GROWTH PROBLEMS OF LAPLACIAN TYPE AND HURWITZ NUMBERS

A.Zabrodin (ITEP, NRU-HSE, Moscow)

Trieste, 17 September 2013

Classical problems of 2D complex analysis such as

- *Inverse potential problem*
- *Riemann mapping problem*
- *Dirichlet boundary value problem*

have <u>integrable structure</u> which is

2D Toda lattice hierarchy in the zero dispersion limit for connected domains or, more generally, Whitham hierarchy

<u>Physical applications</u>: Hele-Shaw flows (Laplacian growth)

<u>Mathematical applications</u>: Hurwitz numbers

Some references

M.Mineev-Weinstein, P.Wiegmann and A.Zabrodin, *Phys. Rev. Lett. 84 (2000) 5106-5109*

I.Krichever, M.Mineev-Weinstein, P.Wiegmann and A.Zabrodin, *Physica D 198 (2004) 1-28*

R.Teodorescu, E.Bettelheim, O.Agam, P.Wiegmann and A.Zabrodin, *Nucl. Phys. B704 (2005) 407-444*

A.Zabrodin, *Physica D 235 (2007) 101-108*

A.Zabrodin, *J. Phys. A: Math. Theor. 46 (2013) 185203, arXiv:1212.6729;*
arXiv: 1306.5168 (2013)

S.Natanzon and A.Zabrodin, *arXiv:1302.7288 (2013)*

A function $U(z,\bar{z})$ in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

such that $\sigma(z,\bar{z}) := \partial\bar{\partial}U(z,\bar{z}) > 0.$

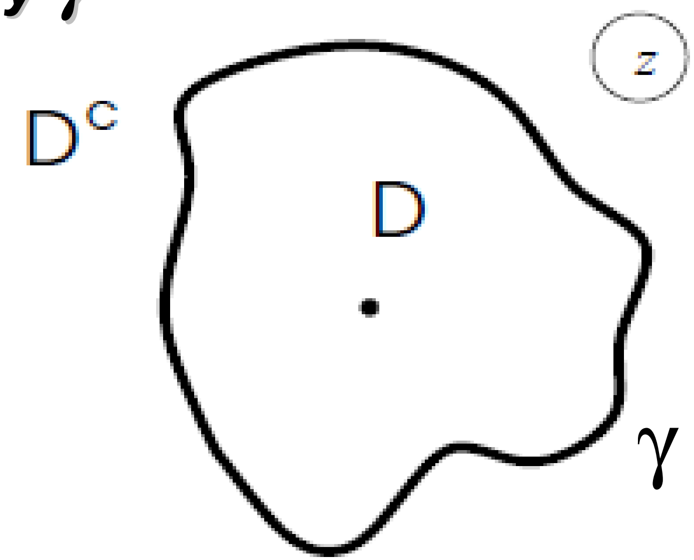
(density, conformal metric, ...)

This function will parametrize solutions of Toda

Example 1 $U(z,\bar{z}) = z\bar{z}\,, \quad \sigma(z,\bar{z}) = 1$

Example 2 $U(z,\bar{z}) = \dfrac{1}{2\beta}\left[\log\dfrac{z\bar{z}}{Q}\right]^2 \qquad \sigma(z,\bar{z}) = \dfrac{1}{\beta\, z\bar{z}}$

A simply-connected compact domain D in the complex plane with smooth boundary γ

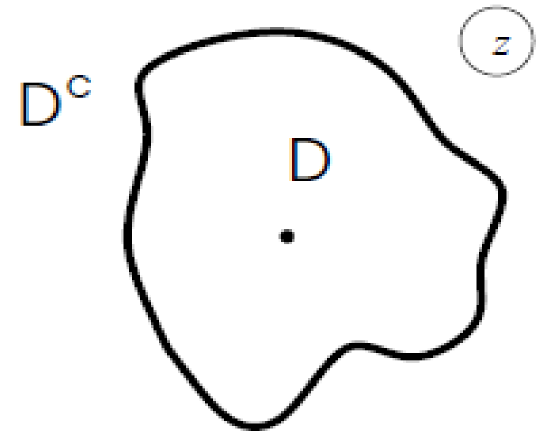


Moments of the exterior:

$$t_k = \frac{1}{2\pi i k}\oint_{\gamma} z^{-k}\partial U(z,\bar z)\,dz = -\frac{1}{\pi k}\iint_{D^c} z^{-k}\sigma(z,\bar z)\,d^2z\,, \quad k \ge 1$$

$$t_0 = \frac{1}{2\pi i}\oint_{\gamma}\partial U(z,\bar z)\,dz = \frac{1}{\pi}\iint_{D}\sigma(z,\bar z)\,d^2z$$

Complimentary set of moments (moments of the interior):



$$v_k = \frac{1}{2\pi i} \oint_\gamma z^k \partial U(z, \bar{z})\, dz = \frac{1}{\pi} \iint_{\mathsf{D}} z^k \sigma(z, \bar{z})\, d^2 z\,, \quad k \geq 1$$

Logarithmic moment:

$$v_0 = \frac{1}{\pi} \iint_{\mathsf{D}} \log|z|^2 \sigma(z, \bar{z})\, d^2 z.$$

Potential created by the charge in D

$$\Phi(z,\bar{z}) = -\frac{2}{\pi}\int_{\mathrm{D}} d^2z' \; \sigma(z',\bar{z}') \log|z-z'|$$

Expansion inside

$$\Phi^{+}(z,\bar{z}) = -U(z,\bar{z}) + v_0 + 2\mathcal{R}e\sum_{k>0} t_k z^k$$

Expansion outside

$$\Phi^{-}(z,\bar{z}) = -2t_0 \log|z| + 2\mathcal{R}e\sum_{k>0} \frac{v_k}{k} z^{-k}$$

