result is obtained (cf. Lemma 4.5 in [10]).

Proposition 4.1 *The submanifold $M_{m,n}$ is a semisimple Frobenius manifold with potential $F_{m,n}$ given in* (2.62). *Let ∂_1, ∂_2 and ∂_3 denote arbitrary tangent vector fields on this manifold.*

(i) The invariant inner product (flat metric) $<\ ,\ >$ and the 3-tensor c are

$$<\partial_1,\partial_2> = \mathrm{res}_{l'(z)=0}\frac{\partial_1 l(z)\cdot\partial_2 l(z)}{l'(z)}dz, \tag{4.2}$$

$$c(\partial_1,\partial_2,\partial_3) = \mathrm{res}_{l'(z)=0}\frac{\partial_1 l(z)\cdot\partial_2 l(z)\cdot\partial_3 l(z)}{l'(z)}dz. \tag{4.3}$$

A system of flat coordinates for the metric $<\ ,\ >$ is given by $\{w^\alpha\}$.

(ii) The unity vector field is $e=\partial/\partial w^1$.

(iii) The Euler vector field is

$$E_{m,n} = \sum_{\alpha=1}^{m} \frac{m-\alpha+1}{m} w^\alpha \frac{\partial}{\partial w^\alpha} + \sum_{\alpha=m+1}^{m+n} \left(\frac{2(m+n-\alpha)+1}{2n} + \frac{1}{2m} \right) w^\alpha \frac{\partial}{\partial w^\alpha}. \tag{4.4}$$

(iv) Suppose $l'(z) = 0$ has $2(m+n)$ pairwise distinct roots $\pm z_1, \pm z_2, \ldots, \pm z_{m+n} \in \mathbb{C}$. Then the canonical coordinates defined by

$$u_i = l(z_i) = l(-z_i), \quad i = 1, 2, \ldots, m+n \tag{4.5}$$

satisfy

$$\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}, \quad < \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} >= \delta_{ij} \frac{2}{l''(z_i)}. \tag{4.6}$$

(iv) The intersection form reads

$$(\partial_1, \partial_2) = \mathrm{res}_{l'(z)=0} \frac{\partial_1 \log l(z) \cdot \partial_2 \log l(z)}{\partial_z \log l(z)} dz \tag{4.7}$$

Observe that the Frobenius manifold $M_{m,n}$ coincides with the one defined on the orbit space of the Coxeter group B_{m+n} by Bertola (see Appendix B in [1]) starting from a superpotential of the form $l(z)$. In particular, the manifold $\mathcal{M}_{m,1}$ is defined on the orbit space $\mathbb{C}^{m+1}/D_{m+1}$ of the Coxeter group D_{m+1}. The Frobenius manifolds $M_{m,n}$ were reconstructed by Zuo [29], whose method is based on the polynomial contravariant metric of the orbit space induced from the Euclidean metric (see for example Lemma 4.1 in [10]) and different choices of unity vector field.

As in [1, 29], one sees that in general the potential $F_{m,n}$ for $M_{m,n}$ is a polynomial in $w^1, w^2, \ldots, w^{m+n}, 1/w^{m+n}$, and is a polynomial in $w^1, w^2, \ldots, w^{m+n}$ if and only if $n = 1$. Following are the potentials for small m and n.

Example 4.2 *For $n = 1$,*

$$F_{1,1} = w^1 \left(\frac{(w^1)^2}{12} + \frac{(w^2)^2}{4} \right), \tag{4.8}$$

$$F_{2,1} = w^1 \left(\frac{w^1 w^2}{8} + \frac{(w^3)^2}{4} \right) + \frac{(w^2)^5}{3840} + \frac{(w^2)^2 (w^3)^2}{32}, \tag{4.9}$$

$$F_{3,1} = w^1 \left(\frac{w^1 w^3 + (w^2)^2}{12} + \frac{(w^4)^2}{4} \right) + \frac{(w^2)^2 (w^3)^3}{1296} - \frac{(w^2)^3 w^3}{216} + \frac{(w^3)^7}{1632960} + \left(\frac{w^2 w^3}{24} + \frac{(w^3)^3}{432} \right) (w^4)^2. \tag{4.10}$$

