1. Random matrices: a brief intro

2. Asymptotics of Taylor polynomials and the normal matrix model.

Ken McLaughlin, Univ. of Arizona

SISSA, September 18, 2013

The basic example of random matrices: $N \times N$ Hermitian matrices:

M_{jj}: Gaussian random variable: Prob $\{M_{jj} < x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt$

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M_{jk}: Complex Gaussian random variable, $M_{jk} = M_{jk}^{(R)} + iM_{jk}^{(I)}$, with

Prob $\left\{ M_{jk}^{(R)} < x \right\} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2} dt$

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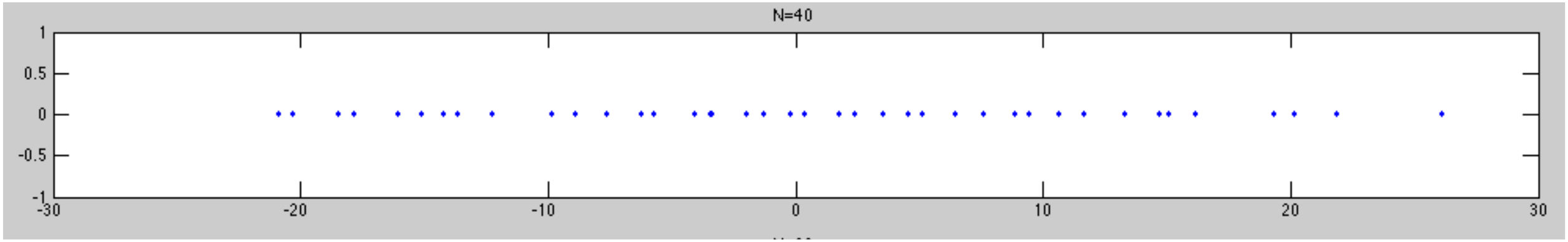
$$\begin{pmatrix} M_{11} & M_{12}^{(R)} + iM_{12}^{(I)} & \cdots & M_{1N}^{(R)} + iM_{1N}^{(I)} \\ M_{12}^{(R)} - iM_{12}^{(I)} & M_{22} & \cdots & M_{2N}^{(R)} + iM_{2N}^{(I)} \\ \vdots & \ddots & \ddots & \vdots \\ M_{1N}^{(R)} - iM_{1N}^{(I)} & M_{2N}^{(R)} - iM_{2N}^{(I)} & \cdots & M_{NN} \end{pmatrix}$$

This is referred to as the Gaussian Unitary Ensemble

Using Matlab, we can generate random matrices and compute their eigenvalues



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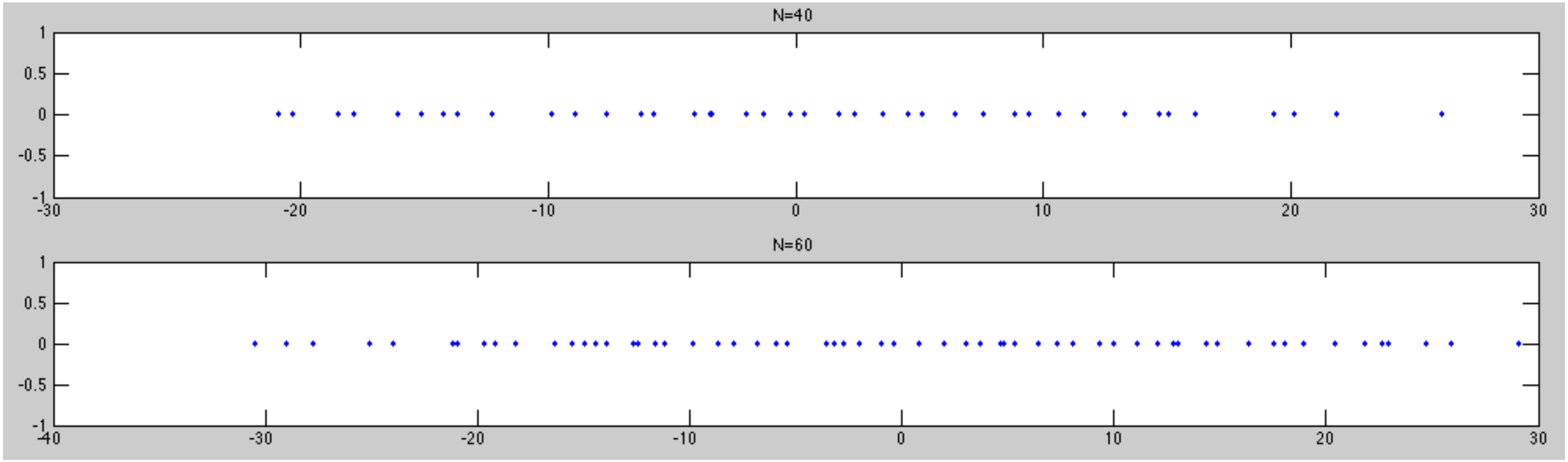

