On a certain generalization of Virasoro constraints for Frobenius manifolds

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1. Motivations

Virasoro constraints for Gromov-Witten invariants

$$L_m Z = 0, \quad Z = e^{\epsilon^{-2}\mathcal{F}_0 + \mathcal{F}_1 + \epsilon^2 \mathcal{F}_2 + \dots}, \quad m \geq -1.$$

Here $\mathcal{F}_g$'s are the generating functions of genus g Gromov-Witten invariants

$$\mathcal{F}_g = \sum_m \sum_{\beta \in H_2(X;\mathbf{Z})} \frac{1}{m!} t^{\alpha_1, p_1} \dots t^{\alpha_m, p_m} \langle \tau_{p_1}(\phi_{\alpha_1}) \dots \tau_{p_m}(\phi_{\alpha_m}) \rangle_{g,\beta}$$

The Virasoro operators are second order linear differential operators of the coupling constants $t^{\alpha,p}, \ \alpha = 1, \dots, n, \ p \geq 0$.

The Virasoro constraints were proposed by
Eguchi, Hori, Xiong, 1997 (modified by S. Katz, 1999)

For an arbitrary Frobenius manifold proposed by
Dubrovin, Z., 1998

Proof of the validity of Virasoro constraints

Genus zero: Liu, Tian, 1998

Genus zero and one for Frobenius manifold: Dubrovin, Z., 1998 (assume semi-simplicity for genus one case)

Genus two: Liu, 2003 (assume semi-simplicity)

General case: Givental formula 2001 & Teleman 2011 (assume semi-simplicity)

Validity of Virasoro constraints reflects important properties of some symmetries – the Virasoro symmetries of the underlined integrable hierarchy of KdV type.

The KdV hierarchy: Gromov-Witten invariant of a point

The extended Toda hierarchy: Gromov-Witten invariants of $\mathbb{CP}^1$

For a Frobenius manifold (with a fixed calibration), one can associate to it an integrable hierarchy of hydrodynamic type, called the Principal Hierarchy

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = K^\alpha_{\beta,q;\gamma}(v) v^\gamma_x, \quad \alpha, \beta = 1, \dots, n, \; p, q \geq 0.$$

$\mathcal{F}_0(t)$: the logarithm of a particular tau function of the hierarchy

$\mathcal{F}_g(t)$: Uniquely determined by $\mathcal{F}_0(t)$ and the Virasoro constraints

$$L_m e^{\epsilon^{-2}\mathcal{F}_0 + \mathcal{F}_1 + \epsilon^2 \mathcal{F}_2 + \dots} = 0, \quad m \geq -1$$

(Dubrovin, Z. 2001)

In this talk, we study a certain deformation of the Virasoro operators L_m:

$$L_m^{(\nu)} = \sum_{k\geq 0} L_{m,k}\nu^k, \quad L_{m,0} = L_m$$

Such deformation appears naturally in the procedure of construction of the Virasoro operators for Frobenius manifold.

We show that for these deformed Virasoro operators the genus zero Virasoro constraints still hold true, and they are related to Virasoro constraints of Hodge integrals.

We also study the properties of the integrable hierarchy underlined the Hodge potentials by using these Virasoro like constraints.

2. The genus zero free energy of a Frobenius manifold

Let $(M^n, <\,,\,>, \cdot\,,\ e\,, E)$ be a Frobenius manifold of dimension n (Dubrovin 1991)

flat metric $<\,,\,>$

multiplication "$\cdot$" on the tangent spaces

the unit vector field e

the Euler vector field E

Let $v^1, \ldots, v^n$ be the local flat coordinates of the flat metric, then the axioms of Frobenius manifold ensures the existence of a potential $F(v)$, so that

$$\frac{\partial}{\partial v^\alpha} \cdot \frac{\partial}{\partial v^\beta} = c^\gamma_{\alpha\beta}(v) \frac{\partial}{\partial v^\gamma}, \quad c^\gamma_{\alpha\beta} = \eta^{\gamma\xi} \frac{\partial^3 F(v)}{\partial v^\alpha \partial v^\beta \partial v^\xi}$$

The unity and the Euler vector field

$$e = \frac{\partial}{\partial v^1}, \quad E = \sum_{\alpha=1}^{n} \left((1 - \frac{d}{2} - \mu_\alpha) v^\alpha + r_\alpha \right) \frac{\partial}{\partial v^\alpha}$$

