Computational Approach to Riemann Surfaces

C. Klein (Dijon)
with
J. Frauendiener (Dunedin)

Theta-functional solutions to the Kadomtsev-Petviashvili equation

- $$3u_{yy} + \partial_x(6uu_x + u_{xxx} - 4u_t) = 0$$

  weakly two-dimensional waves in shallow water

- almost periodic solutions in terms of theta functions on arbitrary compact Riemann surfaces (Krichever 1978)

  $$u = 2\partial_x^2 \ln \Theta(\mathbf{U}x + \mathbf{V}y + \mathbf{W}t + \mathbf{D}) + 2c$$

- $\mathbf{D} \in \mathbb{R}^g$ arbitrary

- Riemann theta function

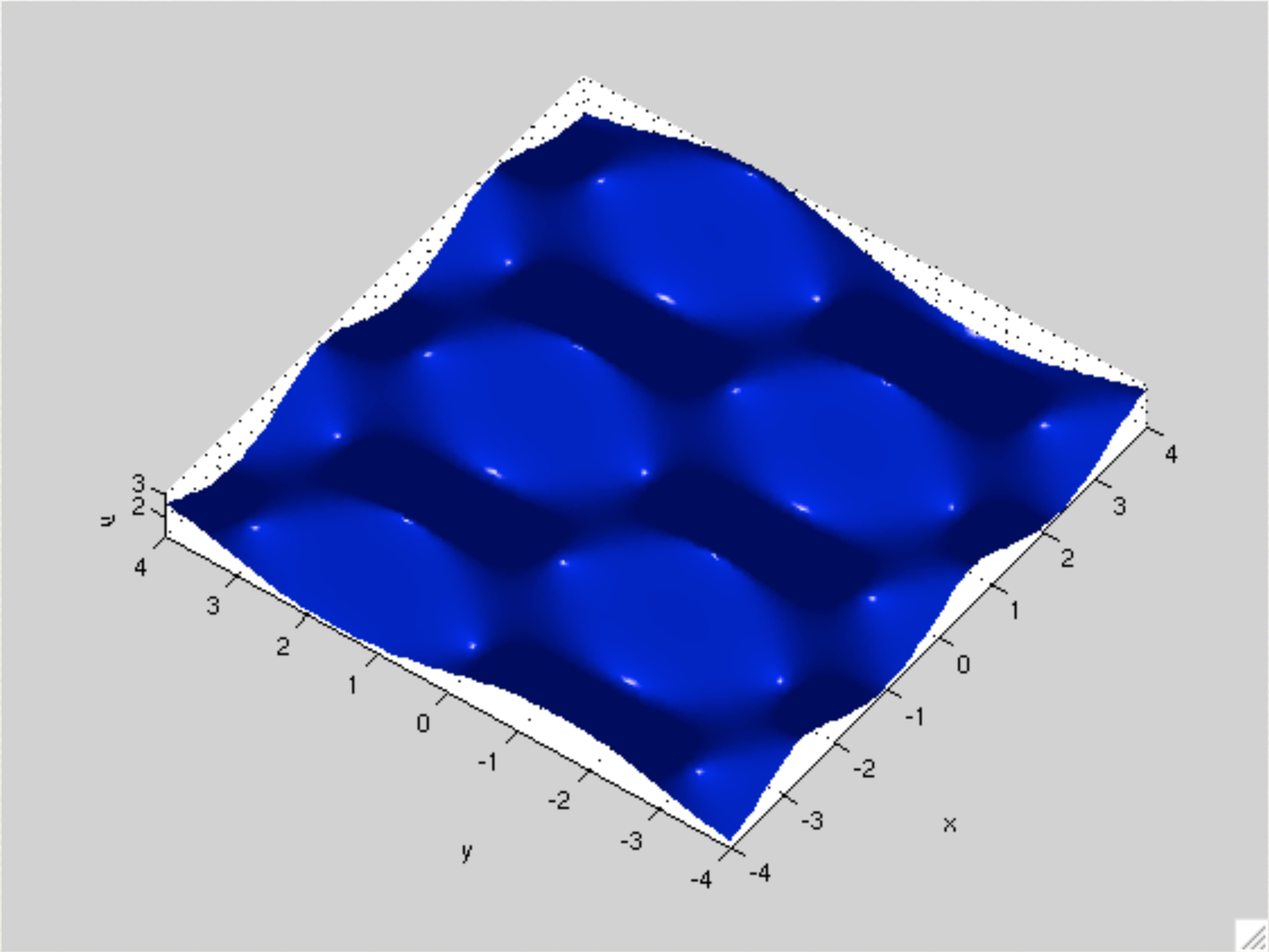
$$\Theta(\mathbf{x}|\mathbf{B}) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp\left\{\mathrm{i}\pi\langle\mathbf{Bn},\mathbf{n}\rangle + 2\pi\mathrm{i}\langle\mathbf{n},\mathbf{x}\rangle\right\}$$

- $\mathbf{B}$ Riemann matrix, matrix of b-periods of the holomorphic differentials

- $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, vectors expressible in terms of derivatives of the holomorphic differentials, c constant expressible in terms of theta functions

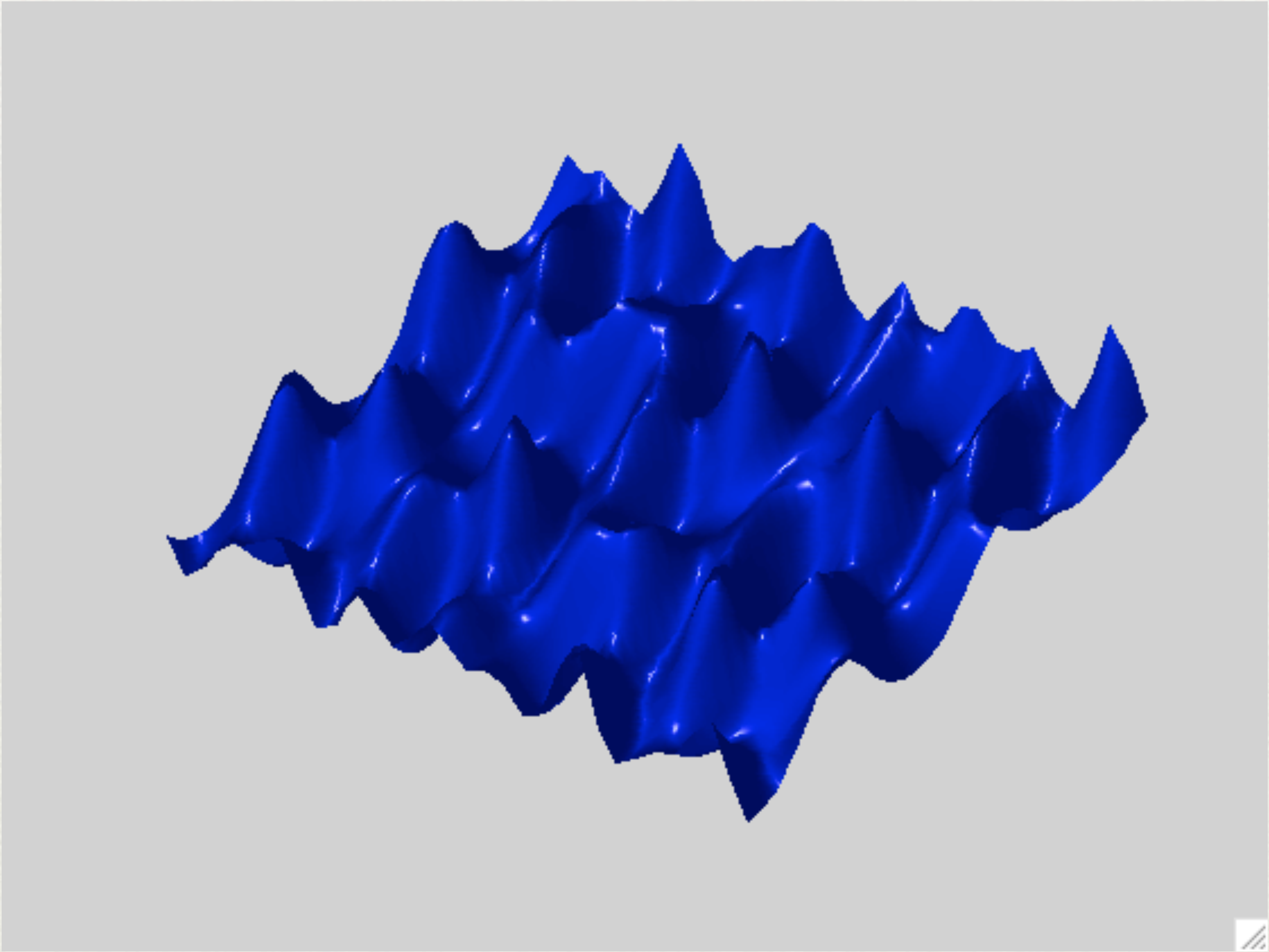
Hyperelliptic solutions (g=2)

Hyperelliptic solutions (g=2)



Hyperelliptic solution (g=4)

Hyperelliptic solution (g=4)

