Dispersionless integrable systems in 3D and Einstein-Weyl geometry

Eugene Ferapontov

Department of Mathematical Sciences, Loughborough University, UK

E.V.Ferapontov@lboro.ac.uk

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Hamiltonian PDEs, Frobenius manifiolds and Deligne-Mumford moduli spaces

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Plan:

- Formal linearisation: applications
- Einstein-Weyl geometry
- Integrability in 3D and Einstein-Weyl geometry
- Integrability in 4D and self-duality

Based on:

E.V. Ferapontov and B. Kruglikov, Dispersionless integrable systems in 3D and Einstein-Weyl geometry, arXiv:1208.2728v3.

Formal linearisation

Given a PDE

$$F(x^i, u, u_{x^i}, u_{x^i x^j}, \dots) = 0,$$

its formal linearisation results upon setting $u \to u + \epsilon v$, and keeping terms of the order ϵ. This leads to a linear PDE for v,

$$\ell_F(v) = 0,$$

where ℓ_F is the operator of formal linearisation,

$$\ell_F = F_u + F_{u_{x^i}} \mathcal{D}_{x^i} + F_{u_{x^i x^j}} \mathcal{D}_{x^i} \mathcal{D}_{x^j} + \dots .$$

Example

dKP equation: $u_{xt} - (uu_x)_x - u_{yy} = 0$.

linearised dKP: $v_{xt} - (uv)_{xx} - v_{yy} = 0$.

Applications of formal linearisation

- *Stability analysis*
- *Symmetries, recursion operators*
- *Contact invariants of ODEs, generalised Laplace invariants of (Darboux integrable) Monge-Ampère equations*
- *Integrability of ODEs can be seen from the monodromy group of linearised equations*

Can one read the integrability of a given PDE off the geometry of its formal linearisation?

Yes, for broad classes of 3D dispersionless second order PDEs.

Types of PDEs studied:

Quasilinear wave equations:

$$f_{11}u_{xx} + f_{22}u_{yy} + f_{33}u_{tt} + 2f_{12}u_{xy} + 2f_{13}u_{xt} + 2f_{23}u_{yt} = 0.$$

Hirota-type equations:

$$F(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt}) = 0.$$

Equations possessing the 'central quadric ansatz':

$$(a(u))_{xx} + (b(u))_{yy} + (c(u))_{tt} + 2(p(u))_{xy} + 2(q(u))_{xt} + 2(r(u))_{yt} = 0.$$

The corresponding formal linearisations are second order linear PDEs. On every solution, their symbols define conformal structures (which depend on a solution). Which conformal geometries correspond to integrable PDEs? Which conformal geometries should be regarded as 'integrable'?

Einstein-Weyl geometry

This is a triple $(\mathbb{D},\ g,\ \omega)$ where $\mathbb{D}$ is a symmetric connection, g is a conformal structure and ω is a covector such that

$$\mathbb{D}_k g_{ij} = \omega_k g_{ij}, \quad R_{(ij)} = \Lambda g_{ij}.$$

Here $R_{(ij)}$ is the symmetrised Ricci tensor of $\mathbb{D}$, and Λ is some function (the first set of equations defines $\mathbb{D}$ uniquely, so it is sufficient to specify g and ω only).

Theorem (E. Cartan, 1941): The triple $(\mathbb{D},\ g,\ \omega)$ satisfies the Einstein-Weyl equations if and only if there exists a two-parameter family of surfaces which are totally geodesic with respect to $\mathbb{D}$, and null with respect to g.

Einstein-Weyl equations are integrable (Hitchin, 1980).

Main results

Theorem 1. A second order PDE is linearisable (by a transformation from the natural equivalence group) if and only if the conformal structure g is conformally flat on every solution (has vanishing Cotton tensor).

Theorem 2. A second order PDE is integrable by the method of hydrodynamic reductions if and only if, on every solution, the conformal structure g satisfies the Einstein-Weyl equations, with the covector $\omega = \omega_s dx^s$ given by the formula

$$\omega_s = 2g_{sj}\mathcal{D}_{x^k}(g^{jk}) + \mathcal{D}_{x^s}(\ln\det g_{ij}).$$

The corresponding two-parameter family of totally geodesic null surfaces is provided by the corresponding dispersionless Lax pair.

Example of dKP

As an illustration let us consider the dKP equation,

$$u_{xt} - (uu_x)_x - u_{yy} = 0.$$

The corresponding Einstein-Weyl structure is provided by the conformal metric $g = 4dxdt - dy^2 + 4udt^2$ and the covector $\omega = -4u_x dt$ (Dunajski, Mason, Tod). One can verify that they satisfy the Einstein-Weyl conditions. The dispersionless Lax pair is given by vector fields

$$X = \partial_y - \lambda\partial_x + u_x\partial_\lambda, \quad Y = \partial_t - (\lambda^2 + u)\partial_x + (u_x\lambda + u_y)\partial_\lambda,$$

such that the commutativity condition, $[X, Y] = 0$, is equivalent to dKP. Projecting integral surfaces of the distribution spanned by X, Y in the extended space x, y, t, λ to the space of independent variables x, y, t, one obtains a two-parameter family of surfaces which are null with respect to g, and totally geodesic in the Weyl connection $\mathbb{D}$ specified by g and ω.

