Symmetric dispersionless 2D Toda hierarchy and double Hurwitz numbers

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Dispersionless 2D Toda hierarchy is equations on a function $F = F(t_0, \mathbf{t}, \bar{\mathbf{t}})$, that depend from a variable t_0 and two infinite sets of variables $\mathbf{t} = \{t_1, t_2, \ldots\}$, $\bar{\mathbf{t}} = \{\bar{t}_1, \bar{t}_2, \ldots\}$ The hierarchy is presented by equations on formal Laurent series from additional variables z and ξ

$$(z-\xi)\exp\bigl(D(z)D(\xi)F\bigr) = z\exp\bigl(-\partial_0 D(z)F\bigr) - \xi\exp\bigl(-\partial_0 D(\xi)F\bigr), \quad (1)$$

$$(\bar{z}-\bar{\xi})\exp\bigl(\bar{D}(\bar{z})\bar{D}(\bar{\xi})F\bigr) = \bar{z}\exp\bigl(-\partial_0 \bar{D}(\bar{z})F\bigr) - \bar{\xi}\exp\bigl(-\partial_0 \bar{D}(\bar{\xi})F\bigr), \quad (2)$$

$$1-\exp\bigl(-D(z)\bar{D}(\bar{\xi})F\bigr) = \frac{1}{z\bar{\xi}}\exp\bigl(\partial_0(\partial_0 + D(z) + \bar{D}(\bar{\xi}))F\bigr), \quad (3)$$

where

$$D(z) = \sum_{k\geq 1}\frac{z^{-k}}{k}\,\partial_k\,, \qquad \bar{D}(\bar{z}) = \sum_{k\geq 1}\frac{\bar{z}^{-k}}{k}\,\bar{\partial}_k\,.$$

$$\partial_k = \partial/\partial t_k,\ \bar{\partial}_k = \partial/\partial \bar{t}_k.$$

We consider *formal solutions* i.e. solutions in form of formal Taylor series

$$F(t_0, \mathbf{t}, \bar{\mathbf{t}}) =$$

$$\sum_{\substack{\mu_1 \geq \mu_2 \geq \dots \geq \mu_\ell > 0 \\ \bar{\mu}_1 \geq \bar{\mu}_2 \geq \dots \geq \bar{\mu}_{\bar{\ell}} > 0}} F(\mu_1, \mu_2, \dots, \mu_\ell | \bar{\mu}_1, \bar{\mu}_2, \dots, \bar{\mu}_{\bar{\ell}} | t_0) t_{\mu_1} t_{\mu_2} \dots t_{\mu_k} \bar{t}_{\bar{\mu}_1} \bar{t}_{\bar{\mu}_2} \dots \bar{t}_{\mu_{\bar{\ell}}}.$$

The indexes $\mu_1 \geq \mu_2 \geq \dots \geq \mu_\ell > 0$ and $\bar{\mu}_1 \geq \bar{\mu}_2 \geq \dots \geq \bar{\mu}_{\bar{\ell}} > 0$ form Yang diagrammes Δ and $\bar{\Delta}$.
Denote by $t_\Delta = t_{\mu_1} t_{\mu_2} \dots t_{\mu_k}$, and $t_{\bar{\Delta}} = \bar{t}_{\bar{\mu}_1} \bar{t}_{\bar{\mu}_2} \dots \bar{t}_{\mu_{\bar{\ell}}}$. Then

$$F(t_0, \mathbf{t}, \bar{\mathbf{t}}) = F(\emptyset|\emptyset|t_0) + \sum_{\Delta} F(\Delta|\emptyset|t_0) t_\Delta + \sum_{\bar{\Delta}} F(\emptyset|\bar{\Delta}|t_0) \bar{t}_{\bar{\Delta}}$$

$$+ \sum_{\Delta, \bar{\Delta}} F(\Delta|\bar{\Delta}|t_0)\, t_\Delta \bar{t}_{\bar{\Delta}},$$

We find all formal *symmetric solutions*, i.e solutions where $\partial_k F|_{t_0} = \bar{\partial}_k F|_{t_0} = 0$ for $k \neq 0$. Here and later $g|_{t_0} = g(t_0, \mathbf{t}, \bar{\mathbf{t}})|_{t_0} = g(t_0, 0, 0)$ means the restriction on t_0.