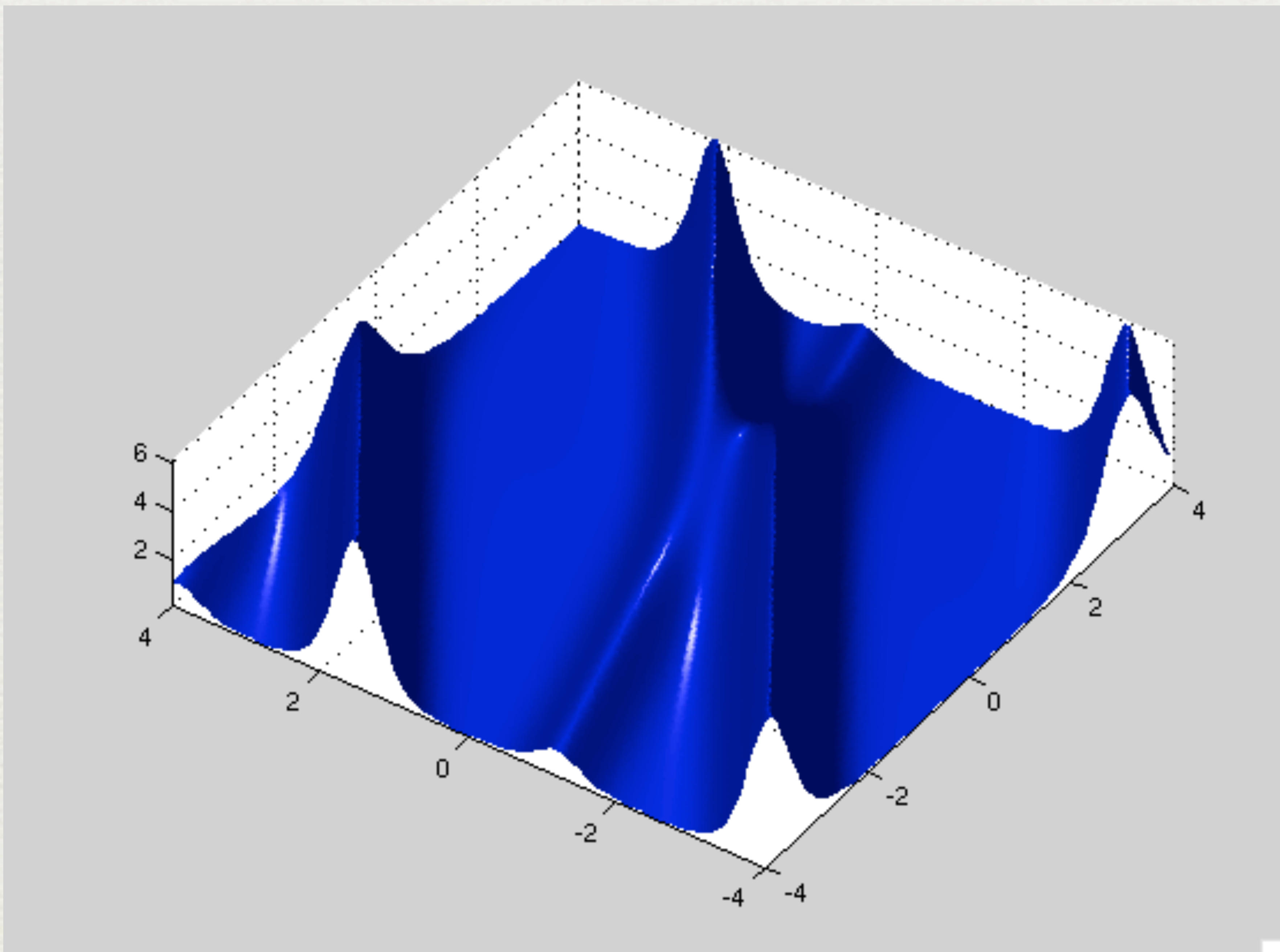


(Line)Solitons (localized in one direction), 2-soliton

- branch points coincide pairwise, surface of genus 0 in the limit

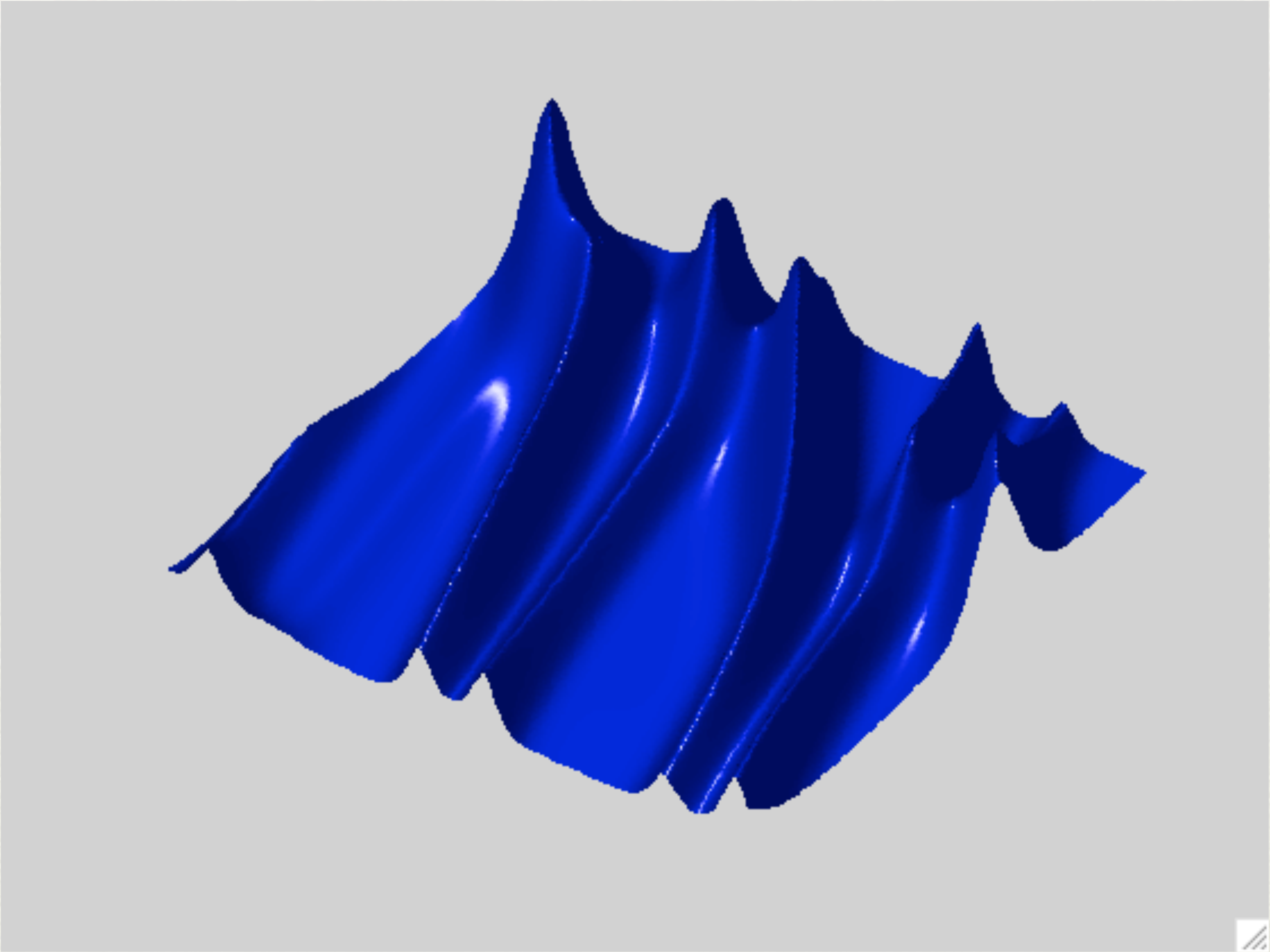
(Line)Solitons (localized in one direction), 2-soliton

- branch points coincide pairwise, surface of genus 0 in the limit



(Line)Solitons, 4-soliton

(Line)Solitons, 4-soliton



Symbolic vs. numerical

- Deconinck, v. Hoeij, Patterson: *algcurves* package in Maple (2001)
- symbolic approach, exact expressions (e.g. RootOf(x^2-2)) manipulated and numerically evaluated, in principle infinite precision
- Frauendiener, K.: fully numeric approach (*floating point)*, hyperelliptic curves (1998), much more rapid, allows study of families of curves and of more complicated curves

Outline

- Riemann surfaces and algebraic curves
- Branch points and singular points
- Monodromy and homology
- Puiseux expansions and holomorphic differentials
- Real Riemann surfaces
- Hyperelliptic surfaces
- Performance tests and examples

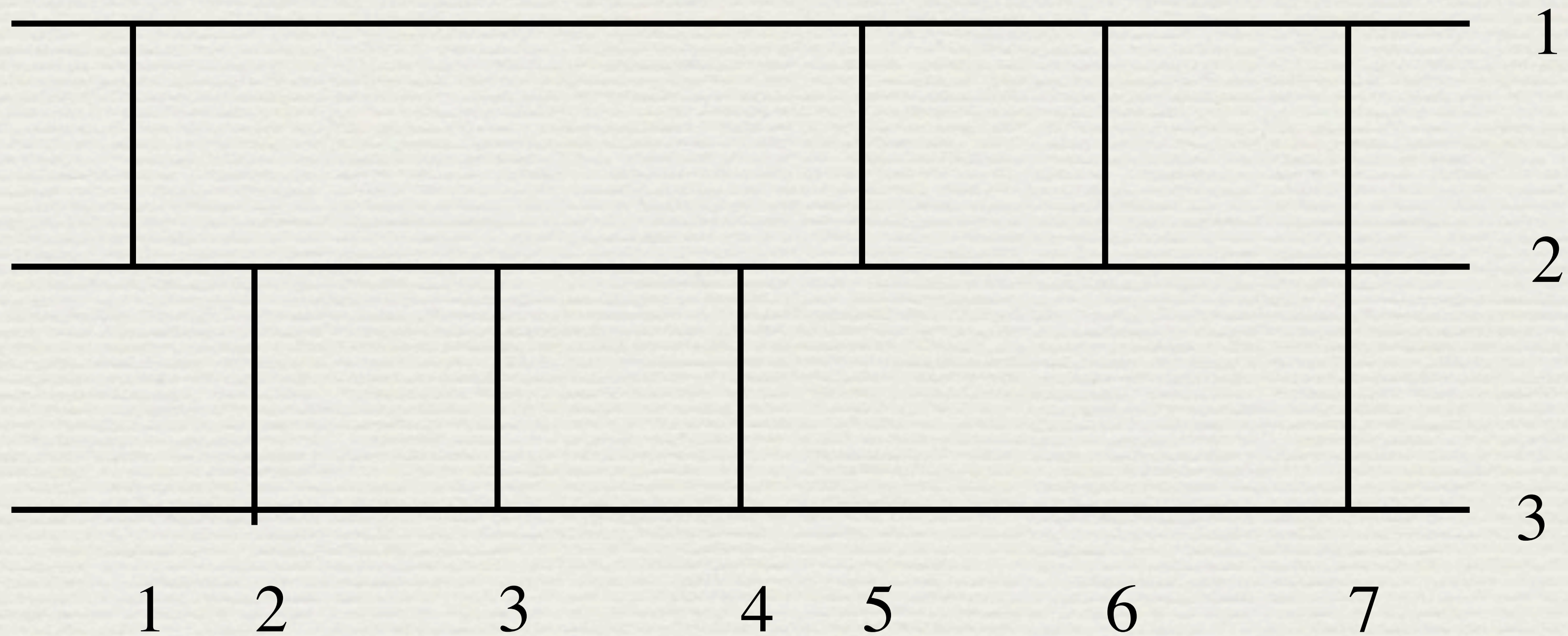
Riemann surfaces

- Definition: A Riemann surface is a connected one-dimensional complex analytic manifold, i.e., a connected two-dimensional real manifold R with a complex structure Σ on it
- Theorem: All compact Riemann surfaces can be described as compactifications of non-singular algebraic curves