Quasi-homogeneity

$$\partial_E F(v) = (3 - d) F(v) + \text{quadratic terms in } v$$

Deformed flat connection on $M \times \mathbb{C}^*$

$$\tilde{\nabla}_a b = \nabla_a b + z\, a \cdot b, \quad \tilde{\nabla}_{\frac{d}{dz}} b = \partial_z b + E \cdot b - \frac{1}{z}\mu b$$

a system of deformed flat coordinates

$$(\tilde{v}_1(v,z), \ldots, \tilde{v}_n(v,z)) = (\theta_1(v,z), \ldots, \theta_n(v,z))\, z^{\mu} z^{R}$$

$$\theta_\alpha(v,z) = \sum_{p \geq 0} \theta_{\alpha,p}(v) z^p, \quad \alpha = 1, \ldots, n.$$

Normalization condition

$$< \nabla\theta_\alpha(v,z), \nabla\theta_\beta(v,-z) >= \eta_{\alpha\beta}$$

The Principal Hierarchy – integrable Hamiltonian hierarchy

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = \eta^{\alpha\gamma} \frac{\partial}{\partial x} \left(\frac{\delta H_{\beta,q}}{\delta v^\gamma} \right), \quad \alpha, \beta = 1, \ldots, n; , \; q \geq 0$$

with Hamiltonians

$$H_{\beta,q} = \int \theta_{\beta,q+1}(v(x)) dx$$

A dense subset of the analytic solutions of the integrable hierarchy can be obtained by the system of equations

$$\sum_{p \geq 0} \tilde{t}^{\alpha,p} \nabla \theta_{\alpha,p}(v) = 0, \quad \tilde{t}^{\alpha,p} := t^{\alpha,p} - c^{\alpha,p}, \; t^{1,0} = x.$$

(Topological solution: $c^{\alpha,p} = \delta_1^\alpha \delta_1^p$)

Define functions $\Omega_{\alpha,p;\beta,q}$ on the Frobenius manifold by

$$\sum_{p,q\geq 0} \Omega_{\alpha,p;\beta,q}(v) z^p w^q = \frac{< \nabla\theta_{\alpha,p}(v,z), \nabla\theta_{\beta,q}(v,w) > -\eta_{\alpha\beta}}{z+w}$$

The genus zero free energy

$$\mathcal{F}_0(t) = \frac{1}{2} \sum_{p,q\geq 0} \tilde{t}^{\alpha,p} \tilde{t}^{\beta,q} \Omega_{\alpha,p;\beta,q}(v(t))$$

It has the property

$$v_\alpha(t) = \frac{\partial^2 \mathcal{F}_0(t)}{\partial t^{1,0} \partial t^{\alpha,0}}, \quad \theta_{\alpha,p}(v(t)) = \frac{\partial^2 \mathcal{F}_0(t)}{\partial t^{1,0} \partial t^{\alpha,p}}$$

$$\Omega_{\alpha,p;\beta,q}(v(t)) = \frac{\partial^2 \mathcal{F}_0(t)}{\partial t^{\alpha,p} \partial t^{\beta,q}}$$

3. Virasoro constraints for Frobenius manifolds

The intersection form

$$g^{\alpha\beta}(v) := (dv^\alpha, dv^\beta) = i_E(dv^\alpha \cdot dv^\beta) = E^\gamma c_\gamma^{\alpha\beta}(v)$$

gives a second flat metric $(g_{\alpha\beta}) = (g^{\alpha\beta})^{-1}$ outside of the discriminant of M. The two flat metrics yield a bihamiltonian structure

$$\mathcal{R}_1 = \eta^{\alpha\beta}\frac{\partial}{\partial x}, \quad \mathcal{R}_2 = g^{\alpha\beta}(v)\frac{\partial}{\partial x} + \Gamma_\gamma^{\alpha\beta}(v)v_x^\gamma$$

and the recursion operator $\mathcal{R} = \mathcal{R}_2\mathcal{R}_1^{-1}$. The integrable hierarchy has the Virasoro symmetries starting from the Galilean symmetry

$$\frac{\partial v^\alpha}{\partial s_{-1}} = \sum_{p\geq 1} t^{\alpha,p}\frac{\partial v^\alpha}{\partial t^{\alpha,p-1}} + \delta_1^\alpha$$