Theorem 1 Any formal symmetric solution of 2D dispersionless Toda hierarchy F is defined by his restriction on $F|_{t_0}$ and is equal

$$F = F|_{t_0} +$$

$$\sum_{|\Delta|=|\bar{\Delta}|} \sum_{\substack{s_1+\ldots+s_m=|\Delta| \\ r_1+\ldots+r_m=m+\ell(\Delta)+\ell(\bar{\Delta})-2}} N_{(\Delta|\bar{\Delta})} \begin{pmatrix} s_1 \ldots s_m \\ r_1 \ldots r_m \end{pmatrix} \partial_0^{r_1}(f^{s_1}) \ldots \partial_0^{r_m}(f^{s_m})\, t_\Delta t_{\bar{\Delta}}, \quad (4)$$

where $f(t_0) = \exp(\partial_0^2 F|_{t_0})$,

$|\Delta|$ is the order of Young diagram Δ and $\ell(\Delta)$ is the number of rows Δ.

Coefficients $N_{(\Delta|\bar{\Delta})} \begin{pmatrix} s_1 \ldots s_m \\ r_1 \ldots r_m \end{pmatrix}$ are found by some recursion relations.

I outline a scheme of proof.
We find, at first, relations on F, that follow from the equation (1)

$$(z-\xi)\exp\bigl(D(z)D(\xi)F\bigr) = z\exp\bigl(-\partial_0 D(z)F\bigr) - \xi\exp\bigl(-\partial_0 D(\xi)F\bigr),$$

The left part is

$$(z-\xi)e^{D(z)D(\xi)F} = (z-\xi)(1+(D(z)D(\xi)F)+\frac{1}{2}(D(z)D(\xi)F)^2+\cdots) =$$

$$(z-\xi)(1+z^{-1}\sum_{j=1}^{\infty}\frac{1}{j}\xi^{-j}\partial_1\partial_j F+\xi^{-1}\sum_{j=1}^{\infty}\frac{1}{j}z^{-j}\partial_1\partial_j F+z^{-2}\xi^{-2}h(z,\xi,t,\bar{t})) =$$

$$(z-\xi)+\sum_{j=1}^{\infty}\frac{1}{j}\xi^{-j}\partial_1\partial_j F-\sum_{j=1}^{\infty}\frac{1}{j}z^{-j}\partial_1\partial_j F+z^{-2}\xi^{-2}(z-\xi)h(z,\xi,t,\bar{t}).$$

The right part is a sum of a function of z and a function of ξ.

Thus $ze^{-\partial_0 D(z)F} = z - \sum_{j=1}^{\infty} \frac{1}{j} z^{-j} \partial_1 \partial_j F \quad$ and $h(z, \xi, t, \bar{t}) = 0$.

The equation $\sum_{j=1}^{\infty} \frac{1}{j} z^{-j} \partial_1 \partial_j F = z - ze^{-\partial_0 D(z)F}$ gives

$$\partial_1 \partial_j F = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m!} \sum_{\substack{k_1 + \cdots + k_m = j+1 \\ k_i > 0}} \frac{j}{k_1 \cdots k_m} \partial_0 \partial_{k_1} F \cdots \partial_0 \partial_{k_m} F.$$

The equation $0 = h(z, \xi, t, \bar{t}) = \sum_{ij} k_{ij}(t, \bar{t}) z^{-i} \xi^{-j}$ means $k_{ij}(t, \bar{t}) = 0$ This is equivalent to

$$\partial_i \partial_j F = \sum_{m=1}^{\infty} \sum_{\substack{s_1 + \cdots + s_m = i+j \\ s_i > 1}} P_{ij}(s_1, ..., s_m) \cdot \partial_1 \partial_{s_1 - 1} F \cdots \partial_1 \partial_{s_m - 1} F$$

Thus

$$\partial_i\partial_j F = \sum_{m=1}^{\infty} \sum_{\substack{s_1+\cdots+s_m=i+j \\ s_i>1}} T_{ij}(p_1,\dots,p_m)\cdot \partial_0\partial_{p_1}F\cdots\partial_0\partial_{p_m}F$$

and

$$\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}F =$$

$$\sum_{m=1}^{\infty} \sum_{\substack{s_1+\cdots+s_m=i_1+\cdots+i_k \\ \ell_1+\cdots+\ell_m=m+k-2 \\ s_j,\ell_j\geqslant 1}} T_{i_1\dots i_k}\binom{s_1\cdots s_m}{\ell_1\cdots\ell_m}\partial_0^{\ell_1}\partial_{s_1}F\cdots\partial_0^{\ell_m}\partial_{s_m}F.$$

Herewith the coefficients $T_{i_1\dots i_k}\binom{s_1\cdots s_m}{\ell_1\cdots\ell_m}$ are calculated during the proof.