Algebraic curves

- Definition: plane algebraic curve C subset in $\mathbb{C}^2$,
  $C = \{(x, y) \in \mathbb{C}^2 | f(x, y) = 0)$,

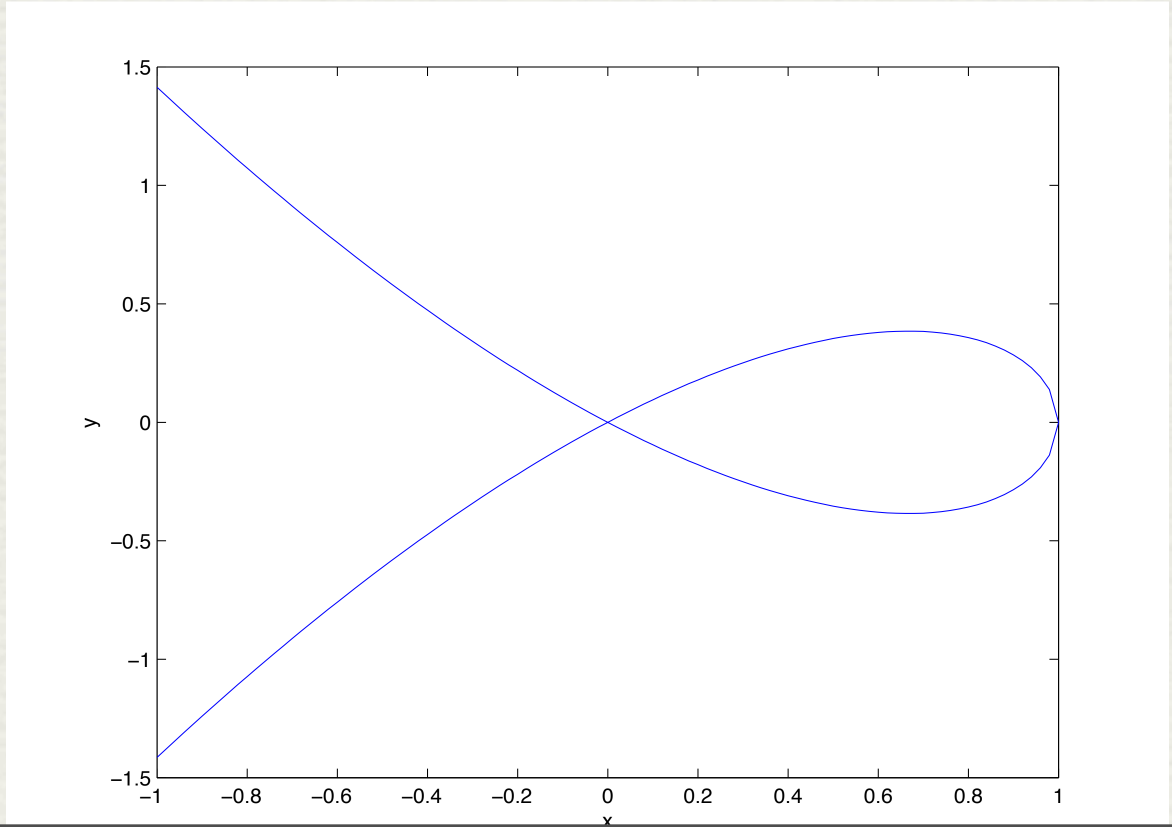
$$f(x,y) = \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} x^i y^j = \sum_{j=0}^{N} a_j(x) y^j$$



Critical points

- general position: N distinct solutions y_n for given x, N sheets of the Riemann surface
- Implicit function theorem: unique solution to $f(x,y)=0$ in vicinity of solution (x_0,y_0) if $f_y(x_0,y_0)\neq 0$
- branch point: $f(x_0,y_0)=f_y(x_0,y_0)=0$, but $f_x(x_0,y_0)\neq 0$
  singular point: $f(x_0,y_0)=f_y(x_0,y_0)=f_x(x_0,y_0)=0$
- critical points given by the resultant $R(x)$ of $Nf - f_y y$ and f_y

simple double point:
$$y^2+x^3-x^2=0$$



Resultant

- resultant of $Nf - f_y y$ and f_y, $2N \times 2N$ Sylvester determinant

$$R(x) =$$

$$\begin{pmatrix}
a_{N-1} & 2a_{N-2} & \dots & Na_0 & 0 & \dots & \dots & 0 \\
0 & a_{N-1} & 2a_{N-2} & \dots & Na_0 & 0 & \dots & 0 \\
\vdots & \ddots & & & & & \ddots & \vdots \\
0 & \dots & \dots & 0 & a_{N-1} & 2a_{N-2} & \dots & Na_0 \\
Na_{N-1} & (N-1)a_{N-2} & \dots & a_1 & 0 & \dots & \dots & 0 \\
0 & Na_{N-1} & (N-1)a_{N-2} & \dots & a_1 & 0 & \dots & 0 \\
\vdots & \ddots & & & & & \ddots & \vdots \\
0 & \dots & \dots & 0 & Na_{N-1} & (N-1)a_{N-2} & \dots & a_1
\end{pmatrix}$$

Numerical root finding

- construct companion matrix (has $R(x)$ as the characteristic polynomial), find eigenvalues with machine precision
- multiple zeros are not found with machine precision, ex. $y^7 = x(x-1)^2$ Klein curve, $R(x) = x^6(x-1)^{12}$, `roots(R(x))` returns the following cluster of roots

```
1.1053 + 0.0297i
1.1053 - 0.0297i
1.0736 + 0.0790i
1.0736 - 0.0790i
1.0224 + 0.1032i
1.0224 - 0.1032i
0.9686 + 0.0980i
0.9686 - 0.0980i
0.9264 + 0.0686i
0.9264 - 0.0686i
0.9037 + 0.0245i
0.9037 - 0.0245i,
```

polynomial root finding

- badly conditioned problem
- Zeng: *multroot package* for multiple roots (Newton iteration, minimize error by choice of multiplicity structure)
- resultant high order polynomial, therefore direct Newton iteration in x and y. Initial iterates from resultant with respect to x and y, pairing
- *en∂game* for higher order zeros

Singularities

- multiple roots are tested for vanishing $f_x(x, y)$
- infinity: homogeneous coordinates X, Y, Z
  via $x = X/Z$, $y = Y/Z$

$$F(X, Y, Z) = Z^d f(X/Z, Y/Z) = 0$$

  infinite points: $Z = 0$, finite points: $Z = 1$

- Singular points at infinity:
  $F_X(X, Y, 0) = F_Y(X, Y, 0) = F_Z(X, Y, 0) = 0$

Example

- curve

$$f(x, y) = y^3 + 2x^3y - x^7 = 0\ ,$$

- finite branch points

```
bpoints =
      -0.3197 - 0.9839i
       0.8370 - 0.6081i
      -1.0346
       0
       0.8370 + 0.6081i
      -0.3197 + 0.9839i
```