Equations possessing the central quadric ansatz

Based on

E.V. Ferapontov, B. Huard and A. Zhang, On the central quadric ansatz: integrable models and Painlevé reductions, J. Phys. A: Math. Theor. **45** *(2012) 195204; arXiv:1201.5061.*

Let us consider PDEs of the form

$$(a(u))_{xx} + (b(u))_{yy} + (c(u))_{tt} + 2(p(u))_{xy} + 2(q(u))_{xt} + 2(r(u))_{yt} = 0.$$

Equations of this type possess solutions $u(x, y, t)$ in implicit form,

$$(x, y, t)M(u)(x, y, t)^T = 1.$$

Level surfaces $u = const$ are central quadrics in the space of independent variables x, y, t. The 3×3 matrix $M(u)$ satisfies an ODE which, in integrable cases, reduces to one of the Painlevé equations.

Which PDEs of the above form are integrable?

Classification result

Given a PDE of the form

$$(a(u))_{xx} + (b(u))_{yy} + (c(u))_{tt} + 2(p(u))_{xy} + 2(q(u))_{xt} + 2(r(u))_{yt} = 0,$$

we introduce the 3×3 symmetric matrix of coefficients,

$$V(u) = \left(\begin{array}{ccc} a' & p' & q' \\ p' & b' & r' \\ q' & r' & c' \end{array} \right).$$

The method of hydrodynamic reductions consists of seeking solutions in the form $u = u(R^1, \dots, R^N)$ where the phases $R^i(x, y, t)$ are required to satisfy a pair of compatible systems of hydrodynamic type,

$$R^i_t = \lambda^i(R) R^i_x, \quad R^i_y = \mu^i(R) R^i_x.$$

The requirement of the existence of such solutions imposes strong constraints on V:

The five canonical forms

Theorem The integrability by the method of hydrodynamic reductions, which is equivalent to the Einstein-Weyl property of the symbol of formal linearisation, implies that $V(u)$ satisfies the constraint

$$V'' = (\ln \det V)' V' + kV,$$

for some scalar function k. This gives five canonical forms of integrable PDEs possessing the central quadric ansatz:

$$u_{xx} + u_{yy} - [\ln(1 - e^u)]_{yy} - [\ln(1 - e^u)]_{tt} = 0,$$

$$u_{xx} + u_{yy} - (e^u)_{tt} = 0,$$

$$(e^u - u)_{xx} + 2u_{xy} + (e^u)_{tt} = 0,$$

$$u_{xt} - (uu_x)_x - u_{yy} = 0,$$

$$(u^2)_{xy} + u_{yy} + 2u_{xt} = 0.$$

Einstein-Weyl structures

Equation 1: $u_{xx} + u_{yy} - (\ln(e^u - 1))_{yy} - (\ln(e^u - 1))_{tt} = 0$

(gauge-invariant dispersionless Hirota equation).

Conformal structure: $g = dx^2 + (1 - e^u)dy^2 + (e^{-u} - 1)dt^2$.

Covector: $\omega = \frac{e^u+1}{e^u-1} u_x dx - u_y dy + u_t dt$.

Equation 2: $u_{xx} + u_{yy} - (e^u)_{tt} = 0$ (BF equation).

Conformal structure: $g = dx^2 + dy^2 - e^{-u}dt^2$.

Covector: $\omega = -u_x dx - u_y dy + u_t dt$.

This Einstein-Weyl structure was obtained by Ward.

Equation 3: $(e^u - u)_{xx} + 2u_{xy} + (e^u)_{tt} = 0$.

Conformal structure: $g = 2dxdy + (1 - e^u)dy^2 + e^{-u}dt^2$.

Covector: $\omega = -u_x dx + (2e^u u_x - u_y)dy + u_t dt$.

Equation 4: $u_{xt} - (uu_x)_x - u_{yy} = 0$ (dKP equation).

Conformal structure: $g = 4dxdt - dy^2 + 4udt^2$.

Covector: $\omega = -4u_x dt$.

This Einstein-Weyl structure was obtained by Dunajski, Mason and Tod.

Equation 5: $(u^2)_{xy} + u_{yy} + 2u_{xt} = 0$.

Conformal structure: $g = 2dxdt + dy^2 - 2udydt + u^2dt^2$.

Covector: $\omega = 2u_x dy + 2(u_y - uu_x)dt$.

Integrability in 4D and self-duality

Integrable equations of Monge-Ampére type in 4D were classified in

B. Doubrov and E.V. Ferapontov, On the integrability of symplectic Monge-Ampère equations, Journal of Geometry and Physics **60** *(2010) 1604-1616.*

- $u_{11} - u_{22} - u_{33} - u_{44} = 0$ (linear wave equation)
- $u_{13} + u_{24} + u_{11}u_{22} - u_{12}^2 = 0$ (second heavenly equation)
- $u_{13} = u_{12}u_{44} - u_{14}u_{24}$ (modified heavenly equation)
- $u_{13}u_{24} - u_{14}u_{23} = 1$ (first heavenly equation)
- $u_{11} + u_{22} + u_{13}u_{24} - u_{14}u_{23} = 0$ (Husain equation)
- $\alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{14}u_{23} = 0$ (general heavenly equation),
  $\alpha + \beta + \gamma = 0$.

Conjecture: A 4D second order dispersionless PDE is integrable if and only if the corresponding conformal structure is self-dual on every solution.

Questions:

- Contact-invariant approach to dispersionless integrability?
- Higher order PDEs and higher Einstein-Weyl geometry?
- Integrability in dimensions higher than four?