Theorem 0

The real parameters $t_0, \ \mathrm{Re}\, t_k, \ \mathrm{Im}\, t_k, \ k \geq 1$

are local coordinates in the space of simply connected domains with smooth boundary

This means:

1. *Any one-parameter deformation* $\mathrm{D}(t)$ *of* $\mathrm{D} = \mathrm{D}(0)$ *with some real parameter* t *such that* $\partial_t t_k = 0, \ k \geq 0$ *is trivial (local uniqueness of domain with given moments)*

2. *These parameters are independent*

In particular, moments v_k *are functions of* t_k

Green's functon of the Dirichlet boundary value problem in the exterior of D

$$G(z,\xi) = \frac{1}{2\pi}\log|z-\xi| + g(z,\xi)$$

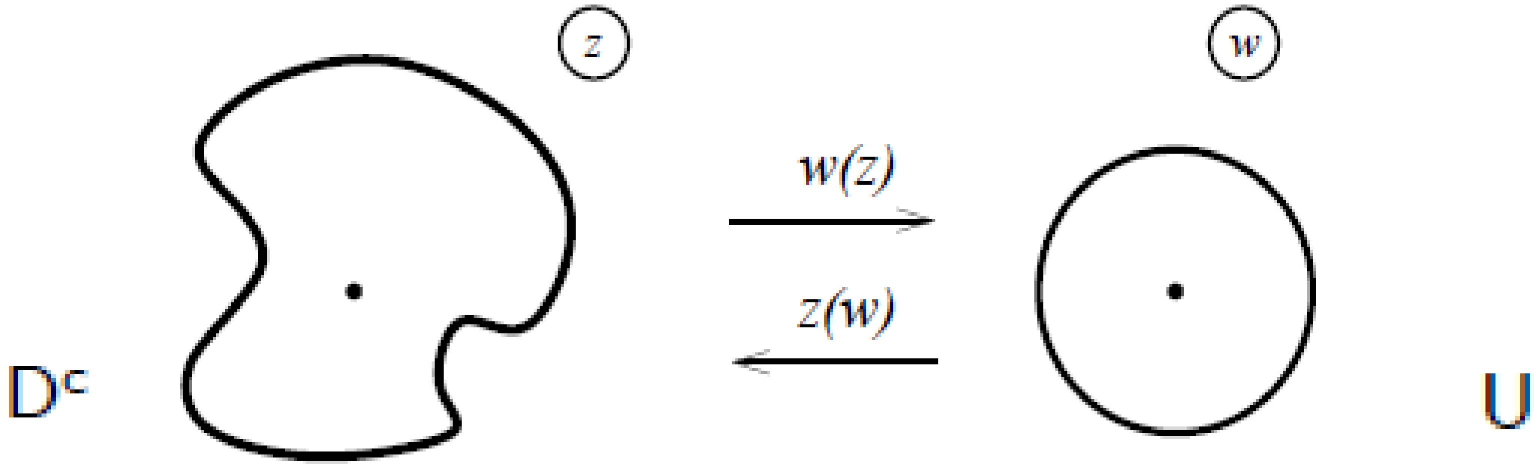
- $G(z,\xi) = G(\xi,z)$ and $G(z,\xi') = 0$ for any $z \in D^c$ and $\xi' \in \gamma$;
- The function $g(z,\xi)$ is harmonic in z for any $\xi \in D^c$

It gives universal solution to the Dirichlet boundary value problem

$$u(z) = -\oint_\gamma u_0(\xi)\partial_{n_\xi} G(z,\xi)|d\xi|$$

(the Poisson formula)

Conformal bijection $w : \mathrm{D}^c \to \mathrm{U}$



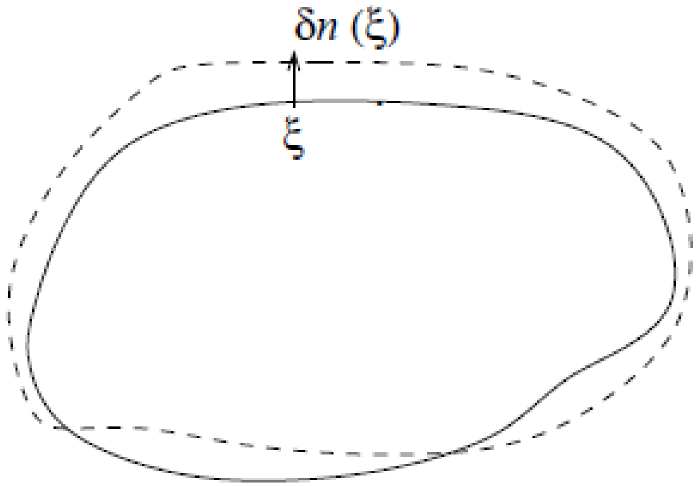
$$G(z,\xi) = \frac{1}{2\pi}\log\left|\frac{w(z)-w(\xi)}{w(z)\overline{w(\xi)}-1}\right|$$

We normalize w(z) by the conditions

$w(\infty) = \infty$ and $w'(\infty)$ is real positive.

$$w(z) = pz + \sum_{j\geq 0} p_j z^{-j}, \text{ where } p > 0$$

Infinitesimal deformations can be described by normal displacement of the boundary



Special deformations

$$\delta_a n(z) = -\frac{\varepsilon\pi}{\sigma(z,\bar{z})}\,\partial_{n_z} G(a,z)\,, \qquad z \in \gamma,\ \varepsilon \to 0.$$

The change of moments under the special deformations:

$$\delta_z t_0 = -\frac{\varepsilon}{2\pi} \oint_\gamma \partial_{n_\xi} G(z,\xi)|d\xi| = \varepsilon$$

$$\delta_z t_k = -\frac{\varepsilon}{2\pi k} \oint_\gamma \xi^{-k} \partial_{n_\xi} G(z,\xi)|d\xi| = \frac{\varepsilon}{k} z^{-k}$$