- singularities,

sing =

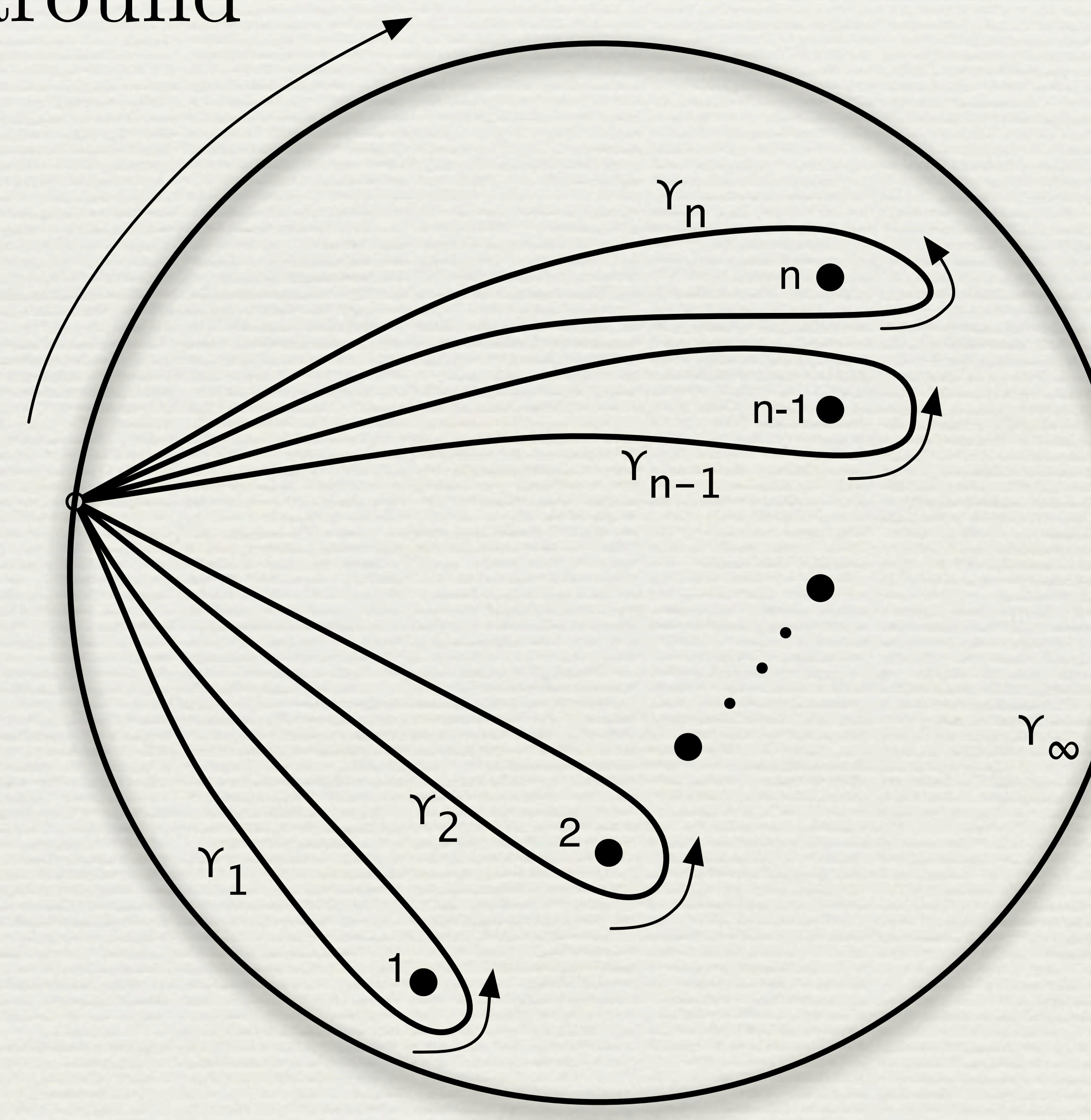
X	Y	Z	
0	0	1	4
0	1	0	9

corresponding to $x = y = 0$ and $Y = 1$, $X = Z = 0$

Fundamental group

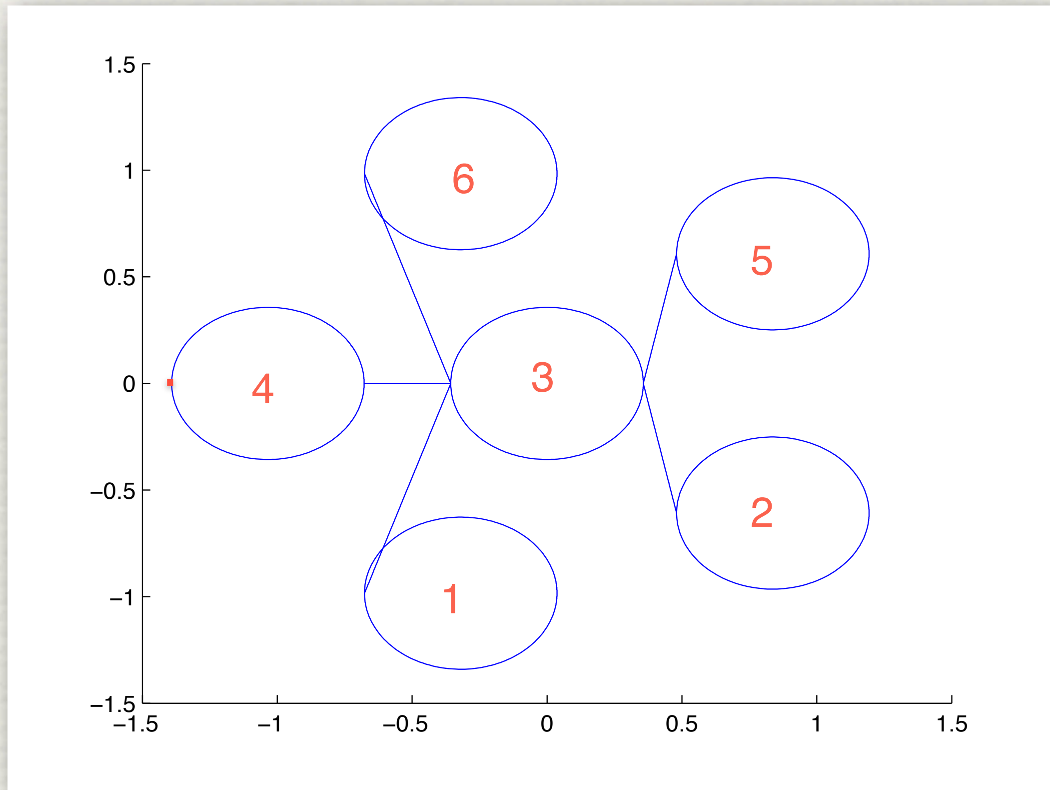
- branching structure at critical points, lift closed contours in the base around points $b_1, \ldots, b_n$ to the covering
- generators $\{\gamma_k\}_{k=1}^n$ of fundamental group $\pi_1(\mathbb{CP}^1 \backslash \{b_1, \ldots, b_n\})$

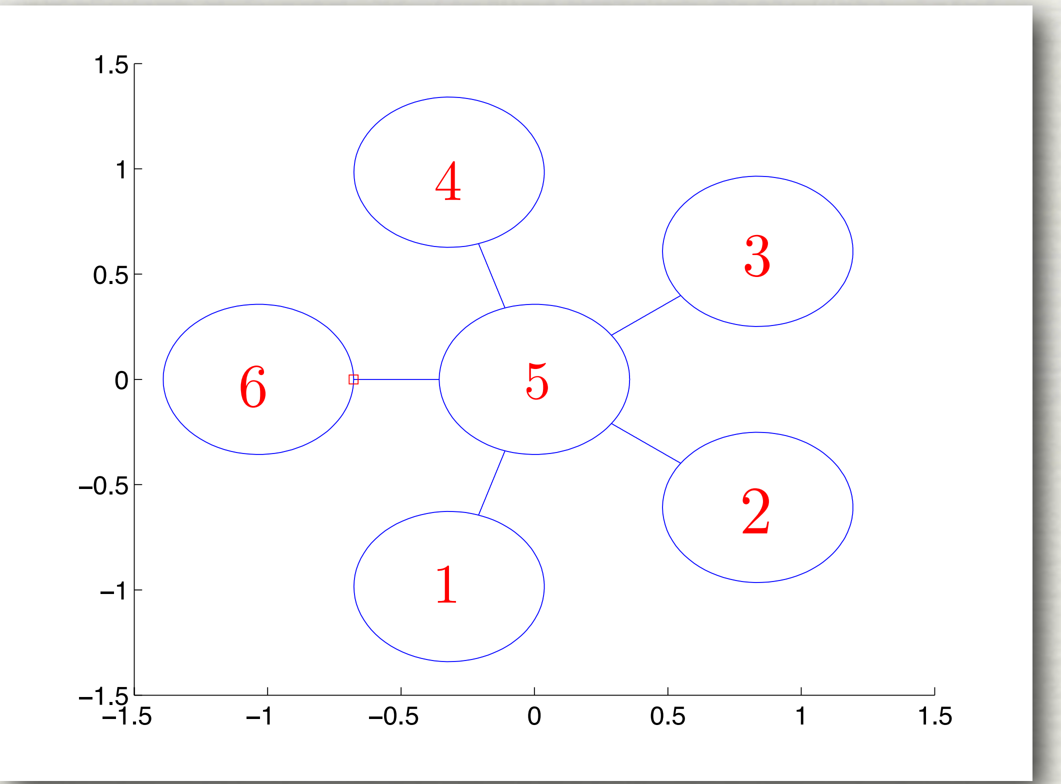
  $\gamma_1 \gamma_2 \ldots \gamma_n \gamma_\infty = \mathrm{id}$



Minimal spanning tree

- Maple: halfcircles around critical points, deformation of connecting paths
- shortest integration paths: start with critical point close to the base, choose point with minimal distance, iterate (Frauendiener, K, Shramchenko 2011)
- ex:





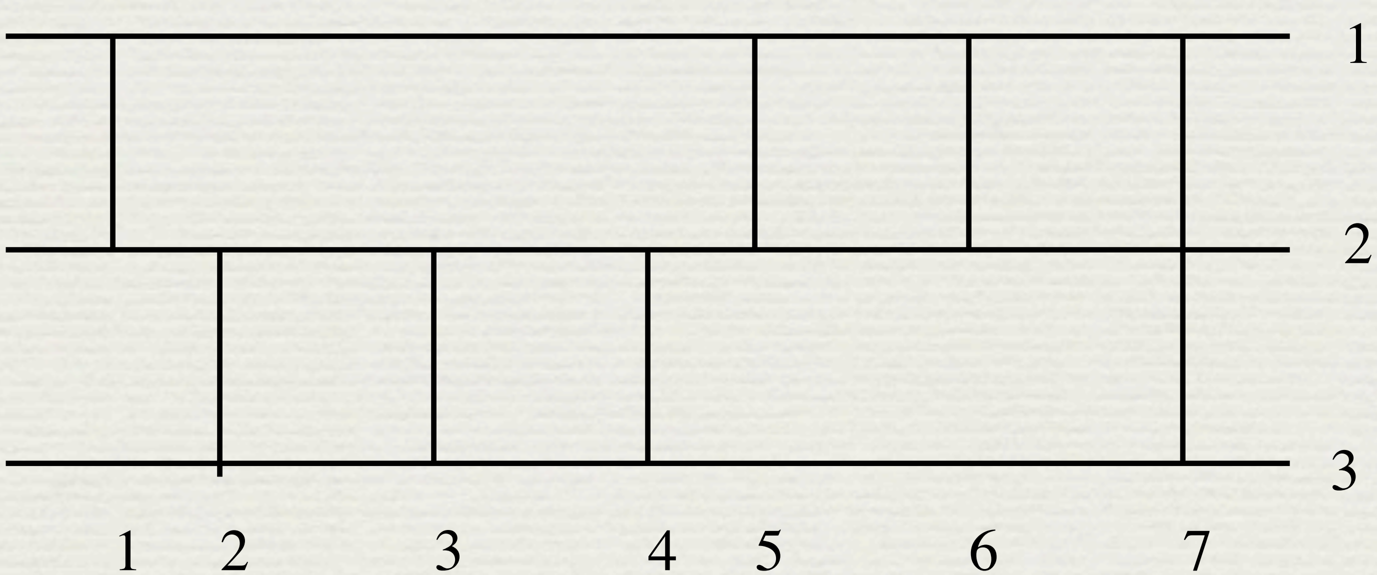
Monodromies

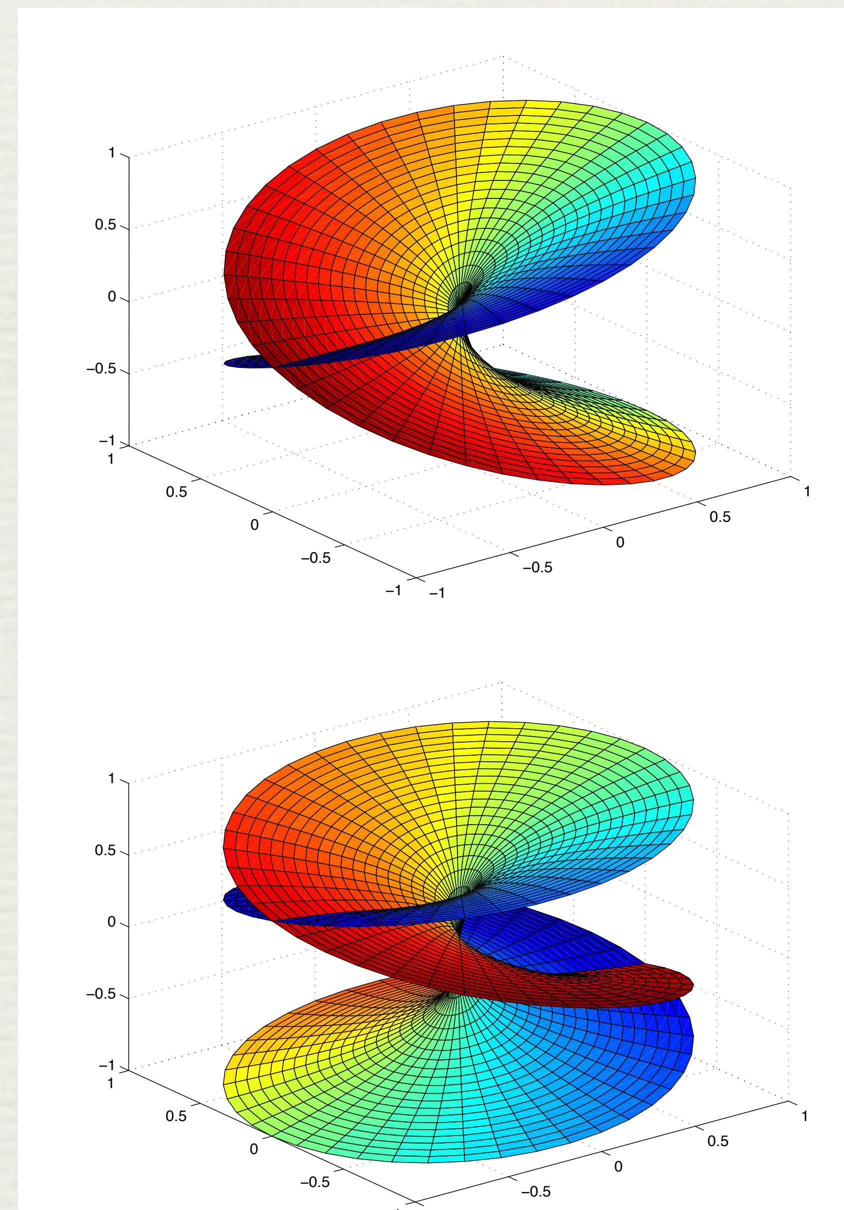
- analytic continuation along a generator: sheets can change
- monodromy at infinity follows from condition on generators

- ex.:

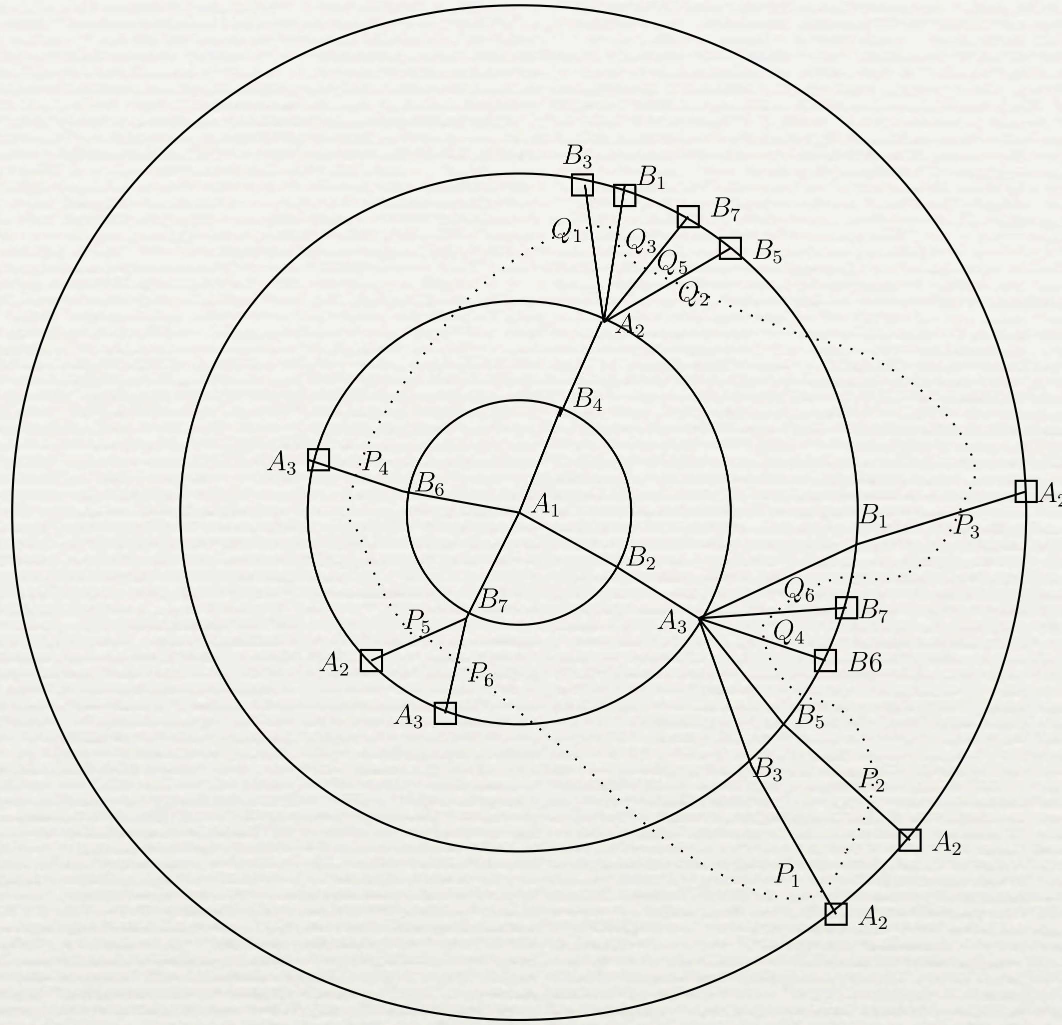
```
Mon =

 2         1         1         1         2         2         2
 1         3         3         3         1         1         3
 3         2         2         2         3         3         1
```





Homology



- Tretkoff-Tretkoff algorithm: Riemann surface connected, planar tree for given monodromies
- *2g+N-1* closed contours built from the generators of the fundamental group, with known intersection numbers
- canonical basis of the homology:

$$a_i \circ b_j = -b_j \circ a_i = \delta_{ij} \quad i, j = 1, \ldots, g$$

Puiseux expansion

- desingularization: atlas of local coordinates to identify all sheets in the vicinity of the singularity

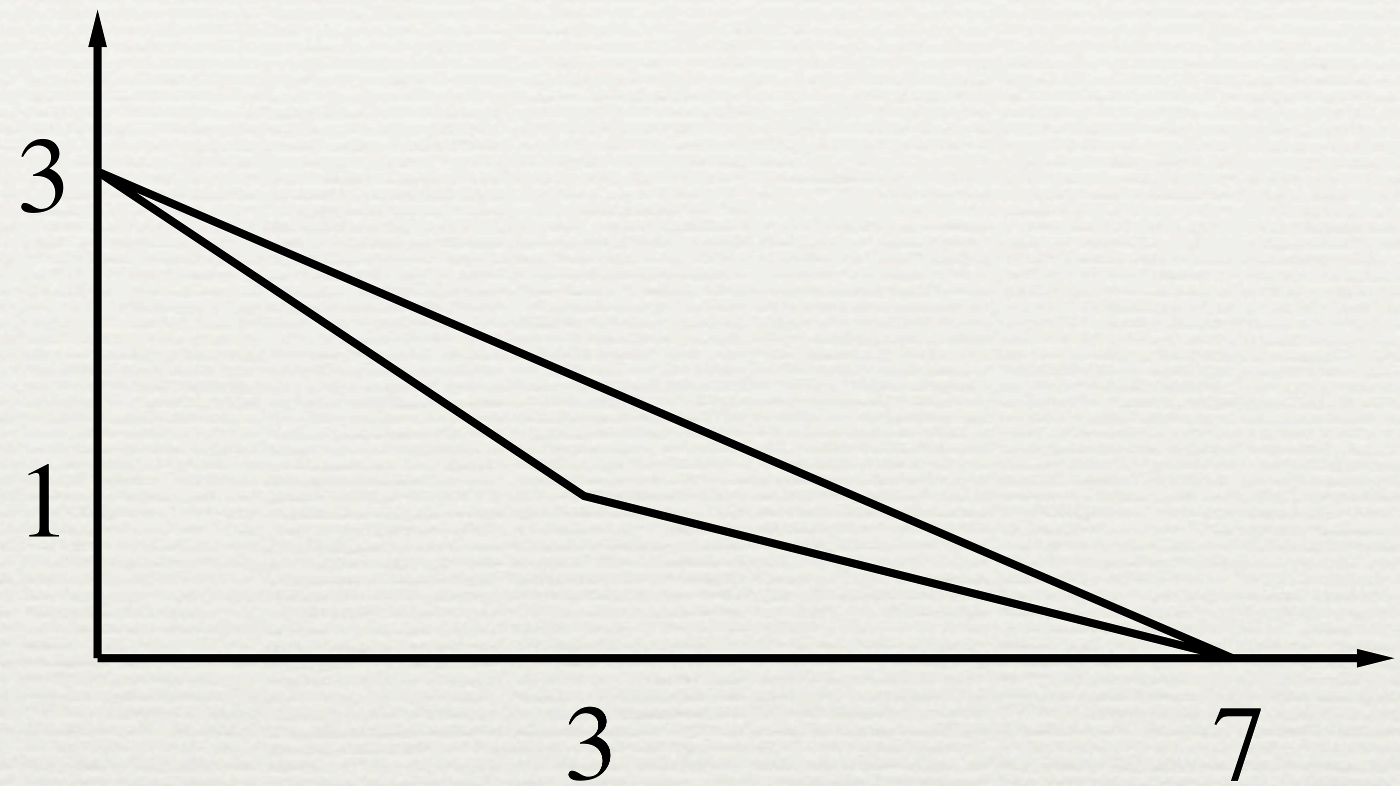
- $y^2 = x$, no Taylor expansion $y(x)$ near $(0,0)$, Puiseux expansion

$$x = t^r \ , \quad y = \alpha_1 t^{s_1} + ...$$

  $r, s_1, \ldots \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$ for $i = 1, 2, \ldots$