For $n > 1$,

$$F_{1,2} = w^1 \left(\frac{(w^1)^2}{12} + \frac{w^2 w^3}{4} \right) + \frac{(w^3)^4}{768} + \frac{(w^2)^3}{6\, w^3}, \tag{4.11}$$

$$F_{2,2} = w^1\left(\frac{w^1w^2}{8}+\frac{w^3w^4}{4}\right)+\frac{(w^2)^5}{3840}+\frac{(w^2)^2w^3w^4}{32}+\frac{w^2(w^4)^4}{768}+\frac{(w^3)^3}{6\,w^4}, \tag{4.12}$$

$$F_{1,3} = w^1\left(\frac{(w^1)^2}{12}+\frac{w^2w^4}{6}+\frac{(w^3)^2}{12}\right)+\frac{w^3(w^4)^3}{648}+\frac{(w^2)^2w^3}{2\,w^4}-\frac{w^2(w^3)^3}{3\,(w^4)^2}+\frac{(w^3)^5}{10\,(w^4)^3}. \tag{4.13}$$

In particular, the potential $F_{m,1}$ coincides with the partition function that solves the D_{m+1}-model in topological field theory [17] up to a rescaling of the flat coordinates.

The projection from $\mathcal{M}_{m,n}$ to its submanifold $M_{m,n}$ can be realized by setting $a(z) = \hat{a}(z)$ so that $\zeta(z) = a(z) - \hat{a}(z)$ vanishes. Such a constraint corresponds to the $(2\,m, 2\,n)$-reduction of the associated bi-Hamiltonian structure (3.6), (3.7), see [27] for details. Hence by using Theorem 1.3 we have the following result.

Proposition 4.3 *The principal hierarchy associated to $M_{m,n}$ is the $(2\,m, 2\,n)$-reduction of the dispersionless two-component BKP hierarchy, i.e.,*

$$\frac{\partial l(z)}{\partial T^{\alpha,p}} = \{A_{\alpha,p}(z),\, l(z)\}, \quad 1 \le \alpha \le m+n \text{ and } p \ge 0, \tag{4.14}$$

where

$$A_{\alpha,p}(z) = \begin{cases} \dfrac{\Gamma\left(\frac{2\alpha-1}{2m}\right)}{2\,m\,\Gamma\left(p+1+\frac{2\alpha-1}{2m}\right)}\left(l(z)^{p+\frac{2\alpha-1}{2m}}\right)_+, & 1 \le \alpha \le m; \\ \\ -\dfrac{\Gamma\left(\frac{2(\alpha-m)-1}{2n}\right)}{2\,n\,\Gamma\left(p+1+\frac{2(\alpha-m)-1}{2n}\right)}\left(l(z)^{p+\frac{2(\alpha-m)-1}{2n}}\right)_-, & m+1 \le \alpha \le m+n \end{cases}$$

with $l(z)^{1/2m} = z + O(z^{-1})$ as $z \to \infty$ and $l(z)^{1/2n} = w^{m+n}\, z^{-1}/2n + O(z)$ as $z \to 0$.

Particularly when $n = 1$, the hierarchy (4.14) is the dispersionless limit of the Drinfeld-Sokolov hierarchy associated to Kac-Moody algebra $D_{m+1}^{(1)}$ and the vertex c_0 of its Dynkin diagram, see [9, 22] for details.

Therefore, Proposition 1.4 is proved.

If we constrain $l(z)$ by $l(z)_- = 0$, then the flows $\partial/\partial T^{\alpha,p}$ in (4.14) with $1 \le \alpha \le m$ are well defined, and they compose the dispersionless limit of the Drinfeld-Sokolov hierarchy of type $(B_m^{(1)}, c_0)$. This observation coincides with the fact that the polynomial potential for the Frobenius manifold on the orbit space of Coxeter group B_m is obtained by omitting the terms containing w^α with $\alpha > m$ in $F_{m,n}$.