Representation of the Virasoro symmetries

Flat pencil of metrics:

$$g^{\alpha\beta} - \lambda \eta^{\alpha\beta}$$

Periods $p(v, \lambda)$ defined by Gauss-Manin system

$$(\nabla^* - \lambda \nabla) dp = 0$$

$$(\lambda - U(v)) \frac{d\phi}{d\lambda} = \left(\frac{1}{2} + \mu\right) \phi, \quad \phi := \nabla p$$

When the spectrum of μ does not contain half integers, a basis

$$(p_1(v;\lambda),\dots,p_n(v;\lambda)) = (\pi_1(v;\lambda),\dots,\pi_n(v;\lambda))\lambda^{-\frac{1}{2}-\mu}\lambda^{-R}$$

of period can be given by taking Laplace type integrals to the deformed flat coordinates $\tilde{v}_\alpha(v;z)$

$$p_\alpha(v;\lambda) = \int_0^\infty e^{-\lambda\, z}\,\tilde{v}_\alpha(v;z)\,\frac{dz}{\sqrt{z}}, \quad \alpha = 1,\dots,n.$$

The Gram matrix is constant

$$G_{\alpha\beta} = (dp_\alpha, dp_\beta)_\lambda = (dp_\alpha, dp_\beta) - \lambda < dp_\alpha, dp_\beta >$$

where

$$G = -2\pi\,\eta\left[e^{\pi\, i\, R}e^{\pi\, i\,\mu} + e^{-\pi\, i\, R}e^{-\pi\, i\,\mu}\right]^{-1}$$

For semi simple Frobenius manifolds, one can regularize the above integral by considering the twisted period

$$p_\alpha^{(\nu)}(v;\lambda) = \int_0^{\infty e^{i\varphi}} e^{-\lambda z}\,\tilde{v}_\alpha(v;z)\,\frac{dz}{z^{\frac{1}{2}-\nu}}$$

by using

$$\tilde{v}_\alpha = \left(\sum_{p\geq 0}\theta_{\gamma,p}(v)z^p\right)\left(z^\mu z^R\right)_\alpha^\gamma, \quad \int_0^\infty e^{-t}t^{s-1}\log^k t\,dt = \partial_s^k\Gamma(s)$$

$$\int_0^{\infty e^{i\varphi}} e^{-\lambda z} z^p z^{\mu+\nu-\frac{1}{2}} z^R dz$$

$$= \sum_q [e^{R\partial_\nu}]_q \Gamma(\mu+p+q+\nu+\frac{1}{2})\,\lambda^{-p-q}\lambda^{-(\mu+\nu+\frac{1}{2})}\lambda^{-R}$$

Note the relation

$$\theta_{\alpha,p}(v(t)) = \frac{\partial^2 \mathcal{F}_0(t)}{\partial x \partial t^{\alpha,p}}$$

we introduce the notation

$$S_\alpha = \int_0^\infty \frac{dz}{z^{\frac{1}{2}-\nu}} e^{-\lambda z} \left[\sum_{p\geq 0} \frac{\partial \mathcal{F}_0}{\partial t^{\gamma,p}} z^p + \sum_{q\geq 0} (-1)^q t_{\gamma,q} z^{-q-1} \right] \left(z^\mu z^R\right)_\alpha^\gamma,$$

Then the Virasoro symmetries an be represented by

$$\frac{\partial \mathcal{F}_0}{\partial s} = \sum_{m\geq -1} \frac{1}{\lambda^{m+2}} \frac{\partial \mathcal{F}_0}{\partial s_m} = -\frac{1}{2} \left[\frac{\partial S_\alpha}{\partial \lambda} G^{\alpha\beta} \frac{\partial S_\alpha}{\partial \lambda} \right]_-$$

Define

$$\phi_\alpha^{(\nu)}(\lambda) = \left(\int_0^\infty \frac{dz}{z^{1-\nu}} \, e^{-\lambda z} \sum_{p \in \mathbf{Z}+\frac{1}{2}} \mathbf{a}_p z^{p+\mu} z^R \right)_\alpha , \quad \alpha = 1, \ldots, n.$$