- $y = 0$ zero of order m for $f(0, y) = 0$, m inequivalent expansions needed to identify all sheets, *singular part*

Newton polygon



- ex.: $f(x,y) = y^3 + 2x^3y - x^7 = 0$

```
PuiExp{1} =
       2.0000             3.0000                   0 + 1.4142i
       2.0000             3.0000                   0 - 1.4142i
       1.0000             4.0000                   0.5000
PuiExp{2} =
       4.0000             7.0000                  -1.0000
       4.0000             7.0000                   0 + 1.0000i
       4.0000             7.0000                   0 - 1.0000i
       4.0000             7.0000                   1.0000
```

`PuiExp`$\{1\}$ for $(0,0)$ ($[0,0,1]$), `PuiExp`$\{2\}$ for infinity ($[0,1,0]$)

Holomorphic 1-forms

- holomorphic in each coordinate chart, g-dimensional space
- Noether:

$$\omega_k = \frac{P_k(x,y)}{f_y(x,y)} dx \ ,$$

adjoint polynomials $P_k(x,y) = \sum_{i+j \leq d-3} c_{ij}^{(k)} x^i y^j$,
degree at most $d-3$ in x and y ($d = \max(i+j)$ for $a_{ij} \neq 0$)

- singular point P: δ_P conditions via Puiseux expansions
- infinity: homogeneous coordinates
- ex.: $f(x,y) = y^3 + 2x^3 y - x^7 = 0$

$$\omega_1 = \frac{x^3}{3y^2 + 2x^3}, \quad \omega_2 = \frac{xy}{3y^2 + 2x^3}$$

Cauchy integral approach

- numerical problem: cancellation errors, ex. $\frac{e^x-1}{x}$ for $x \to 0$
- Cauchy formula

$$f(t) = \frac{1}{2\pi i} \int_\gamma \frac{f(t')}{t'-t} dt'$$

- closed contours around critical points identified via monodromy group
- series in t for holomorphic f $(|t| < |t'|)$

$$f(t) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} t^n \int_\gamma f(t') \frac{dt'}{(t')^{n+1}}$$

- Puiseux series: $f = y$;
  holomorphicity condition for differentials (no negative powers)
- infinity: express γ_∞ in terms of the γ_i

Numerical integration

- Gauss-Legendre integration: expansion of integrand in terms of Legendre polynomials $\mathcal{F}(x_l) = \sum_{k=0}^{N_l} a_k \mathcal{P}_k(x_l)$, $l = 0, \ldots, N_l$

$$\int_{-1}^{1} \mathcal{F}(x)dx \sim \sum_{k=0}^{N_l} a_k \int_{-1}^{1} \mathcal{P}_k(x)dx$$

- integration:

$$\int_{-1}^{1} \mathcal{F}(x)dx \sim \sum_{k=0}^{N_l} \mathcal{F}(x_k)\mathcal{L}_k$$

- analytic continuation of y_j along the γ_i on the collocation points x_l, integration of the holomorphic differentials

Riemann matrix

- a- and b-periods

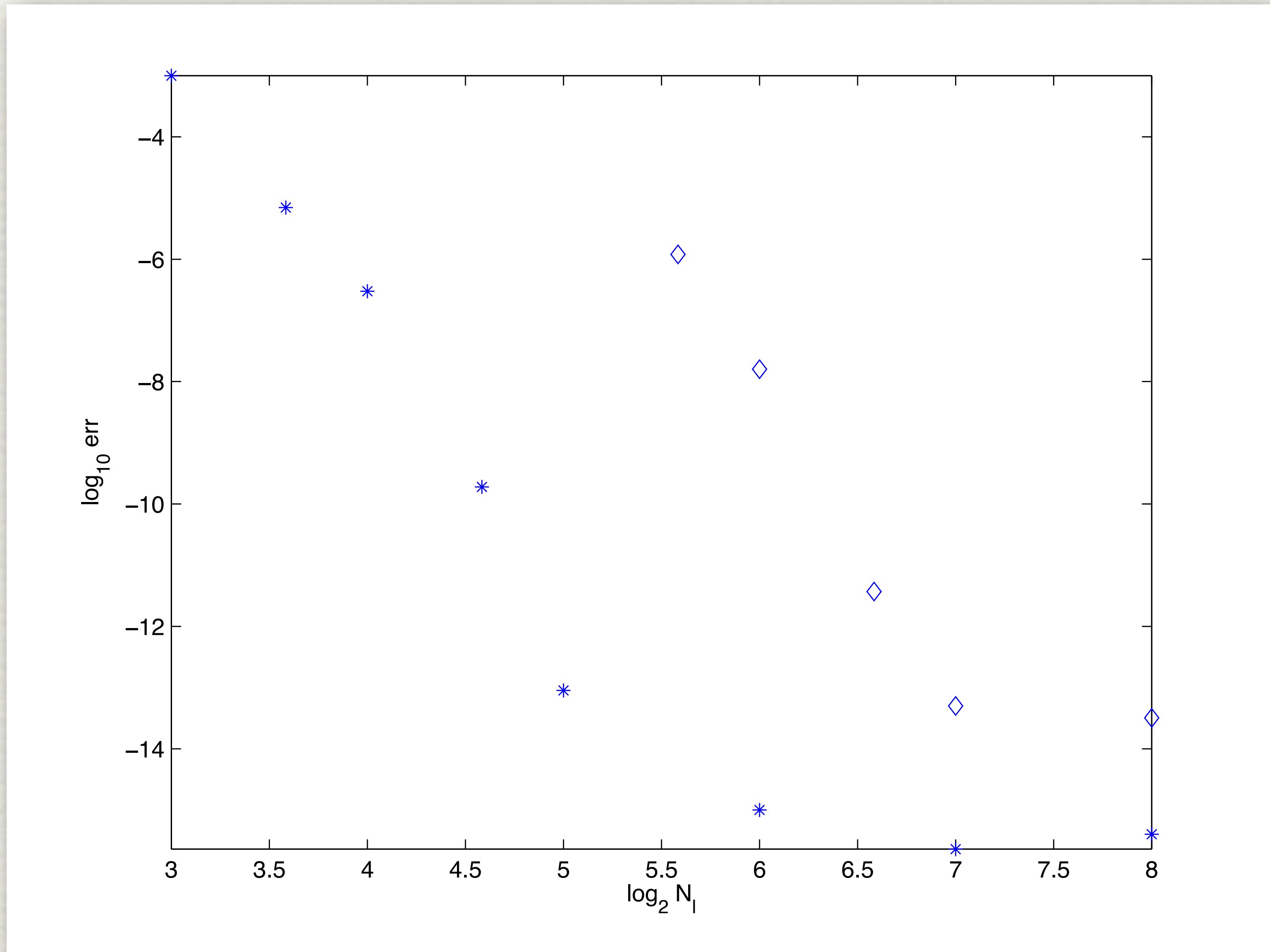
$$\mathbf{B} = \mathcal{A}^{-1}\mathcal{B}$$

- numerical asymmetry of Riemann matrix as test
- ex.:

```
RieMat =
   0.3090 + 0.9511i   0.5000 - 0.3633i
   0.5000 - 0.3633i  -0.3090 + 0.9511i.
```

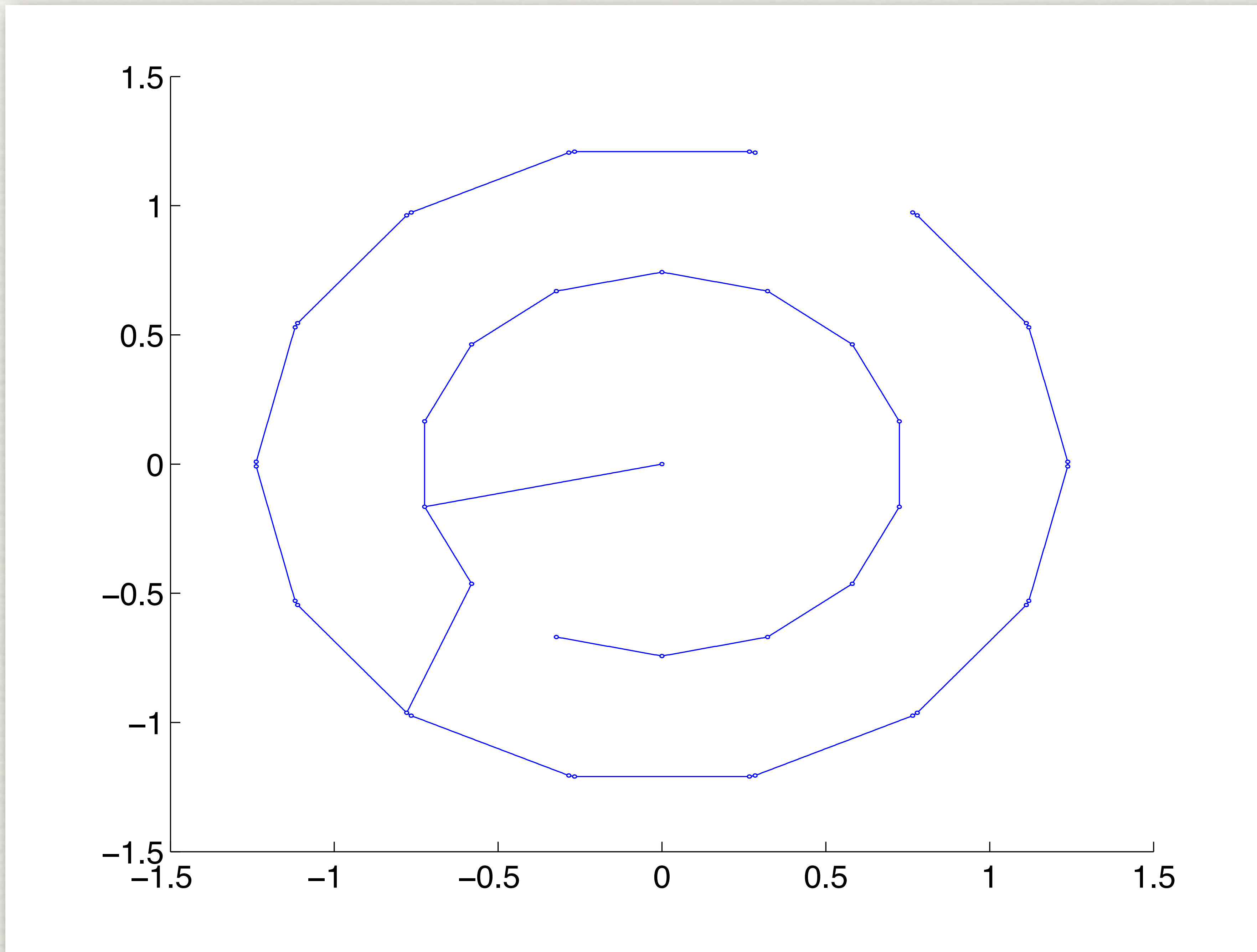
Performance

- error: asymmetry of the Riemann matrix and periods of cycles homologous to 0 for
  $f(x,y) = y^3 + 2x^3y - x^7 = 0$ (stars) and
  $f(x,y) = y^9 + 2x^2y^6 + 2x^4y^3 + x^6 + y^2 = 0$ (diamonds),
  spectral convergence