Remark 4.4 *It is easy to see that the infinite-dimensional submanifold $\mathcal{M}^0_{m,n}$ of $\mathcal{M}_{m,n}$ fulfills all conditions to define a Frobenius manifold except the existence of a unity vector field. A manifold of this kind is called a* quasi-Frobenius manifold *following [16] (see Section 3.2 there). Such manifolds will be studied in a follow-up work.*

5 Conclusion and outlook

For every pair of positive integers m and n, we have constructed an infinite-dimensional semisimple Frobenius manifold $\mathcal{M}_{m,n}$ on the space that consists of pairs of meromorphic even functions $\big(a(z), \hat{a}(z)\big)$ satisfying the conditions (M1)-(M3). Such Frobenius manifolds have rich geometric properties described in a similar way as in the finite-dimensional case.

The bi-Hamiltonian structure (3.13), (3.14) of hydrodynamic type associated to the Frobenius manifold $\mathcal{M}_{m,n}$ is just the one for the dispersionless two-component BKP hierarchy. Based on this fact, we have obtained the Lax representation (1.21) of the principal hierarchy for $\mathcal{M}_{m,n}$, which is a wide extension of the dispersionless two-component BKP hierarchy. Note that the definition of the complementary flows $\partial/\partial T^{t^i,p}$ relies on the analyticity property of the functions $a(z)$ and $\hat{a}(z)$. How to deform these flows, such as from $\partial/\partial T^{h^j,p}$ and $\partial/\partial T^{\hat{h}^k,p}$ to the full flows of the two-component BKP hierarchy, has not been considered. From another point of view, a tau function of the principal hierarchy can be defined by using the tau-symmetric Hamiltonian densities $\theta_{\alpha,p}$ in (3.16). Whether the tau function has a topological deformation, which might be governed by an analogue of the universal loop equation in [16], will be studied in subsequent publications.

We have shown that $\mathcal{M}_{m,n}$ contains an $(m+n)$-dimensional Frobenius submanifold $M_{m,n}$, which coincides with the semisimple Frobenius manifold defined on the orbit space of the Coxeter group B_{m+n} (or D_{m+1} when $n = 1$). Moreover, the principal hierarchy for $M_{m,n}$ is shown to be the $(2\,m, 2\,n)$-reduction of the dispersionless two-component BKP hierarchy. In particular, when $n = 1$, the principal hierarchy is just the dispersionless limit of Drinfeld-Sokolov hierarchy of type $(D^{(1)}_{m+1}, c_0)$. It is interesting to look for infinite-dimensional Frobenius manifolds that contain Frobenius submanifolds defined on the orbit space of Coxeter groups besides types B and D, and relate them to integrable systems such as Drinfeld-Sokolov hierarchies.

Finally, we hope that the present result would be helpful to achieving a normal form for infinite-dimensional Frobenius manifolds underlying $2+1$ integrable systems. For instance, the technique here probably helps to generalize the result in [3] such that there is also a class of Frobenius manifolds for the dispersionless 2D Toda hierarchy, and they possess submanifolds of finite dimension associated with the extended bigraded Toda hierarchy [4, 2].

Acknowledgments. The authors are grateful to Guido Carlet, Boris Dubrovin, Si-Qi Liu and Youjin Zhang for helpful discussions. The first named author acknowledges the support of the Young SISSA Scientist Grant "FIMA.645". The second named author is partially supported by the National Basic Research Program of China (973 Program) No.2007CB814800, the NSFC No.10801084 and 11071135; he is also supported by the State Scholarship Fund of China No.2010621203 and would like to express thanks for the hospitality of SISSA where part of the work was done.