Here ν is an arbitrary complex parameter, $\mathbf{a}^q = \left(a^{1,q}, \ldots, a^{n,q} \right)^T$

$$a_{\alpha,p} = \epsilon \frac{\partial}{\partial t^{\alpha, p-\frac{1}{2}}}, \;\; p > 0, \quad a_{\alpha,p} = \epsilon^{-1} (-1)^{p+\frac{1}{2}} \eta_{\alpha\beta} t^{\beta, -p-\frac{1}{2}}, \;\; p < 0.$$

$$T^{(\nu)}(\lambda) = \sum_{m \in \mathbf{Z}} \frac{L_m^{(\nu)}}{\lambda^{m+2}} = -\frac{1}{2} : \partial_\lambda \phi_\alpha^{(\nu)} \, G^{\alpha\beta}(\nu) \partial_\lambda \phi_\beta^{(-\nu)} : + \frac{1}{4\,\lambda^2} \mathrm{tr} \left(\frac{1}{4} - \mu^2 \right).$$

To represent the Virasoro symmetries in terms of the action of Virasoro operators on the partition function $Z = e^{\epsilon^{-2} \mathcal{F}_0 + \mathcal{F}_1 + \cdots}$.

Definition of the Virasoro operators

$$L_m = \lim_{\nu\to 0} L_m^{(\nu)}$$

$$= \epsilon^2 \sum a_m^{\alpha,p;\beta,q} \frac{\partial^2}{\partial t^{\alpha,p}\partial t^{\beta,q}} + \sum b_{m;\alpha,p}^{\beta,q} t^{\alpha,p} \frac{\partial}{\partial t^{\beta,q}}$$

$$+\epsilon^{-2} \sum c_{\alpha,p;\beta,q} t^{\alpha,p} t^{\beta,q} + c\delta_{m,0}; \qquad m \geq -1$$

they satisfy

$$[L_i, L_j] = (i-j)L_{i+j}, \quad i,j \geq -1.$$

Genus zero Virasoro constraints

$$L_m|_{t^{\alpha,p}\to\tilde{t}^{\alpha,p}} e^{\epsilon^{-2}\mathcal{F}_0} = \mathcal{O}(\epsilon^0) e^{\epsilon^{-2}\mathcal{F}_0}$$

4. Deformations of the Virasoro operators

Theorem

1. *The operators* $L_m^{(\nu)}$ *have the expansion*

$$L_m^{(\nu)} = \sum_{k=0}^{\left[\frac{m+1}{2}\right]} L_{m,2k}\,\nu^{2k}$$

2. *The operators* $L_m^{(\nu)}\,(m \geq -1, \nu \geq 0)$ *form a Lie algebra*
3. *The genus zero Virasoro-like constraints hold true*

$$L_m^{(\nu)}\Big|_{t^{\alpha,p}\to\tilde{t}^{\alpha,p}} e^{\epsilon^{-2}\mathcal{F}_0} = \mathcal{O}(\epsilon^0)e^{\epsilon^{-2}\mathcal{F}_0}$$

Examples of $L_m^{(\nu)}$:

$$L_{m,0} = \text{the usual Virasoro operators}$$

$$L_{2k-1,2k} = \sum_{p\geq 0} t^{\alpha,p} \frac{\partial}{\partial t^{\alpha,p+2k-1}} - \frac{\epsilon^2}{2} \sum_{q=0}^{2k-2} (-1)^q \eta^{\alpha\gamma} \frac{\partial^2}{\partial t^{\alpha,q} \partial t^{\gamma,2k-2-q}}$$

$$L_{2k,2k} = \frac{1+2k}{2} \left(\epsilon^2 \sum_{p=0}^{2k-1} (-1)^p (p + \frac{1}{2} + \mu_\alpha - k) \eta^{\alpha\beta} \frac{\partial^2}{\partial t^{\alpha,p} \partial t^{\beta,2k-1-p}} \right.$$

$$\left. +2 \sum_{p\geq 0} (p + \frac{1}{2} + \mu_\alpha + k) t^{\alpha,p} \frac{\partial}{\partial t^{\alpha,2k+p}} + \dots \right)$$

In general, we have

$$L_m^{(\nu)} = \frac{1}{2} \sum_{\substack{p+q+r=m \\ p,q\in\mathbb{Z}+\frac{1}{2},r\geq 0}} : a_p N_q^p(r,\nu) a^q : + \frac{\delta_{m,0}}{4} tr\left(\frac{1}{4} - \mu^2\right)$$

with

$$N_q^p(r,\nu) := \frac{1}{\pi}[e^{R\partial_\nu}]_r \Big(\Gamma(\mu+\nu+p+r+1)\cos\pi(\mu+\nu)\Gamma(-\mu-\nu+q+1)\Big)$$