In particulary, $\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}F|_{t_0}=0$ for all symmetric solutions.

If F is a symmetric solution , then

$$\exp(\partial_0(\partial_0 + D(z) + \bar{D}(\bar{\xi}))F)\Big|_{t_0} = \exp(\partial_0^2 F|_{t_0}) = \exp(F|''_{t_0}) = f(t_0).$$

Thus the equation (3)

$$1 - e^{-D(z)\bar{D}(\bar{\xi})F} = z^{-1}\bar{\xi}^{-1}e^{\partial_0(\partial_0 + D(z) + \bar{D}(\bar{\xi}))F}$$

is equivalent

$$-D(z)\bar{D}(\bar{\xi})F|_{t_0} = \ln(1 - z^{-1}\bar{\xi}^{-1}f(t_0)) = -\sum_{k=1}^{\infty} k^{-1}z^{-1}\bar{\xi}^{-1}f^k(t_0)$$

and therefore

$$\partial_i\bar{\partial}_j F|_{t_0} = \begin{cases} 0 & \text{for } i \neq j\,, \\ if^i(t_0) & \text{for } i = j\,. \end{cases}$$

This relation together with the relation

$$\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}F =$$

$$\sum_{m=1}^{\infty}\sum_{\substack{s_1+\cdots+s_m=i_1+\cdots+i_k\\ \ell_1+\cdots+\ell_m=m+k-2\\ s_j,\ell_j\geqslant 1}} T_{i_1\cdots i_k}\binom{s_1\cdots s_m}{\ell_1\cdots\ell_m}\partial_0^{\ell_1}\partial_{s_1}F\cdots\partial_0^{\ell_m}\partial_{s_m}F.$$

gives a representation

$$\partial_{i_1}\cdots\partial_{i_k}\bar{\partial}_{\bar{i}_1}\cdots\bar{\partial}_{\bar{i}_{\bar{k}}}F\Big|_{t_0} =$$

$$\sum_{m=1}^{\infty}\sum_{\substack{s_1+\ldots+s_m=i_1+\ldots+i_k=\bar{i}_1+\ldots+\bar{i}_{\bar{k}}\\ r_1+\cdots+r_m=m+k+\bar{k}-2}} \tilde{N}_{\binom{i_1\ \ldots\ i_k}{\bar{i}_1\ \ldots\ \bar{i}_{\bar{k}}}}\binom{s_1\ \ldots\ s_m}{r_1\ \ldots\ r_m}\partial_0^{r_1}f^{s_1}\ldots\partial_0^{r_m}f^{s_m},$$

This representation for $\partial_{i_1} \cdots \partial_{i_k} \bar{\partial}_{\bar{i}_1} \cdots \bar{\partial}_{\bar{i}_{\bar{k}}} F \Big|_{t_0}$ is equivalent of theorem 1, where

$$N_{(\Delta|\bar{\Delta})} \begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix} = \frac{1}{\sigma(\Delta)\sigma(\bar{\Delta})} \tilde{N}_{\left(\begin{smallmatrix} i_1 \dots i_k \\ \bar{i}_1 \dots \bar{i}_{\bar{k}} \end{smallmatrix}\right)} \begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix},$$

$$\Delta = [i_1 \dots i_k],\ \bar{\Delta} = [\bar{i}_1 \dots \bar{i}_{\bar{k}}]$$

and $\sigma([\mu_1 \dots \mu_k]) = m_1! \dots m_r!$ for $m_i = |\{j|\mu_j = i\}|$.

The proof gives also an algorithm of calculation of the constats

$$N_{(\Delta|\bar{\Delta})} \begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix}.$$

Example 1 If $\Delta = [\mu_1, \dots, \mu_l]$, $\bar{\Delta} = [\bar{\mu}_1, \dots, \bar{\mu}_{\bar{l}}]$ and $d = |\Delta| = |\bar{\Delta}|$, then

$$N_{(\Delta|\bar{\Delta})}\begin{pmatrix} d \\ \ell(\Delta) + \ell(\bar{\Delta}) - 2 \end{pmatrix} = \frac{\rho(\Delta)\rho(\bar{\Delta})}{d\sigma(\Delta)\sigma(\bar{\Delta})},$$

where $\rho(\Delta) = \mu_1 \cdots \mu_l$. For other cases $N_{(\Delta|\bar{\Delta})}\begin{pmatrix} s \\ r \end{pmatrix} = 0$