Appendix: Proof of Lemma 2.10

Proof of Lemma 2.10 For any $u, v, w, s \in \boldsymbol{h} \cup \hat{\boldsymbol{h}}$, let

$$C_{uvw} = -\big(\mathrm{res}_\infty + \mathrm{res}_0\big)\frac{\partial_u l(z)\cdot\partial_v l(z)\cdot\partial_w l(z)}{l'(z)}dz, \tag{A.1}$$

then

$$\begin{aligned}\partial_s C_{uvw} = &-\Big(\mathrm{res}_\infty + \mathrm{res}_0\Big)\frac{\partial_s\partial_u l(z)\cdot\partial_v l(z)\cdot\partial_w l(z) + \text{c.p.}(u,v,w)}{l'(z)}dz\\ &+\Big(\mathrm{res}_\infty + \mathrm{res}_0\Big)\frac{\partial_u l(z)\cdot\partial_v l(z)\cdot\partial_w l(z)\cdot\partial_z\partial_s l(z)}{l'(z)^2}dz.\end{aligned} \tag{A.2}$$

Here c.p. denotes the cyclic permutation. To show the lemma, it only needs to check that (A.2) is symmetric with respect to u, v, w and s. We will employ the formulae (2.37), (2.38) and (2.40) to do this.

First, let us compute $(2m)^2\partial_{h^{j_4}}C_{h^{j_1}h^{j_2}h^{j_3}}$. By using (2.37) and (2.40) we have

$$\frac{\partial l(z)}{\partial h^j} = \Big(\chi(z)^{2m-2j}\chi'(z)\Big)_+,\quad \frac{\partial^2 l(z)}{\partial h^{j_1}\partial h^{j_2}} = \frac{1}{2m}\partial_z\Big(\chi(z)^{2m-2j_1-2j_2+1}\chi'(z)\Big)_+. \tag{A.3}$$

In this case the first term on the right hand side of (A.2) yields

$$\begin{aligned}&-(2m)^2\big(\mathrm{res}_\infty + \mathrm{res}_0\big)\frac{\partial_{h^{j_4}}\partial_{h^{j_1}}l(z)\cdot\partial_{h^{j_2}}l(z)\cdot\partial_{h^{j_3}}l(z)}{l'(z)}dz\\
=&-\mathrm{res}_\infty\frac{\partial_z\big(\chi^{2m-2j_1-2j_4+1}\chi'\big)_+\cdot\big(\chi^{2m-2j_2}\chi'\big)_+\big(\chi^{2m-2j_3}\chi'\big)_+}{\chi^{2m-1}\chi'}dz\\
=&-\mathrm{res}_\infty\frac{\partial_z\big(\chi^{2m-2j_1-2j_4+1}\chi'\big)_+}{\chi^{2m-1}\chi'}\Big(\chi^{4m-2j_2-2j_3}(\chi')^2\\
&\quad-\big(\chi^{2m-2j_2}\chi'\big)_\oplus\big(\chi^{2m-2j_3}\chi'\big)_- - \big(\chi^{2m-2j_2}\chi'\big)_-\big(\chi^{2m-2j_3}\chi'\big)_\oplus\Big)_\oplus dz\\
=&-\mathrm{res}_\infty\partial_z\big(\chi^{2m-2j_1-2j_4+1}\chi'\big)_+\cdot\chi^{2m-2j_2-2j_3+1}\chi'dz\\
&-\mathrm{res}_\infty\partial_z\big(\chi^{2m-2j_1-2j_4+1}\chi'\big)_\oplus\cdot\Big(-\chi^{-2j_2+1}\big(\chi^{2m-2j_3}\chi'\big)_- - \big(\chi^{2m-2j_2}\chi'\big)_-\chi^{-2j_3+1}\Big)dz\\
=&-\mathrm{res}_\infty\partial_z\big(\chi^{2m-2j_1-2j_4+1}\chi'\big)_+\chi^{2m-2j_2-2j_3+1}\chi'dz\\
&-\mathrm{res}_\infty\Big(\chi^{2m-2j_1-2j_2-2j_4+2}\chi'\partial_z\big(\chi^{2m-2j_3}\chi'\big)_- + \chi^{2m-2j_1-2j_3-2j_4+2}\chi'\partial_z\big(\chi^{2m-2j_2}\chi'\big)_-\Big)dz\\
&-\mathrm{res}_\infty\Big((-2j_2+1)\chi^{2m-2j_1-2j_2-2j_4+2}(\chi')^2\big(\chi^{2m-2j_3}\chi'\big)_-\\
&\quad+(-2j_3+1)\chi^{2m-2j_1-2j_3-2j_4+2}(\chi')^2\big(\chi^{2m-2j_2}\chi'\big)_-\Big)dz.\end{aligned} \tag{A.4}$$