(by virtute of the Poisson formula for the solution of the Dirichlet problem)

Consider the function

$$H_k(\xi) = -i \oint_{\infty} z^k \partial_z G(z,\xi)\, dz$$

and the deformations

$$\delta n(\xi) = \varepsilon \mathrm{Re}\, (\partial_{n_\xi} H_k(\xi)) \text{ and } \delta n(\xi) = \varepsilon \mathrm{Im}\, (\partial_{n_\xi} H_k(\xi))$$

They change $x_k = \mathrm{Re}\, t_k$ ***and*** $y_k = \mathrm{Im}\, t_k$ ***only***

$$\delta_\infty n(\xi) = -\frac{\varepsilon\pi}{\sigma(\xi,\bar{\xi})}\partial_{n_\xi} G(\infty,\xi) \qquad \text{changes } t_0 \text{ only.}$$

Introduce differential operators

$$D(z) = \sum_{k\geq 1} \frac{z^{-k}}{k} \partial_k \,, \qquad \bar{D}(\bar{z}) = \sum_{k\geq 1} \frac{\bar{z}^{-k}}{k} \bar{\partial}_k$$

where

$$\partial_k = \partial/\partial t_k, \; \bar{\partial}_k = \partial/\partial \bar{t}_k$$

and the operator

$$\nabla(z) = \partial_0 + D(z) + \bar{D}(\bar{z})$$

Lemma 1. ***Let X be any functional on the set of domains* D *regarded as a function of*** $t_0, \{t_k\}, \{\bar{t}_k\}$, ***then for any z in the exterior of* D *we have***

$$\delta_z X = \varepsilon \nabla(z) X$$

Lemma 2. ***Let X be a functional of the form***

$$X = \int_{\mathrm{D}} \Psi(\zeta, \bar{\zeta})\, \sigma(\zeta, \bar{\zeta})\, d^2\zeta$$

with an arbitrary integrable function Ψ , ***then***

$$\nabla(z) X = \pi \Psi^H(z)$$

where $\Psi^H(z)$ ***is the harmonic extension of*** Ψ

***from the boundary to the exterior of* D**

The dispersionless tau-function

$$\boxed{F = -\frac{1}{\pi^2}\iint_{\mathsf{D}}\iint_{\mathsf{D}} \sigma(z,\bar{z}) \log\left|z^{-1}-\zeta^{-1}\right| \sigma(\zeta,\bar{\zeta})\, d^2z d^2\zeta}$$

$$\nabla(z)F = -\frac{2}{\pi}\iint_{\mathsf{D}} \log\left|z^{-1}-\zeta^{-1}\right| \sigma(\zeta,\bar{\zeta})\, d^2\zeta$$

Therefore

$$v_0 = \partial_0 F, \qquad v_k = \partial_k F, \qquad \bar{v}_k = \bar{\partial}_k F, \qquad k \geq 1$$

Theorem 1.

$$G(z,\zeta) = \frac{1}{2\pi}\log\left|z^{-1} - \zeta^{-1}\right| + \frac{1}{4\pi}\nabla(z)\nabla(\zeta)F.$$

Corollary. *The conformal map w(z) is given by*

$$w(z) = z\exp\left(\left(-\frac{1}{2}\partial_0^2 - \partial_0 D(z)\right)F\right)$$

Theorem 2. ***The function F satisfies the equations***

$$(z-\xi)e^{D(z)D(\xi)F} = ze^{-\partial_0 D(z)F} - \xi e^{-\partial_0 D(\xi)F}$$

$$(\bar{z}-\bar{\xi})e^{\bar{D}(\bar{z})\bar{D}(\bar{\xi})F} = \bar{z}e^{-\partial_0 \bar{D}(\bar{z})F} - \bar{\xi} e^{-\partial_0 \bar{D}(\bar{\xi})F}$$

$$1 - e^{-D(z)\bar{D}(\bar{\xi})F} = \frac{1}{z\bar{\xi}} e^{\partial_0(\partial_0 + D(z) + \bar{D}(\bar{\xi}))F}$$

(equations of dispersionless 2D Toda hierarchy in the Hirota form)

Proof: **Substitute** $w(z) = z\exp\left(\left(-\frac{1}{2}\partial_0^2 - \partial_0 D(z)\right)F\right)$ ***into***

$$\log\left|\frac{w(z)-w(\xi)}{1-w(z)\bar{w}(\xi)}\right| = \log\left|\frac{1}{z}-\frac{1}{\xi}\right| + \frac{1}{2}\nabla(z)\nabla(\xi)F.$$

and separate holomorphic and antiholomorphic parts

Important comment:

Although the definitions of the moments and the function F

depend on the background density, the formulas for the

Green function and the conformal map do not.

This means that the conformal maps can be described

by any non-degenerate solution of the Toda hierarchy.

Physical applications to Hele-Shaw flows (Laplacian growth)

$$U(z,\bar{z}) = z\bar{z}\,, \quad \sigma(z,\bar{z}) = 1$$

Then the vector field in the space of domains corresponding to the special deformation with the Green function G(z,a) is the