Example 2

$$\tilde{N}_{\left(\substack{i_1 i_2 \\ \bar{i}_1 \bar{i}_2}\right)}\begin{pmatrix} \bar{i}_1 \bar{i}_2 \\ 11 \end{pmatrix} = \tilde{N}_{\left(\substack{i_1 i_2 \\ \bar{i}_1 \bar{i}_2}\right)}\begin{pmatrix} \bar{i}_2 \bar{i}_1 \\ 11 \end{pmatrix} = -\frac{i_1 i_2}{2\sigma([i_1, i_2])\sigma([\bar{i}_1, \bar{i}_2])}\min\{i_1, i_2, \bar{i}_1, \bar{i}_2\})$$

. For other cases $\tilde{N}_{\left(\substack{i_1 i_2 \\ \bar{i}_1 \bar{i}_2}\right)}\begin{pmatrix} s_1 s_2 \\ r_1 r_2 \end{pmatrix} = 0.$

Double Hurwitz numbers

Consider a meromorphic function $\varphi : \Omega \to \overline{\mathbb{C}}$ of degree d on a connected surface Ω. The preimage $\varphi^{-1}(z)$ of $z \in \overline{\mathbb{C}}$ consists of $\ell(z) \leq d$ точек. Ramification ordering in these points form a Young diagram Δ_z of degree $|\Delta_z| = d$ with a number of rows $\ell(\Delta_z) = \ell(z)$. We call it *topological type* of the value z.
Topological type $[1, 1, \dots, 1]$ is called trivial. Values $z \in \overline{\mathbb{C}}$ of non-trivial topological type are called *critical values* of φ. There are only finite number of critical values. Values of type $\Delta = [2, 1, \dots, 1]$ re called *simple critical values*.

A **Hurwitz number** $H(\Delta_1, \dots, \Delta_k)$ of degree d is a weighted number of meromorphic functions of degree d, having topological types $\Delta_1, \dots, \Delta_k$ in ponts $z_1, \dots, z_k$ and have not critical values in other points. These numbers is don't depend from positions of z_i.

A Hurwitz number of degree d

$$H_{d,l}(\Delta|\bar{\Delta}) = H(\Delta, \bar{\Delta}, \underbrace{[2,1,\dots,1],\dots,[2,1,\dots,1]}_{l})$$

is called *double Hurwitz number*.
Consider a generating function for double Hurwitz numbers:

$$\Phi(\beta, Q, \mathbf{t}, \bar{\mathbf{t}}) = \sum_{l\geq 0} \frac{\beta^l}{l!} \sum_{d\geq 1} Q^d \sum_{|\Delta|=|\bar{\Delta}|=d} H_{d,l}(\Delta, \bar{\Delta}) \prod_{i=1}^{\ell(\Delta)} \mu_i t_{\mu_i} \prod_{i=1}^{\ell(\bar{\Delta})} \bar{\mu}_i \bar{t}_{\bar{\mu}_i} \,,$$

where $\Delta = [\mu_1, \dots, \mu_{\ell(\Delta)}]$, $\bar{\Delta} = [\bar{\mu}_1, \dots, \bar{\mu}_{\ell(\bar{\Delta})}]$.

Add a new variable $\hbar$ and put

$$\Phi(\hbar;\beta,Q,\mathbf{t},\bar{\mathbf{t}}):=\hbar^2\Phi(\hbar\beta,Q,\frac{\mathbf{t}}{\hbar},\frac{\bar{\mathbf{t}}}{\hbar})$$

. This function has a representation

$$\Phi(\hbar;\beta,Q,\mathbf{t},\bar{\mathbf{t}})=\sum_{g\geq 0}\hbar^{2g}\Phi_g(\beta,Q,\mathbf{t},\bar{\mathbf{t}})),$$

where

$$\Phi_g(\beta,Q,\mathbf{t},\bar{\mathbf{t}}))=$$

$$\sum_{d\geq 1}\sum_{|\Delta|=|\bar{\Delta}|=d}\frac{Q^d\,\beta^{\ell(\Delta)+\ell(\bar{\Delta})+2g-2}}{(\ell(\Delta)+\ell(\bar{\Delta})+2g-2)!}H_{d,\ell(\Delta)+\ell(\bar{\Delta})+2g-2}(\Delta,\bar{\Delta})\prod_{i=1}^{\ell(\Delta)}\mu_i t_{\mu_i}\prod_{i=1}^{\ell(\bar{\Delta})}\bar{\mu}_i\bar{t}_{\bar{\mu}}$$

is the part of the series $\Phi(\beta,Q,\mathbf{t},\bar{\mathbf{t}}))$, generated by meromorphic functions of genus g.