Here and below the subscript "$\oplus$" means that the operation of taking the nonnegative part of the series can be omitted. On the right hand side of (A.4) the first term can be rewritten to

$$\mathrm{res}_\infty\chi^{2m-2j_1-2j_4+1}\chi'\partial_z\big(\chi^{2m-2j_2-2j_3+1}\chi'\big)_-dz$$

$$=\mathrm{res}_\infty \chi^{2m-2j_1-2j_4+1}\chi'\partial_z\big(\chi^{2m-2j_2-2j_3+1}\chi' - \big(\chi^{2m-2j_2-2j_3+1}\chi'\big)_+\big)dz,$$

hence

$$\begin{aligned}
&-\mathrm{res}_\infty \partial_z\big(\chi^{2m-2j_1-2j_4+1}\chi'\big)_+\chi^{2m-2j_2-2j_3+1}\chi' \\
=&\frac{1}{2}\mathrm{res}_\infty\Big((2m-2j_2-2j_3+1)\chi^{2m-2j_1-2j_2-2j_3-2j_4+1}(\chi')^3 \\
&\quad+\chi^{2m-2j_1-2j_2-2j_3-2j_4+2}\chi'\chi''\Big)dz \\
&-\frac{1}{2}\mathrm{res}_\infty\Big(\partial_z\big(\chi^{2m-2j_1-2j_4+1}\chi'\big)_+\cdot\chi^{2m-2j_2-2j_3+1}\chi' \\
&\quad+\chi^{2m-2j_1-2j_4+1}\chi'\partial_z\big(\chi^{2m-2j_2-2j_3+1}\chi'\big)_+\Big)dz. \qquad \text{(A.5)}
\end{aligned}$$