Some commutation relations

$$[L_{m,2k}, L_{-1,0}] = (m+1)L_{m-1,2k} - \frac{n}{2}\,\delta^m_{-1}\delta^k_1,$$

$$[L_{m,2k}, L_{0,0}] = m\,L_{m,2k},$$

$$[L_{m,2k}, L_{1,0}] = (m-1)L_{m+1,2k} + \frac{2}{(m+2)(2k-1)} \sum_{h=k}^{[\frac{m+2}{2}]} (2h-4k+3)\binom{2h}{2k-2} L_{m+1,2h},$$

$$[L_{m,2k}, L_{1,2}] = -\frac{2}{m+2} \sum_{h=k+1}^{\infty} \left(\begin{array}{c} 2h \\ 2k \end{array} \right) L_{m+1,2h} + \delta_{m,-1}\delta_{k,-1}\frac{N}{2}.$$

The above commutation relation are important to prove the theorem, one need to prove some identities involving the Stirling numbers.

To prove the genus zero Virasoro-like constraints, we first prove the validity of the constraints for the operators $L_{2k-1,2k}$ by using recursion relations for the functions $\Omega_{\alpha,p;\beta,q}$ and the definition of the genus zero free energy. Then by using the known Virasoro constraints for the operators L_m and the commutation relation $[L_{2k-1,2k}, L_{1,0}]$ and $[L_{2k,2k}, L_{1,2}]$ to prove inductively the validity of the Virasoro-like constraints.

Remark

- *The operators $L_{1,2}, L_{2,2}$ were derived by Eguchi, Hori, Xiong in their Viarsoro paper, they denote them by $\tilde{L}_1, \tilde{L}_2$, they also conjectured the existence of $\tilde{L}_n$ for $n \geq 3$. The validity of genus zero constraints for $\tilde{L}_1, \tilde{L}_2$ was proved by Liu, Tian.*
- *For Gromov-Witten invariants of $\mathbb{P}^1$, the constraints correspond to $L_{2k-1,2k}, L_{2k,2k}$ together with the dilation equation*

$$\frac{\partial \mathcal{F}_0}{\partial t^{1,1}} = \sum_{p\geq 0} t^{\alpha,p} \frac{\partial \mathcal{F}_0}{\partial t^{\alpha,p}} - 2\mathcal{F}_0$$

determine uniquely the genus zero free energy.

- *The Virasoro-like constraints for higher genus free energy no longer hold true.*

5. Virasoro constraints for Hodge integrals

Genus g free energy for Hodge integrals

$$\mathcal{H}_g(t;s) = \sum_{\beta\in H_2(X,\mathbb{Z})} \langle e^{\sum s_{2k-1} ch_{2k-1}(\mathbb{E})} e^{\sum t^{\alpha,p}\tau_p(\phi_\alpha)}\rangle_{g,\beta}$$

$$\langle \prod_{i=1}^{l} \mathrm{ch}_{k_i}(\mathbb{E}) \prod_{j=1}^{m} \tau_{p_j}(\phi_{\alpha_j})\rangle_{g,\beta}$$
$$= \int_{[X_{g,m,\beta}]^{\mathrm{vir}}} \prod_{i=1}^{l} \mathrm{ch}_{k_i}(\mathbb{E}) \wedge \prod_{j=1}^{m} ev_j^*(\phi_{\alpha_j}) \wedge c_1^{p_j}(\mathcal{L}_j),$$

The partition function for the Hodge integrals

$$Z_{\mathbb{E}} = e^{\epsilon^{-2}\mathcal{H}_0+\mathcal{H}_1+\epsilon^2\mathcal{H}_2+\dots}$$

satisfies (Faber, Pandharipande 2000)

$$\frac{\partial Z_{\mathbb{E}}}{\partial s_{2k-1}} = \frac{B_{2k}}{(2k)!}\left(\frac{\partial}{\partial t^{1,2k}} - \sum t^{\alpha,p}\frac{\partial}{\partial t^{\alpha,2k-1+p}}\right.$$
$$\left. + \frac{\epsilon^2}{2}\sum_{p=0}^{2k-2}(-1)^p \eta^{\alpha\beta}\frac{\partial^2}{\partial t^{\alpha,p}\partial t^{\beta,2k-2-p}}\right) Z_{\mathbb{E}}$$