In particulary

$$\Phi_0 = \sum_{d\geq 1} \sum_{|\Delta|=|\bar\Delta|=d} \frac{Q^d\, H^0(\Delta,\bar\Delta)}{\beta^2(\ell(\Delta)+\ell(\bar\Delta)-2)!} \prod_{i=1}^{\ell(\Delta)} (\beta\mu_i t_{\mu_i}) \prod_{i=1}^{\ell(\bar\Delta)} (\beta\bar\mu_i \bar t_{\bar\mu_i})$$

where $H^0(\Delta,\bar\Delta) = H_{d,\ell(\Delta)+\ell(\bar\Delta)-2}(\Delta,\bar\Delta)$ is the number of meromorphic functions $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with topological type divisors $(\Delta,\bar\Delta)$.

It is follow from Okounkov theorem, the function

$$\tau_n(\mathbf{t},\bar{\mathbf{t}}) = e^{\frac{1}{12}\beta n(n+1)(2n+1)} Q^{\frac{1}{2}n(n+1)} \exp\Big(\Phi(\beta, e^{\beta(n+\frac{1}{2})}Q, \mathbf{t}, \bar{\mathbf{t}}\Big)$$

is an τ-function of 2D Toda hierarchy for any β and Q.
The logarithm of $\tau_n(\mathbf{t},\bar{\mathbf{t}})$ is a solution of 2D Toda hierarchy. The renormalisation $t_k \to t_k/\hbar, \quad \beta \to \hbar\beta$ and limit $\hbar \to 0$ change the solution of full 2D Toda hierarchy to solution of dispersionless 2D Toda hierarchy.

$$F(\beta, Q, t_0, \mathbf{t}, \bar{\mathbf{t}}) = \frac{\beta t_0^3}{6} + \frac{t_0^2}{2}\log Q + \Phi_0(\beta, Qe^{\beta t_0}, \mathbf{t}, \bar{\mathbf{t}}) = \frac{\beta t_0^3}{6} + \frac{t_0^2}{2}\log Q +$$

$$\sum_{d\geq 1}\sum_{|\Delta|=|\bar{\Delta}|=d} \frac{(Qe^{\beta t_0})^d H^0(\Delta,\bar{\Delta})}{\beta^2(\ell(\Delta)+\ell(\bar{\Delta})-2)!} \prod_{i=1}^{\ell(\Delta)}(\beta\mu_i t_{\mu_i}) \prod_{i=1}^{\ell(\bar{\Delta})}(\beta\bar{\mu}_i\bar{t}_{\bar{\mu}_i}) \qquad (5)$$

These are symmetric solution for any β and Q. Thus they coincide with the series from Theorem 1.

A comparison with Theorem 1 gives

Theorem 2 Double Hurwitz numbers of genus 0 are

$$H^0(\Delta|\bar{\Delta}) = \frac{(\ell(\Delta) + \ell(\bar{\Delta})-2)!}{\rho(\Delta)\,\rho(\bar{\Delta})} \sum s_1^{r_1} \dots s_m^{r_m} N_{(\Delta|\bar{\Delta})} \begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix}$$

where the sum is carried over all matrixces $\begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix}$ such that $s_1 + \dots + s_m = |\Delta|$ and $r_1 + \dots + r_m = m + \ell(\Delta) + \ell(\bar{\Delta}) - 2$.

Thus the algorithm of calculation of coefficients $N_{(\Delta|\bar{\Delta})} \begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix}$ generate an algorithm of calculation of double Hurwitz numbers of genus 0.

Our Examples 1 and 2 of calculations of the coefficients gives, in particular:

Corollary 1

The double Hurwitz numbers of polynomials are

$$H^0(\Delta|[n]) = \frac{(\ell(\Delta)-1)!}{\sigma(\Delta)}\, n^{\ell(\Delta)-2}$$

Corollary 2

The double Hurwitz numbers of simplest Laurent polynomials are

$$H^0([i_1,i_2]|[\bar{i}_1,\bar{i}_2]) = 2\frac{d-\min\{i_1,i_2,\bar{i}_1,\bar{i}_2\}}{(1+\delta_{i_1 i_2})(1+\delta_{\bar{i}_1\bar{i}_2})}$$

These formulas were at first found by integration on communication of moduli spaces of algebraic curves (S.Lando,D.Zvonkine 2007; I.Goulden, D.Jackson, Vakil 2005; S.Shadrin, M.Shapiro, A.Wainshtein 2008)