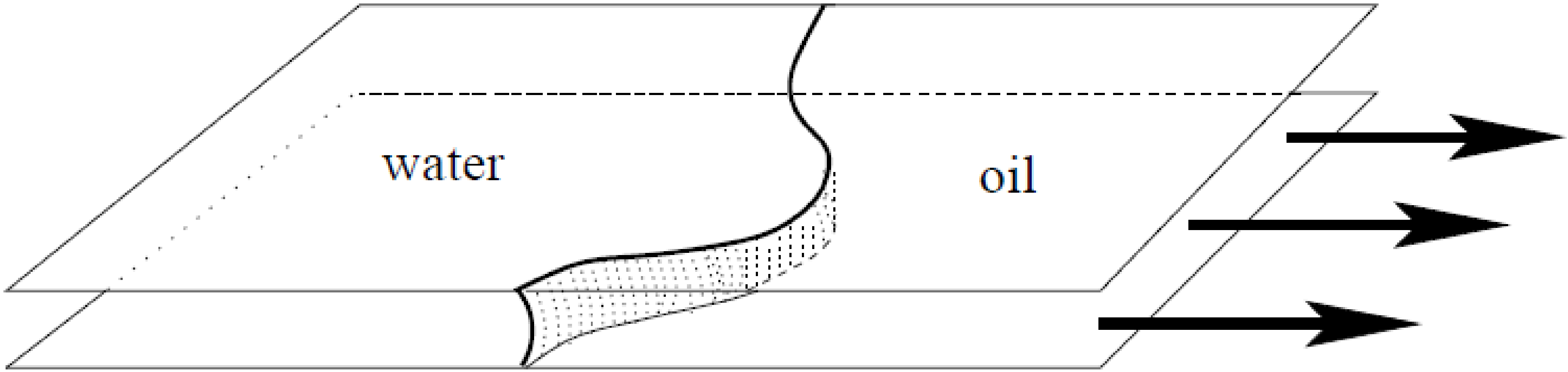
<u>Hele-Shaw flow</u>

(with zero surface tension) with a sink at the point a

$$V_n(z) \propto \partial_{n_z} G(z,a)$$

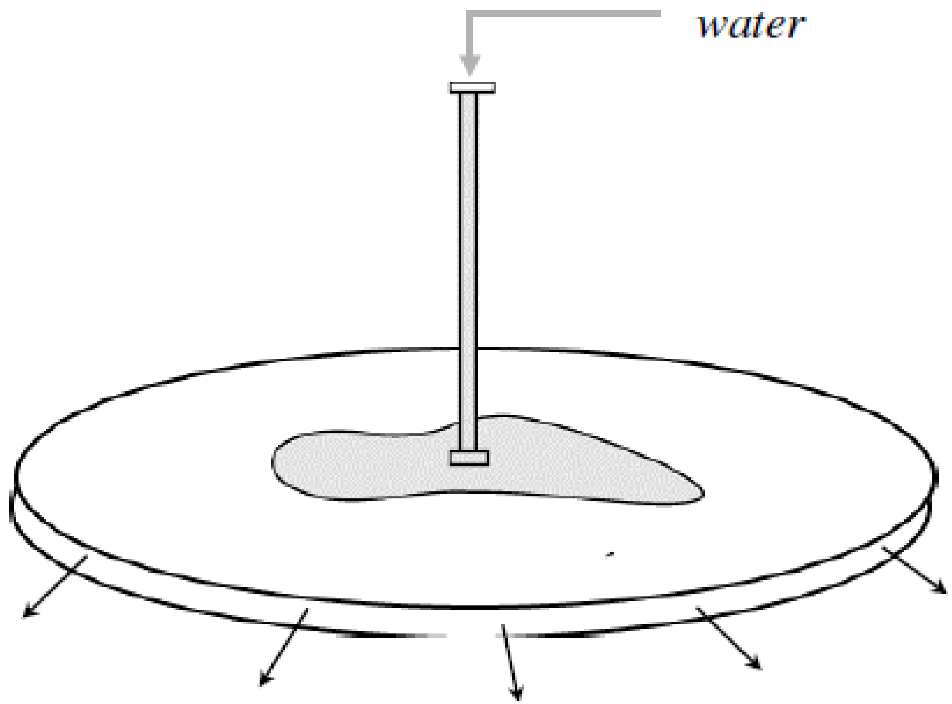
In particular, $a \to \infty, \quad V_n \propto \partial_n \log|w(z)|$

The Hele-Shaw cell



The Darcy law: $\vec{V} = -\vec{\nabla}\Phi$ $\qquad \Delta\Phi(Z) = 0$

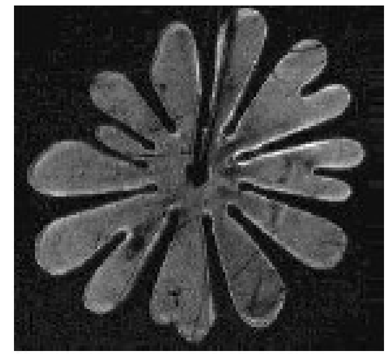
At the interface: $V_n(Z) = -\partial_n \Phi(Z)$



Radial Hele-Shaw cell, schematic view

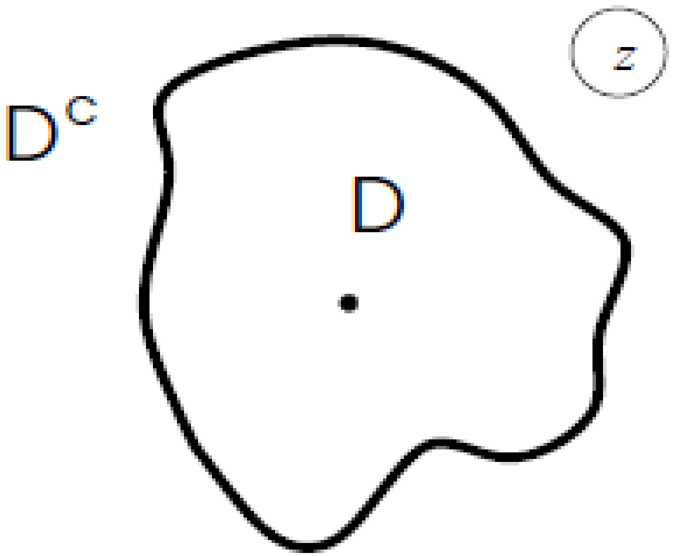
Experimental patterns

Large flux,
small surface tension
(after Swinney)



Small flux,
large surface
tension

Laplacian growth and inverse potential problem



(S.Richardson, 1972)

$$t_k = \frac{1}{2\pi i k} \oint_\gamma z^{-k} \bar{z} dz$$

(exterior harmonic moments) are conserved

t_0 = Area (D)/π = time

The LG process is changing the area keeping the harmonic moments constant.