Let us denote $-\frac{B_{2k}}{(2k)!}s_{2k-1}$ by s_k, then we have

$$\frac{\partial Z_{\mathbb{E}}}{\partial s_k} = (L_{2k-1,2k}|_{t^{1,1}\to t^{1,1}-1})\, Z_{\mathbb{E}}, \quad Z_{\mathbb{E}}(t,0) = Z(t)$$

In what follows, we will denote the time shifted operators $L_{m,k}(t)|_{t^{1,1}\to t^{1,1}-1}$ simply by $L_{m,k}$.

Then we have

$$Z_{\mathbb{E}}(t;s) = e^{U(s)}Z(t), \quad U = \sum_{k\geq 1} s_k L_{2k-1,2k}$$

We define the operators

$$L_m(s) = e^{U(s)}L_{m,0}e^{-U(s)}, \quad m \geq -1$$

Theorem

1. *The operators* $L_m(s)$ *are linear combonations of the operators* $L_{2k+m-1,2k}, L_{2k+m-2,2k}, \ldots, L_{2k-1,2k} (k \geq 1)$ *and* $L_{m,0}$*, they satisfy the Virasoro commutation relation*

$$[L_i(s), L_j(s)] = (i-j)L_{i+j}(s)$$

2. *The partition function of the Hodge integrals* $Z_{\mathbb{E}}$ *satisfies the Virasoro constraints*

$$L_m(s)Z_{\mathbb{E}} = 0, \quad m \geq -1$$

Examples of the operators $L_m(s)$

$$L_{-1}(s) = L_{-1,0} - \frac{s_1}{2}$$

$$L_0(s) = L_{0,0} + \sum_{k\geq 1} \frac{s_k}{k} \left(\begin{array}{c} 2k \\ 2 \end{array} \right) L_{2k-1,2k}.$$

$$L_1(s) = L_{1,0} + \sum_{k\geq 1} \frac{s_k}{k} \left(\begin{array}{c} 2k \\ 2 \end{array} \right) L_{2k,2k}$$

$$+ \sum_{k_1,k_2\geq 1} \prod_{i=1}^{2} \frac{s_{k_i}}{k_i} \left\{ \begin{array}{c} 2(k_1+k_2) \\ 2k_1-1, 2k_2-1 \end{array} \right\}_{stirling} L_{2(k_1+k_2)-1,2(k_1+k_2)}.$$

6. Hodge integrals and integrable hierarchies

Using the genus zero Virasoro like constraints, we obtain a simple algorithm to solve the the equations

$$\frac{\partial Z_{\mathbb{E}}}{\partial s_k} = \left(L_{2k-1,2k}|_{t^{1,1}\to t^{1,1}-1}\right) Z_{\mathbb{E}}, \quad Z_{\mathbb{E}}(t,0) = Z(t)$$

to obtain

$$\mathcal{H}_0(t;s) = \mathcal{F}_0(t),$$

$$\mathcal{H}_1(t;s) = \mathcal{F}_1(t) - \frac{1}{2}\, s_1\, \eta^{\alpha\beta}\Omega_{\alpha,0;\beta,0}(v(t)),$$

$$\mathcal{H}_g(t;s) = \mathcal{F}_g(t) + \text{polynomial in } s_1,\dots,s_g$$

where the coefficients of the polynomials are rational functions of $v^\alpha(t) = \frac{\partial^2\mathcal{F}_0(t)}{\partial x\partial t_{\alpha,0}}, v^\alpha_x(t),\dots,\partial_x^{3g-3}v^\alpha(t)$.

One then obtain a quasi-Miura transformation

$$u^\alpha = \frac{\partial^2 \left(\mathcal{H}_0(t;s) + \epsilon^2 \mathcal{H}_1(t;s) + \dots\right)}{\partial x \partial t_{\alpha,0}}$$
$$= v^\alpha + \text{rational function of } v^\gamma, v^\gamma_x, \dots$$

The Principal Hierarchy satisfied by v^γ leads to the integrable hierarchy satisfied by $u^1, \dots, u^n$.

When the Frobenius manifold comes from the Gromov-Witten invariants of a smooth projective variety, the above mentioned algorithm (to represent the Hodge potential in terms of the Gromov-Witten potential) do no need the semi-simplicity condition.