- $$f(x, y) = y^9 + 2x^2y^6 + 2x^4y^3 + x^6 + y^2 = 0$$

  genus 16, 42 finite branch points, two singular points $(0, 0, 1)$ and (1,0,0), minimal distance between branch points 0.018



Theta functions

- theta series approximated as sum

$$\Theta(\mathbf{z}|\mathbf{B}) \approx \sum_{N_1=-N_\theta}^{N_\theta} \ldots \sum_{N_g=-N_\theta}^{N_\theta} \exp\left\{ i\pi \left\langle \mathbf{B}\vec{N}, \vec{N} \right\rangle + 2\pi i \left\langle \vec{z}, \vec{N} \right\rangle \right\}$$

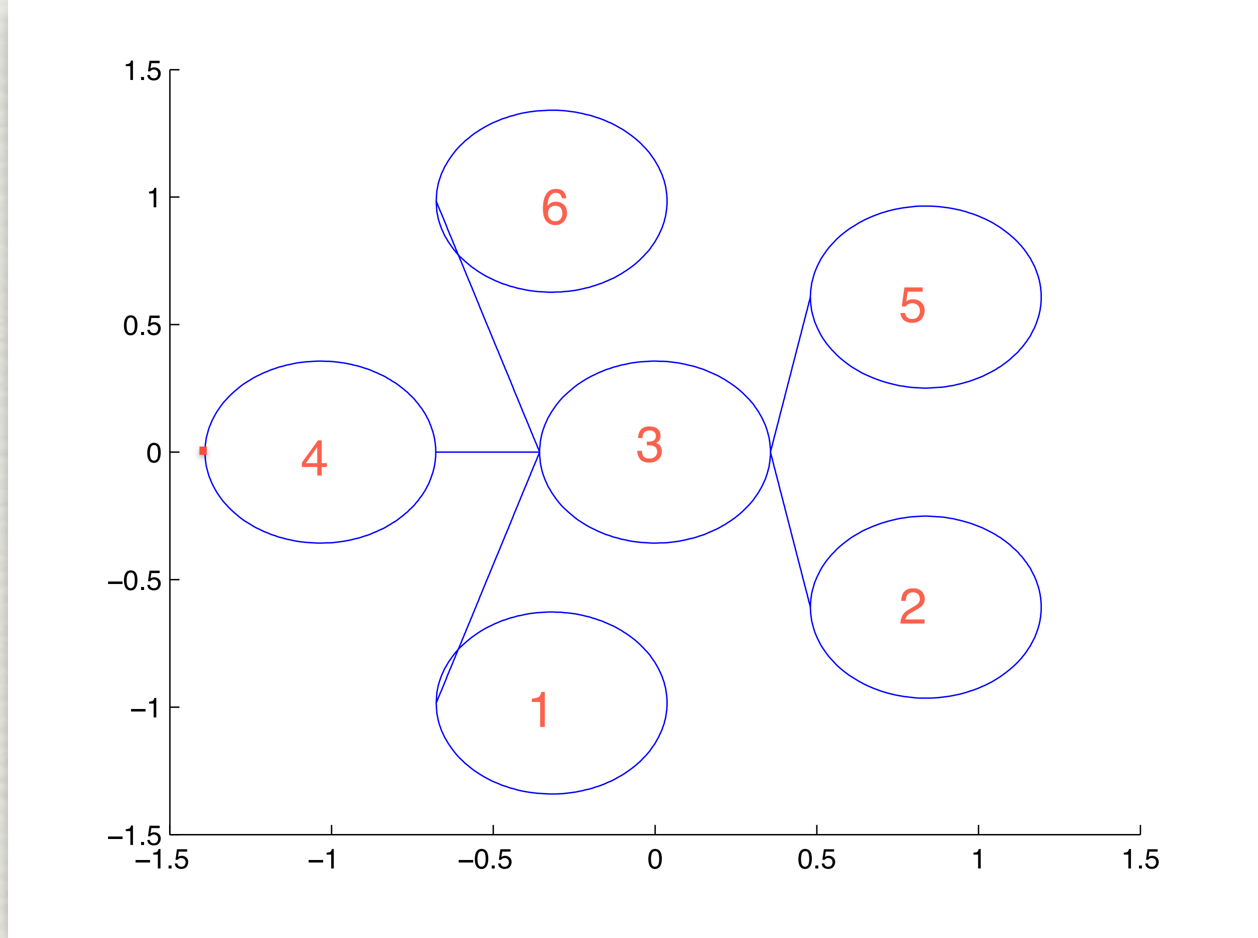
- use periodicity properties of theta function to have argument in the fundamental cell, λ_1 smallest eigenvalue of the imaginary part of the Riemann matrix

$$N_\theta > \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\ln \epsilon}{\pi \lambda_1}}$$

- symplectic transformation to Siegel fundamental domain (approximately) (Deconinck et al. 2002)

Abel map

- *A(P):* determine closest marked point to *P*, analytic continuation of *y* from there and integration as before.
- critical points, infinity: substitution as indicated by Puiseux expansions



Real Riemann surfaces

- in applications, solutions to PDEs in terms of theta functions must satisfy reality and smoothness conditions
- real Riemann surfaces: anti-holomorphic involution, convenient form of the homology basis
- smoothness: study of the theta divisor (zeros of the theta function) (Dubrovin, Natanzon, Vinnikov)

Davey-Stewartson equations

$$\mathrm{i}\,\psi_t + \psi_{xx} - \alpha^2\,\psi_{yy} + 2\,(\Phi + \rho\,|\psi|^2)\,\psi = 0,$$

$$\alpha = \mathrm{i}, 1, \quad \rho = \pm 1, \quad \Phi_{xx} + \alpha^2\,\Phi_{yy} + 2\rho\,|\psi|^2_{xx} = 0,$$

- model the evolution of weakly nonlinear water waves traveling predominantly in one direction, wave amplitude slowly modulated in two horizontal directions, plasma physics, ...
- completely integrable, theta-functional solutions (Malanyuk 1994, Kalla 2011)
- algorithm to transform computed homology basis to \`Vinnikov' basis (K, Kalla 2011)

Trott curve

- M-curve, $g = 3$, real simple branch points $(s = (1, -1, -1))$

$$144\,(x^4 + y^4) - 225\,(x^2 + y^2) + 350\,x^2y^2 + 81 = 0$$

$\mathrm{DS1}^+$, $\lambda(a) = -0.2$, $\lambda(b) = 0.2$ $\qquad \alpha = \mathrm{i}$, $\rho = 1$

$t \in [-2, 2]$

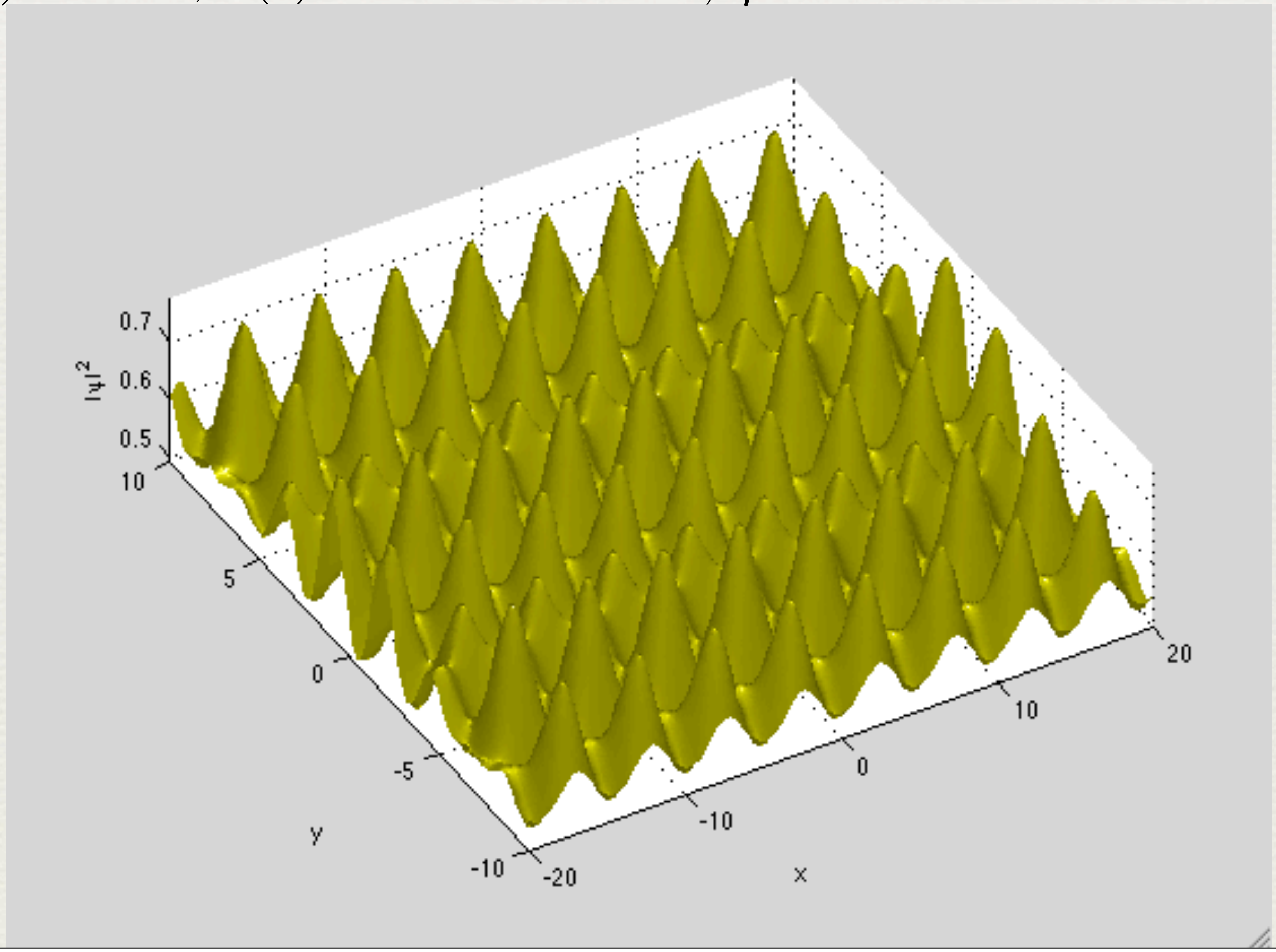
Trott curve

- M-curve, $g = 3$, real simple branch points ($s = (1, -1, -1)$)