Integrability of the radial LG problem

(M.Mineev-Weinstein, P.Wiegmann, A.Z., 1999)

Dispersionless tau-function $F = F(t_0, \{t_k\}, \{\bar{t}_k\})$

Conformal map from the domain to the exterior of the unit disk

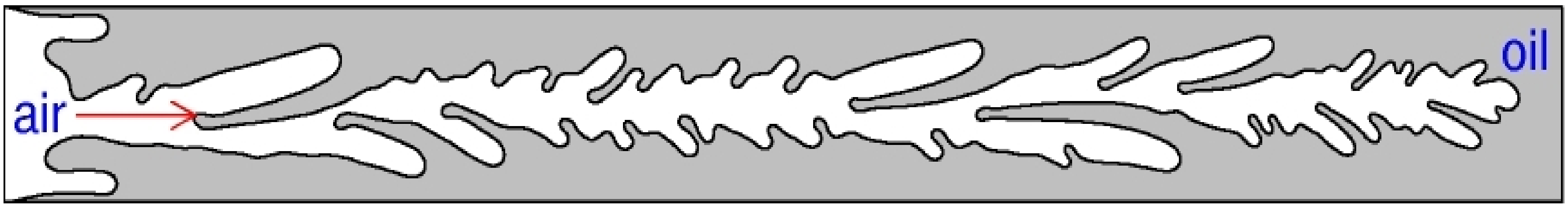
$$w(z) = z \exp\Big(- \frac{1}{2} \partial_{t_0}^2 F_0 - \partial_{t_0} D(z) F \Big)$$

The Green's function

$$G(z, \zeta) = \log \left| z^{-1} - \zeta^{-1} \right| + \frac{1}{2} \nabla(z, \bar{z}) \nabla(\zeta, \bar{\zeta}) F,$$

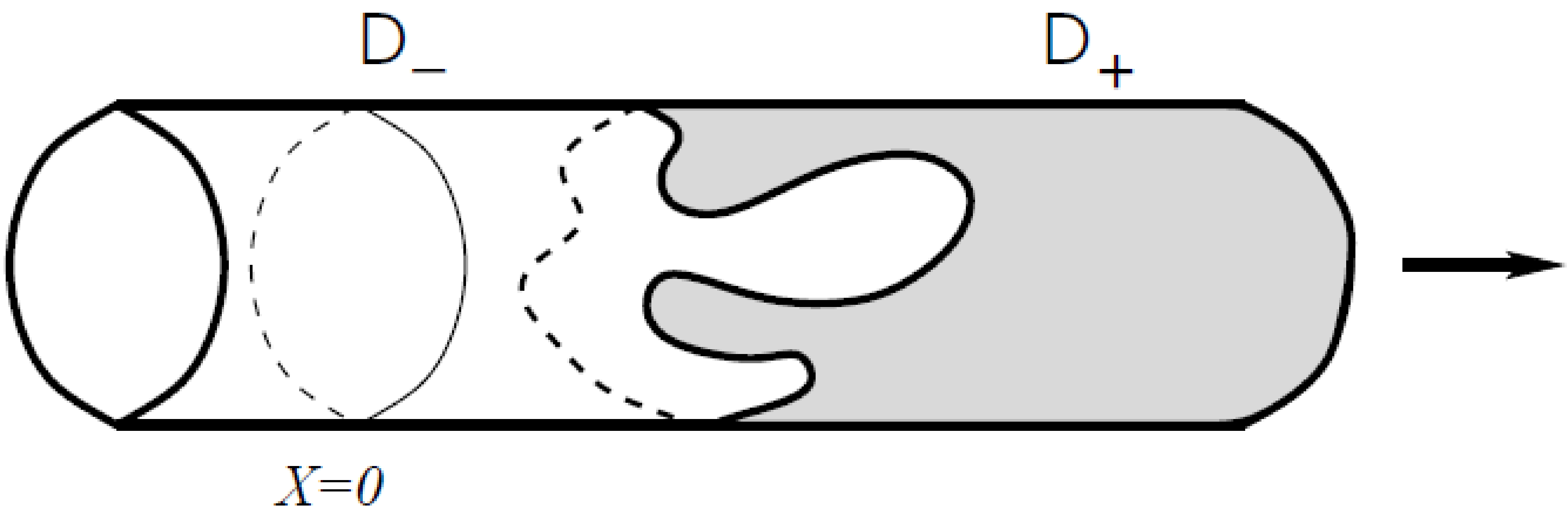
$$\nabla(z, \bar{z}) := \partial_{t_0} + \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_{t_k} + \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \partial_{\bar{t}_k} = \partial_{t_0} + D(z) + \bar{D}(\bar{z})$$

The Hele-Shaw problem in a channel



- is integrable (can be embedded in the same Toda hirarchy)