Example: the target space X is a point

$$x = t^{1,0}, \quad v(t) = \frac{\partial^2 \mathcal{F}_0(t)}{\partial x \partial x}, \quad v_m = \partial_x^m v(t)$$

Then we have

$$\mathcal{H}_0(t; s) = \mathcal{F}_0(t)$$

$$\mathcal{H}_1(t; s) = \mathcal{F}_1(t) - \frac{1}{2} s_1 v = \frac{1}{24} \log v_x - \frac{1}{2} s_1 v$$

$$\mathcal{H}_2(t; s) = \mathcal{F}_2(t) + \left(\frac{11 v_2^2}{480 v_1^2} - \frac{v_3}{40 v_1} \right) s_1 + \frac{7}{40} v_2 s_1^2 - \frac{1}{10} v_1^2 s_1^3 - \frac{1}{48} v_1^2 s_2$$

$$\mathcal{H}_3(t, s) = \mathcal{F}_3 + \text{polynomial in } s_1, s_2, s_3$$

The deformed KdV equation

$$u(t) = \frac{\partial^2}{\partial x^2}\left(\mathcal{H}_0 + \epsilon^2\mathcal{H}_1 + \epsilon^4\mathcal{H}_2 + \dots\right)$$

$$= v + \epsilon^2\left(-\frac{1}{2}v_2 s_1 - \frac{v_2^2}{24 v_1^2} + \frac{v_3}{24 v_1}\right)$$

$$+\epsilon^4\left[\left(\frac{11 v_2^4}{80 v_1^4} - \frac{67 v_2^2 v_3}{240 v_1^3} + \frac{17 v_3^2}{240 v_1^2} + \frac{23 v_2 v_4}{240 v_1^2} - \frac{v_5}{40 v_1}\right) s_1 + \dots\right]$$

Denote $u_k = \partial_x^k u(t)$, then u satisfies the PDE

$$\frac{\partial u}{\partial t_1} = u u_x + \epsilon^2\left(\frac{u_{xxx}}{12} - s_1 u_x u_{xx}\right)$$

$$+\epsilon^4\left[-\frac{s_1}{60}u_5 + s_1^2\left(u_2 u_3 + \frac{1}{5}u_1 u_4\right) + s_1^3\left(-\frac{8}{5}u_1 u_2^2 - \frac{4}{5}u_1^3 u_3\right)\right.$$

$$\left. + s_2\left(-\frac{1}{3}u_1 u_2^2 - \frac{1}{6}u_1^2 u_3\right)\right] + \dots$$

The deformed KdV is a Hamiltonian system

$$\frac{\partial u}{\partial t_1} = \left(\partial_x - s_1\,\partial_x^3 + \frac{3}{5}\,s_1^2\,\partial_x^5 + \dots\right)\frac{\delta H}{\delta u}$$

The Hamiltonian

$$H = \int h(u, u_x, u_{xx}, \dots)dx$$

has the density

$$h = \frac{1}{6}u^3 + \epsilon^2\left(-\frac{1}{24} - \frac{1}{2}s_1\,u\right)u_x^2$$

$$+\epsilon^4\left[\left(-\frac{1}{5}s_1^3 - \frac{1}{24}s_2\right)uu_x^2u_{xx} + \frac{1}{30}\left(s_1 + 6s_1^2u\right)u_{xx}^2\right] + \dots$$

The normal form of the Hamiltonian system

Perform a Miura type transformation

$$w = u + \frac{1}{2}\epsilon^2 s_1 u_{xx}$$

$$+\epsilon^4 \left[\frac{1}{5}s_1^3\left(u_{xx}^2 + u_x u_{xxx}\right) + \frac{3}{42}s_1^2 u_{xxxx} + \frac{1}{24}s_1^2\left(u_{xx}^2 + u_x u_{xxx}\right)\right] + \dots$$

The deformed KdV is transformed to the form

$$\frac{\partial w}{\partial t_1} = \frac{\partial}{\partial x}\frac{\delta \tilde{H}}{\delta w}$$

$$\tilde{H} = \int \left(\frac{1}{6}w^3 - \frac{\epsilon^2}{24}w_x^2 - \frac{\epsilon^4}{120}s_1 w_{xx}^2 + \dots\right) dx$$

Conjecture

The integrable hierarchy that is associated to the Hodge integrals is equivalent, up to a Miura type transformation, to the normal form of the integrable deformation of the dispersionless KdV hierarchy that possesses a Hamiltonian structure and tau symmetry property.