$$144\,(x^4 + y^4) - 225\,(x^2 + y^2) + 350\,x^2y^2 + 81 = 0$$

DS1$^+$, $\lambda(a) = -0.2$, $\lambda(b) = 0.2$ $\quad \alpha = \mathrm{i},\ \rho = 1$

$t \in [-2, 2]$

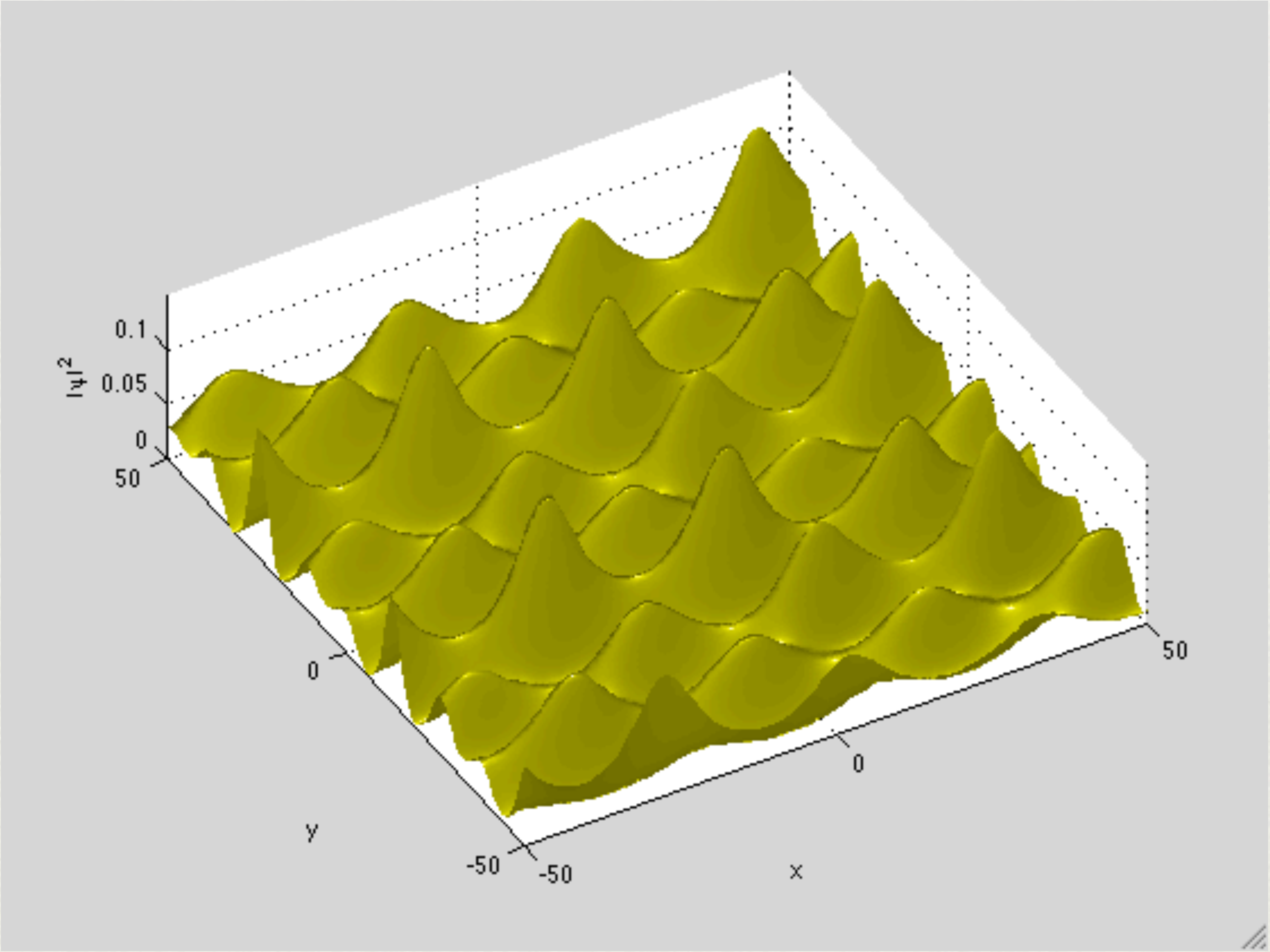


DS on Fermat curve g=3

DS2$-$, $\lambda(a) = -1.5 + \mathrm{i}$, $\lambda(a) = -1.5 - \mathrm{i}$ $\qquad t \in [-5, 5]$

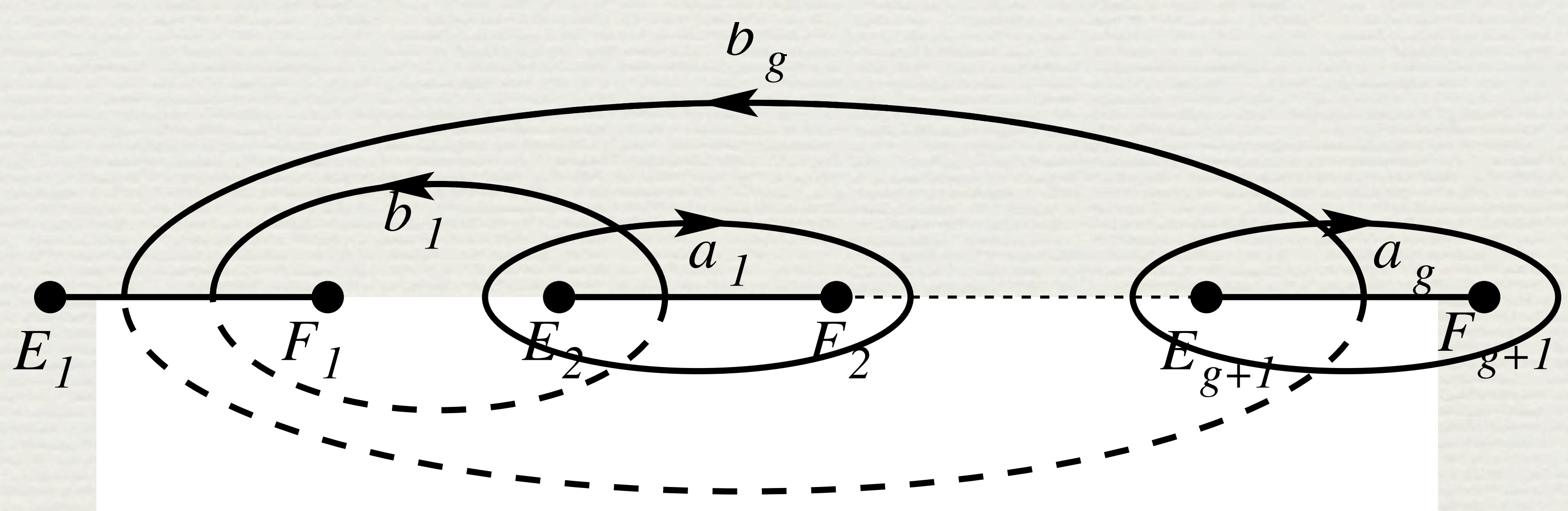
DS on Fermat curve g=3

DS2$-$, $\lambda(a) = -1.5 + \mathrm{i}$, $\lambda(a) = -1.5 - \mathrm{i}$ $\qquad t \in [-5, 5]$



Hyperelliptic surfaces

- general surface: analytic continuation most time consuming
- y: square root of polynomial in x, holomorphic differentials known, $y^2 + \prod_{i=1}^{2g+2}(x - x_i) = 0$
- analytic continuation of the root trivial (square root, correct unwanted sign changes)
- branch points prescribed, can almost collapse
- homology can be chosen a priori



Outlook

- more efficient determination of critical points, homotopy tracing, endgame
- Abel map via Cauchy formula
- Siegel transformation of the Riemann matrix to fundamental domain
- parallelization of theta functions

References

- C. Kalla and C. Klein, *Computation of the topological type of a real Riemann surface* (2012) arXiv:1204.4826

- C. Kalla and C. Klein, *On the numerical evaluation of algebro-geometric solutions to integrable equations*, Nonlinearity Vol. 25 569-596 (2012).

- J. Frauendiener, C. Klein and V. Shramchenko, *Efficient computation of the branching structure of an algebraic curve*, Comput. Methods Funct. Theory Vol. 11 (2011), No. 2, 527–546.

- J. Frauendiener and C. Klein, *Algebraic curves and Riemann surfaces in Matlab*, in A. Bobenko and C. Klein (ed.), *Computational Approach to Riemann Surfaces*, Lecture Notes in Mathematics Vol. 2013 (Springer) (2011).

- J. Frauendiener and C. Klein, *Hyperelliptic theta functions and spectral methods: KdV and KP solutions*, Lett. Math. Phys., Vol. 76, 249-267 (2006).

- J. Frauendiener and C. Klein, *Hyperelliptic theta functions and spectral methods*, J. Comp. Appl. Math., Vol. 167, 193 (2004).