- is related to algebraic geometry of ramified coverings

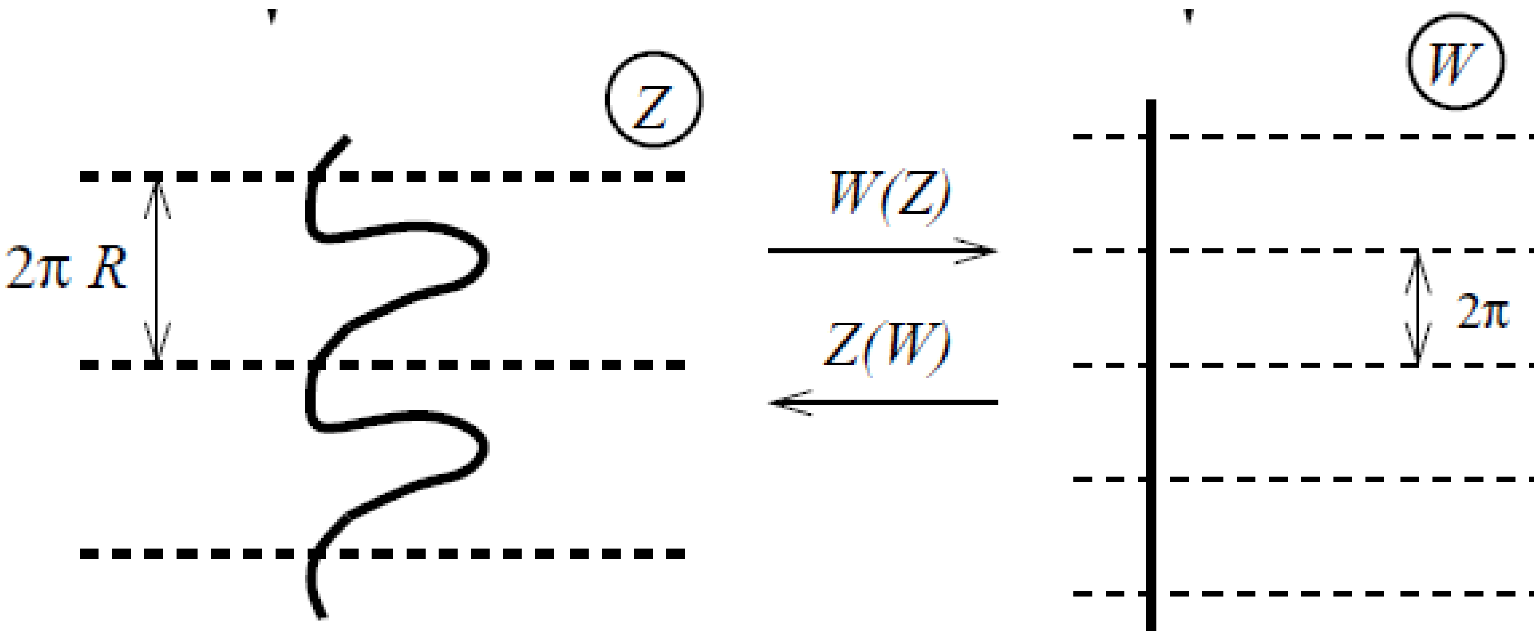


$$
\begin{cases}
\Delta\Phi(Z) = 0 \quad \text{in} \;\; \mathsf{D}_+ \\[2ex]
\Phi(Z + 2\pi i R) = \Phi(Z) \\[2ex]
\Phi(Z) = 0, \qquad Z \in \Gamma \\[2ex]
\Phi(Z) = -\frac{1}{2}\,\mathcal{R}eZ + \ldots \quad \text{as} \quad \mathcal{R}eZ \to +\infty
\end{cases}
$$

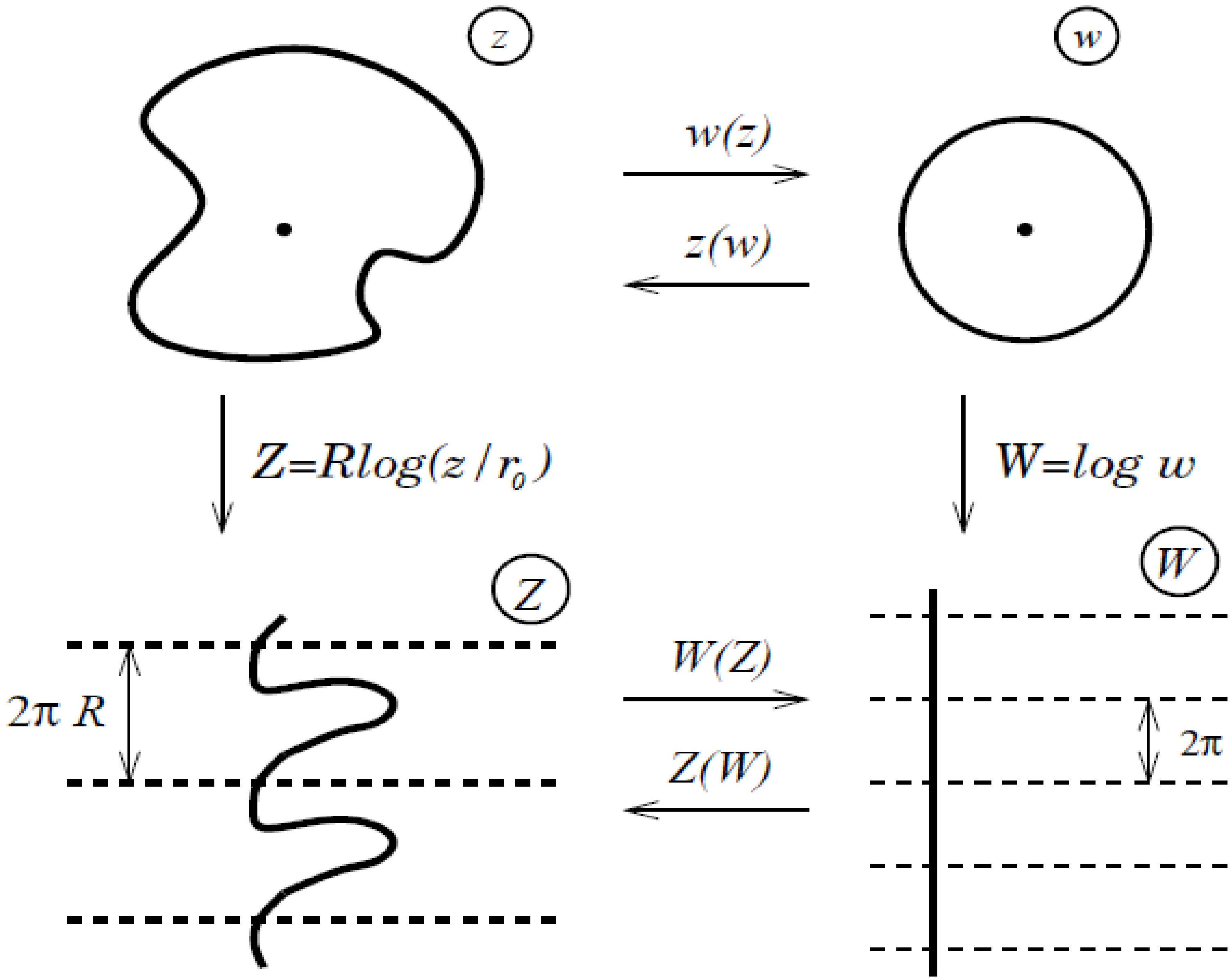
Conformal map: $$W(Z) = Z/R + \sum_{k\geq 0} c_k e^{-kZ/R}$$