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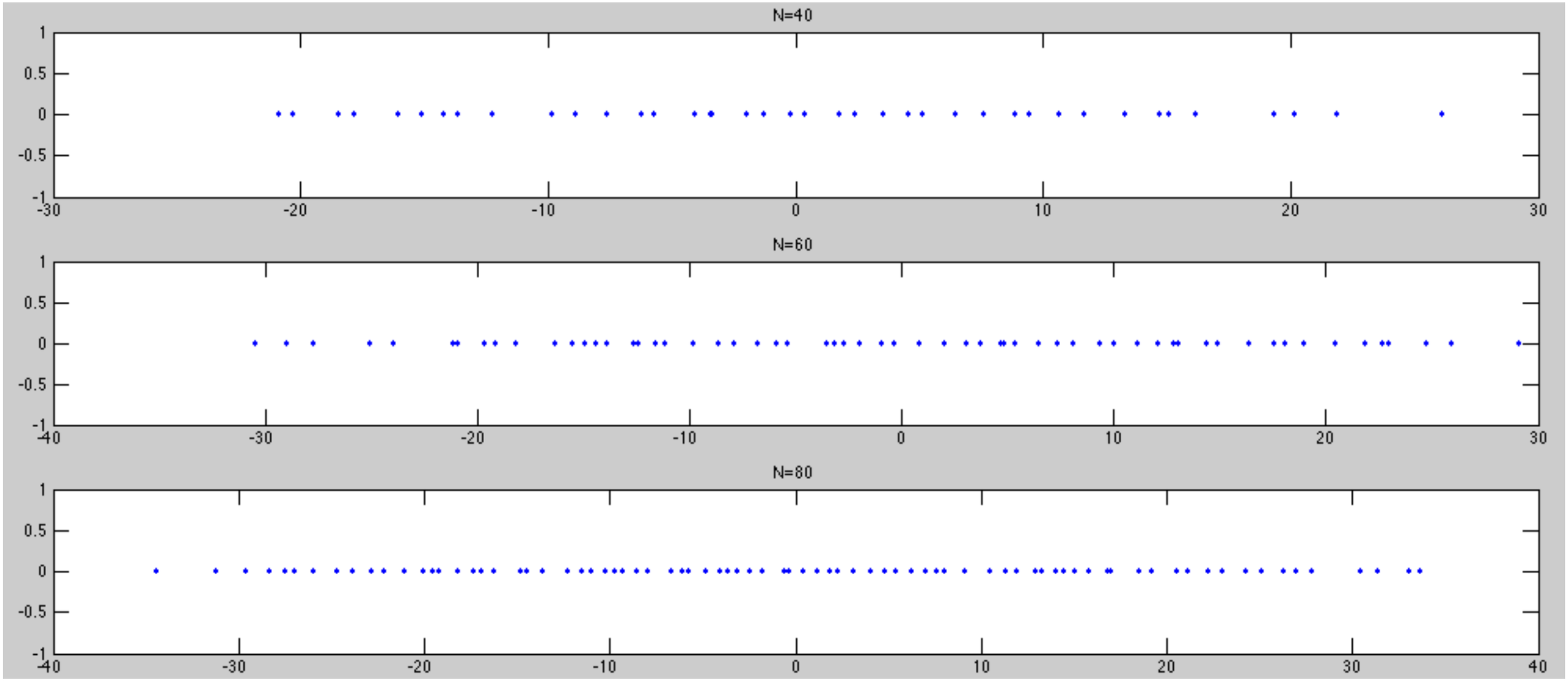