On the other hand, $(2m)^2$ times of the last term of (A.2) equals

$$\begin{aligned}
&\mathrm{res}_\infty\frac{\big(\chi^{2m-2j_1}\chi'\big)_+\big(\chi^{2m-2j_2}\chi'\big)_+\big(\chi^{2m-2j_3}\chi'\big)_+\partial_z\big(\chi^{2m-2j_4}\chi'\big)_+}{(\chi^{2m-1}\chi')^2}dz \\
=&\mathrm{res}_\infty\frac{\big(\chi^{2m-2j_3}\chi'\big)_+\partial_z\big(\chi^{2m-2j_4}\chi'\big)_+}{(\chi^{2m-1}\chi')^2}\Big(\chi^{4m-2j_1-2j_2}(\chi')^2 \\
&\quad-\big(\chi^{2m-2j_1}\chi'\big)_\oplus\big(\chi^{2m-2j_2}\chi'\big)_- - \big(\chi^{2m-2j_1}\chi'\big)_-\big(\chi^{2m-2j_2}\chi'\big)_\oplus\Big)_\oplus dz \\
=&\mathrm{res}_\infty\chi^{-2j_1-2j_2+2}\big(\chi^{2m-2j_3}\chi'\big)_+\partial_z\big(\chi^{2m-2j_4}\chi'\big)_+dz \\
&-\mathrm{res}_\infty\frac{\big(\chi^{2m-2j_3}\chi'\big)_\oplus\partial_z\big(\chi^{2m-2j_4}\chi'\big)_+}{\chi^{2m-1}\chi'}\Big(\chi^{-2j_1+1}\big(\chi^{2m-2j_2}\chi'\big)_- \\
&\quad+\big(\chi^{2m-2j_1}\chi'\big)_-\chi^{-2j_2+1}\Big)dz \\
=&\mathrm{res}_\infty\chi^{-2j_1-2j_2+2}\Big(\chi^{2m-2j_3}\chi'\partial_z\big(\chi^{2m-2j_4}\chi'\big) \\
&\quad-\big(\chi^{2m-2j_3}\chi'\big)_\oplus\partial_z\big(\chi^{2m-2j_4}\chi'\big)_- - \big(\chi^{2m-2j_3}\chi'\big)_-\partial_z\big(\chi^{2m-2j_4}\chi'\big)_\oplus\Big)_\oplus dz \\
&+\mathrm{res}_\infty\partial_z\Big(\chi^{-2j_1-2j_3+2}\big(\chi^{2m-2j_2}\chi'\big)_- + \chi^{-2j_2-2j_3+2}\big(\chi^{2m-2j_1}\chi'\big)_-\Big)\cdot\chi^{2m-2j_4}\chi'dz \\
=&\mathrm{res}_\infty\Big((2m-2j_4)\chi^{4m-2j_1-2j_2-2j_3-2j_4+1}(\chi')^3+\chi^{4m-2j_1-2j_2-2j_3-2j_4+2}\chi'\chi''\Big)dz \\
&-\mathrm{res}_\infty\chi^{2m-2j_1-2j_2-2j_3+2}\chi'\partial_z\big(\chi^{2m-2j_4}\chi'\big)_-dz \\
&+\mathrm{res}_\infty\Big(\chi^{2m-2j_2-2j_3-2j_4+2}\chi'\partial_z\big(\chi^{2m-2j_1}\chi'\big)_- + \text{c.p.}(j_1,j_2,j_3)\Big)dz \\
&+\mathrm{res}_\infty\Big((-2j_2-2j_3+2)\chi^{2m-2j_2-2j_3-2j_4+1}(\chi')^2\big(\chi^{2m-2j_1}\chi'\big)_- + \text{c.p.}(j_1,j_2,j_3)\Big)dz. \qquad \text{(A.6)}
\end{aligned}$$

Substituting (A.4)–(A.6) into (A.2) we obtain

$$\begin{aligned}
&(2m)^2\partial_{h^{j_4}}C_{h^{j_1}h^{j_2}h^{j_3}} \\
&=\mathrm{res}_\infty\Big(\Big(5m-2j_1-2j_2-2j_3-2j_4+\frac{3}{2}\Big)\chi^{4m-2j_1-2j_2-2j_3-2j_4+1}(\chi')^3
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{5}{2}\chi^{4m-2j_1-2j_2-2j_3-2j_4+2}\chi'\chi''\Big)dz \\
&- \mathrm{res}_\infty\Big(\chi^{2m-2j_2-2j_3-2j_4+2}\chi'\partial_z\big(\chi^{2m-2j_1}\chi'\big)_- + \text{c.p.}(j_1,j_2,j_3,j_4)\Big)dz \\
&- \frac{1}{2}\mathrm{res}_\infty\Big(\partial_z\big(\chi^{2m-2j_1-2j_4+1}\chi'\big)_+ \cdot \chi^{2m-2j_2-2j_3+1}\chi' \\
&\quad + \chi^{2m-2j_1-2j_4+1}\chi'\partial_z\big(\chi^{2m-2j_2-2j_3+1}\chi'\big)_+ + \text{c.p.}(j_1,j_2,j_3)\Big)dz, \qquad \text{(A.7)}
\end{aligned}
$$

which is indeed symmetric with respect to $j_1, j_2, j_3, j_4 \in \{1, 2, \ldots, m\}$.

For $u, v, w, s \in \hat{\boldsymbol{h}}$ it is almost the same. With similar method, the other cases are easy to check by virtue of the fact $\partial_{h^j}\partial_{\hat{h}^k} l(z) = 0$. Therefore Lemma 2.10 is proved.

□

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