Solution: $$\Phi(Z) = -\tfrac{R}{2}\,\mathcal{R}e W(Z)$$

$$V_n(Z) = \frac{R}{2}\,|W'(Z)|$$



Physical plane and auxiliary physical plane



We can express the normal velocity in the physical plane through the normal velocity in the auxiliary physical plane:

$$V_n^{(Z)} = \left|\frac{dZ}{dz}\right| V_n^{(z)} = \frac{R}{|z|} V_n^{(z)}$$

If $U(z,\bar{z}) = \frac{R}{2}\left[\log\frac{z\bar{z}}{r_0^2}\right]^2.$ ***then***

$$V_n^{(z)}(z) = \frac{|z|^2}{2R}\,|w'(z)|\,, \quad z \in \gamma$$

$$V_n^{(Z)}(Z) = \frac{R}{|z|} V_n^{(z)}(z) = \frac{R}{2}\,|W'(Z)|$$

The dispersionless tau-function

$$F_0 = -\frac{R^2}{\pi^2}\iint_{\mathsf{D}\setminus\mathsf{B}(r_0)}\iint_{\mathsf{D}\setminus\mathsf{B}(r_0)} \log\left|z^{-1}-\zeta^{-1}\right| \frac{d^2z\,d^2\zeta}{|z\,\zeta|^2}$$

$$= -\frac{1}{\pi^2 R^2}\iint_{\mathsf{D}_-^{(0)}}\iint_{\mathsf{D}_-^{(0)}} \log\left|e^{-Z/R}-e^{-Z'/R}\right| d^2Z\,d^2Z' - t_0^2\log r_0$$

- $$2F_0 = R\partial_R F_0 + t_0\partial_{t_0}F_0 + \sum_{k>1}\left(t_k\partial_{t_k}F_0 + \bar{t}_k\partial_{\bar{t}_k}F_0\right)$$

$\beta = 1/R,$

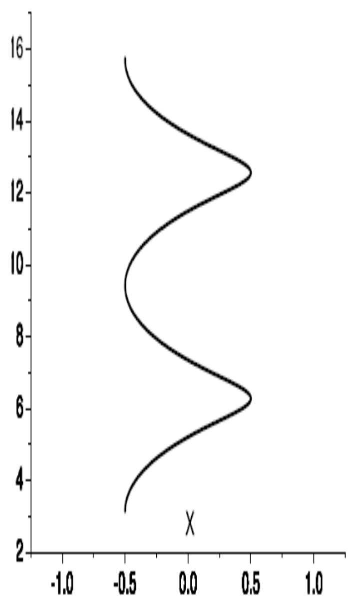
- $$\partial_\beta F_0 = \frac{t_0^3}{6} + t_0\sum_{k>1} k t_k \partial_{t_k} F_0 + \frac{1}{2}\sum_{k,l\geq 1}\left(kl t_k t_l \partial_{t_{k+l}} F_0 + (k+l) t_{k+l}\partial_{t_k}F_0\partial_{t_l}F_0\right)$$

(the cut-and-join operator)

Example

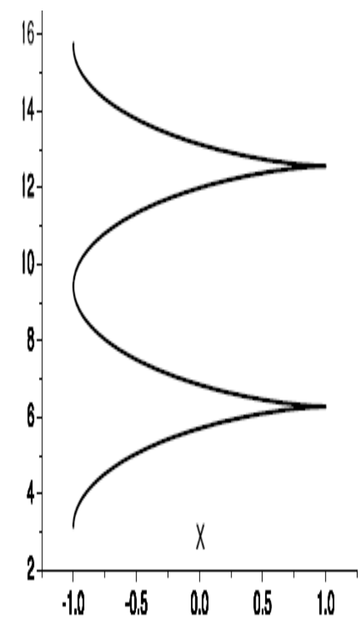
$$t_0 = t,\ t_1 \neq 0,\ t_k = 0 \text{ at } k \geq 2.$$