KP hierarchy for $s_k = \frac{B_{2k}}{2k(2k-1)} c^{2(2k-1)}$, Kazarian 2008

Deformation of KdV up to ϵ^4 for special choice of s_1, s_2, Brini 2012

Under the same assumption on s_k, Buryak (2013) gave a nice description of the deformed KdV hiearchy .

Deformation of the dispersionless KdV hierarchy

The dispersionless KdV hierarchy

$$\frac{\partial w}{\partial t^p} = \frac{1}{p!} w^p w_x, \quad p \geq 0$$

The KdV hierarchy is a particular deformation

$$\frac{\partial w}{\partial t_0} = w_x, \quad \frac{\partial w}{\partial t_1} = w w_x + \frac{\epsilon^2}{12} w_{xxx}\,,$$

$$\frac{\partial w}{\partial t_p} = \frac{1}{2p+1}\left(\frac{1}{4}\partial_x^2 + 2w + w_x \partial_x^{-1}\right)\frac{\partial w}{\partial t_{p-1}}, \quad p \geq 2$$

Hamiltonian deformations of the dispersionless KdV hierarchy

The dispersion KdV hierarchy is Hamiltonian

$$\frac{\partial w}{\partial t_p} = \frac{\partial}{\partial x}\frac{\delta H_p}{\delta w}, \quad \text{with } H_p = \int \frac{w^{p+2}(x)}{(p+2)!}\,dx, \quad p \geq 0$$

Hamiltonians deformations

$$\frac{\partial w}{\partial t_p} = \frac{\partial}{\partial x}\frac{\delta \hat{H}_p}{\delta w}, \quad \text{with } \hat{H}_p = \int h_p(w, w_x, \ldots)dx, \quad p \geq 0$$

Here h_p have the form ($h_{p,g}$ are polynomials in $w_x, w_{xx}, \ldots$)

$$h_p = \frac{1}{(p+2)!}w^{p+2} + \sum_{g\geq 1}\epsilon^{2g}h_{p,g}(w, w_x, \ldots, w^{(2g)})$$

Requirement on the deformation

- The deformed flows are commutative, i.e.

$$\frac{\partial}{\partial t_p}\left(\frac{\partial w}{\partial t_q}\right) = \frac{\partial}{\partial t_q}\left(\frac{\partial w}{\partial t_p}\right), \quad p \neq q \geq 0$$

- The deformed hierarchy has the tau symmetry property

$$\frac{\partial h_{p-1}}{\partial t_q} = \frac{\partial h_{q-1}}{\partial t_p}, \quad p \neq q \geq 0$$

- The Hamiltonian density can be normalized to the form

$$h_1 = \frac{w^3}{6} - \frac{\epsilon^2}{24} a_0(w) w_1^2 + \epsilon^4 a_1(w) w_2^2 + \epsilon^6 (a_2(w) w_2^3 + a_4(w) w_3^2)$$

$$+\epsilon^8 \left(a_3(w) w_2^4 + a_5(w) w_2 w_3^2 + a_6(w) w_4^2\right) + \dots$$

Conjecture

The deformations of the dispersionless KdV hierarchy that satisfies the above requirement if only if the functions $a_1(w), a_2(w), a_3(w), a_5(w), \ldots$ as coefficients of $\partial_x^{2g} w$ are constants, and other functions $a_k(w)$ are uniquely determined by these constant parameters by the relations

$$a_4 = -\frac{240\, a_1^2}{7\, a_0}, \quad a_5 = -\frac{2376\, a_1\, a_2}{7\, a_0},$$

$$a_6 = \frac{387072\, a_0^3\, a_2 + 16721510400\, a_1^3}{13547520\, a_0^2}, \ldots$$

Note that we can assume $a_0 \neq 0$, and by rescaling ϵ we can assume that $a_0 = 1$.

The relation between the parameters s_i and a_i

$$s_1 = -120a_1,$$

$$s_2 = 8294400a_1^3 - 1728a_2,$$

$$s_3 = -\frac{34398535680000}{7}a_1^5 + \frac{11943936000}{7}a_1^2 a_2 - 34560a_3.$$

Thanks