$$Z(W) = RW + u_0 + u_1 e^{-W}$$



Trochoid

$$\xrightarrow{t.}$$



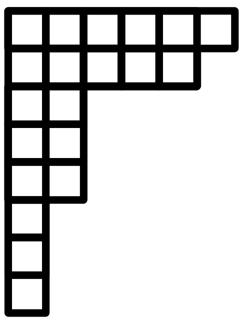
cusp

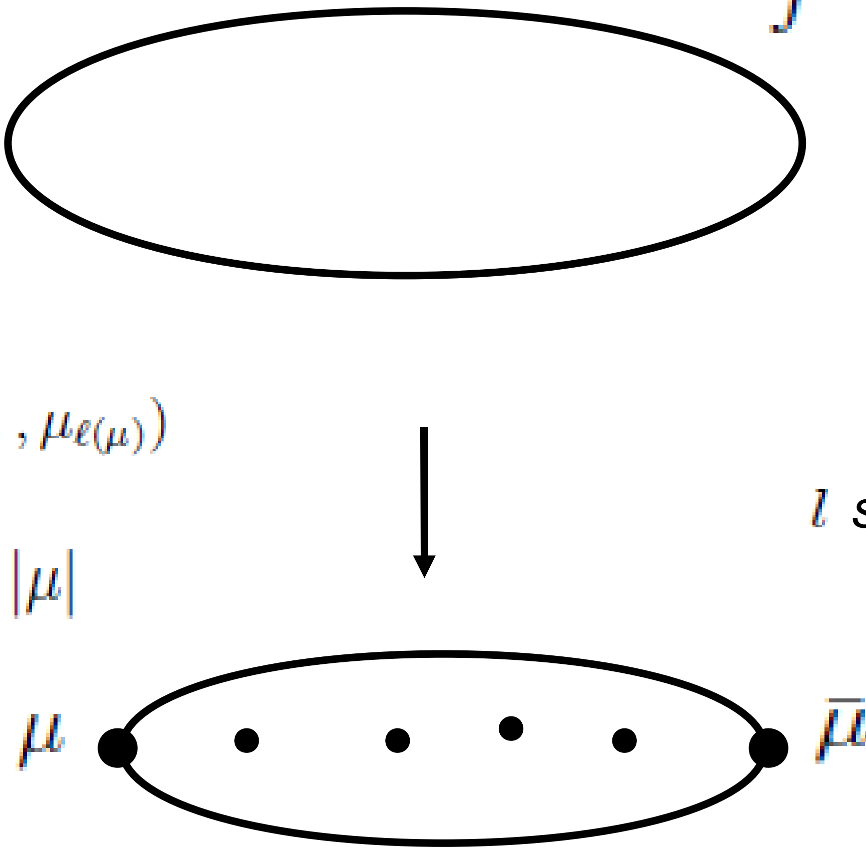
Cycloid

Hurwitz numbers

degree d covering

$$f\ :\ \Sigma\ \longrightarrow\ \mathbb{CP}^1$$





$$\mu = (\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)})$$

$$d = \sum_{i=1}^{\ell(\mu)} \mu_i := |\mu|$$

l *simple ramification points*

$$|\mu| = |\bar{\mu}| = d$$

$$double\ \ Hurwitz\ \ numbers\ \ H_{d,l}(\mu, \bar{\mu})$$

Generating function of the double Hurwitz numbers for connected coverings

$$\mathrm{t} = \{t_1, t_2, \ldots\},\ \bar{\mathrm{t}} = \{\bar{t}_1, \bar{t}_2, \ldots\}$$

$$F^{(H)}(\beta, Q, \mathrm{t}, \bar{\mathrm{t}}) = \sum_{l\geq 0} \frac{\beta^l}{l!} \sum_{d\geq 1} Q^d \sum_{|\mu|=|\bar{\mu}|=d} H_{d,l}(\mu, \bar{\mu}) \prod_{i=1}^{\ell(\mu)} \mu_i t_{\mu_i} \prod_{i=1}^{\ell(\bar{\mu})} \bar{\mu}_i \bar{t}_{\bar{\mu}_i}$$

$$\tau_n(\mathrm{t}, \bar{\mathrm{t}}) = e^{\frac{1}{12}\beta n(n+1)(2n+1)} Q^{\frac{1}{2}n(n+1)} \exp\Big(F^{(H)}(\beta, e^{\beta(n+\frac{1}{2})}Q, \mathrm{t}, \bar{\mathrm{t}})\Big)$$

is the tau-function of the 2D Toda lattice hierarchy

(A.Okounkov, 2000)

Genus expansion

$$t_k \to t_k/\hbar, \ \beta \to \hbar\beta$$

$$F^{(H)}(\hbar;\beta,Q,\mathrm{t},\bar{\mathrm{t}}) := \hbar^2 F^{(H)}(\hbar\beta,Q,\mathrm{t}/\hbar,\bar{\mathrm{t}}/\hbar)$$

$$F^{(H)}(\hbar;\beta,Q,\mathrm{t},\bar{\mathrm{t}}) = \sum_{g\geq 0} \hbar^{2g} F_g^{(H)}(\beta,Q,\mathrm{t},\bar{\mathrm{t}})$$

Riemann-Hurwitz formula

$$2g-2 = l - \ell(\mu) - \ell(\bar{\mu})$$

The generating function of double Hurwitz numbers for connected genus 0 coverings

$$F_0^{(H)} = \sum_{d\geq 1} \sum_{|\mu|=|\bar{\mu}|=d} \frac{Q^d\, H_{d,\ell(\mu)+\ell(\bar{\mu})-2}(\mu,\bar{\mu})}{\beta^2(\ell(\mu)+\ell(\bar{\mu})-2)!} \prod_{i=1}^{\ell(\mu)}(\beta\mu_i t_{\mu_i}) \prod_{i=1}^{\ell(\bar{\mu})}(\beta\bar{\mu}_i \bar{t}_{\bar{\mu}_i})$$

Relation to the dispersionless tau-function for LG on a cylinder

$$\beta = 1/R,\ Q = r_0^2$$

$$\boxed{F_0 = \frac{\beta t_0^3}{6} + t_0^2 \log r_0 + F_0^{(H)}(\beta, r_0^2 e^{\beta t_0}, \mathrm{t}, \bar{\mathrm{t}})}$$

Conclusion:

Conformal maps of plane domains and

connected genus 0 ramified coverings of the sphere

are governed by the same "master function"

which is a special solution to

the dispersionless Toda hierarchy