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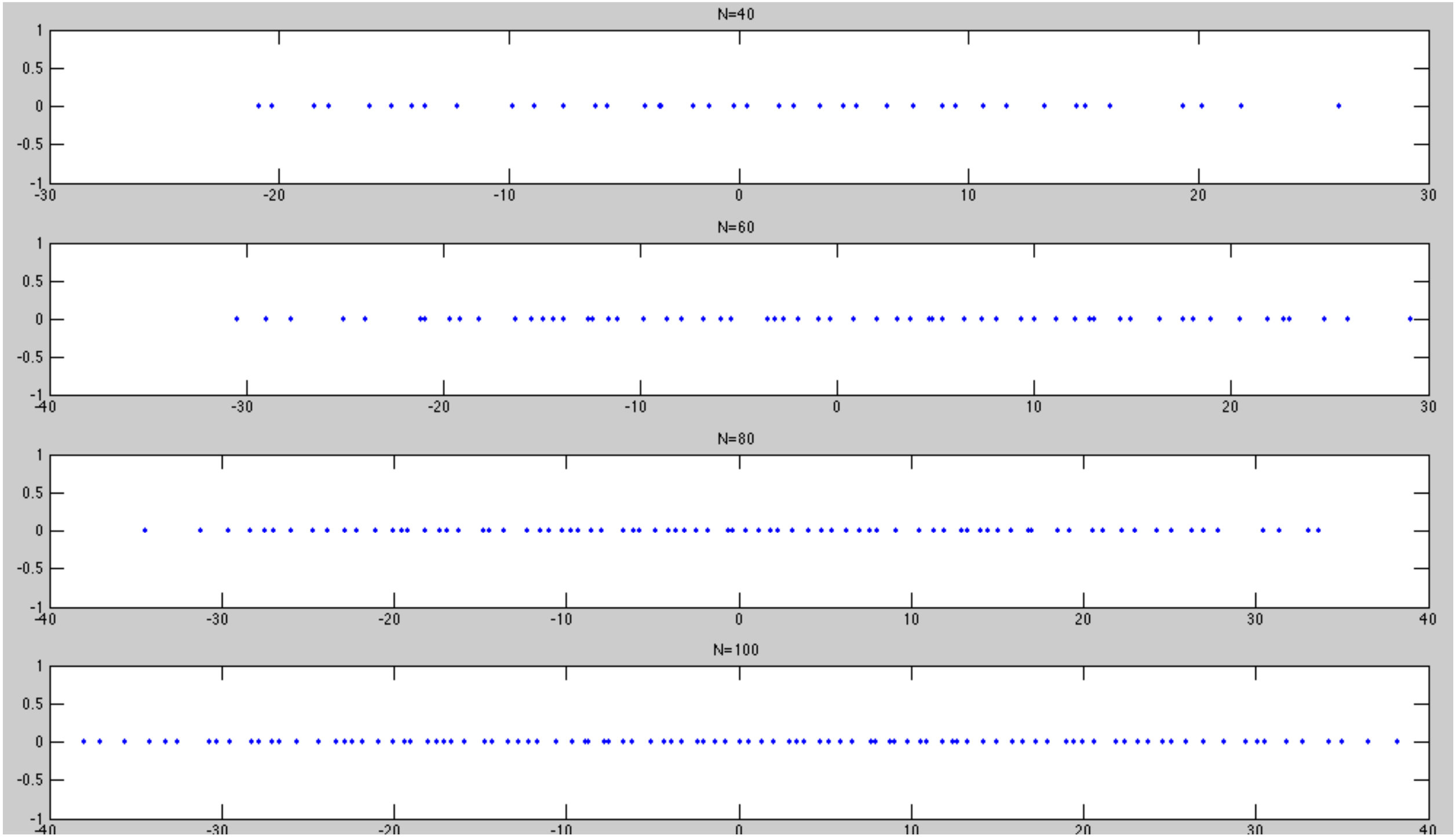

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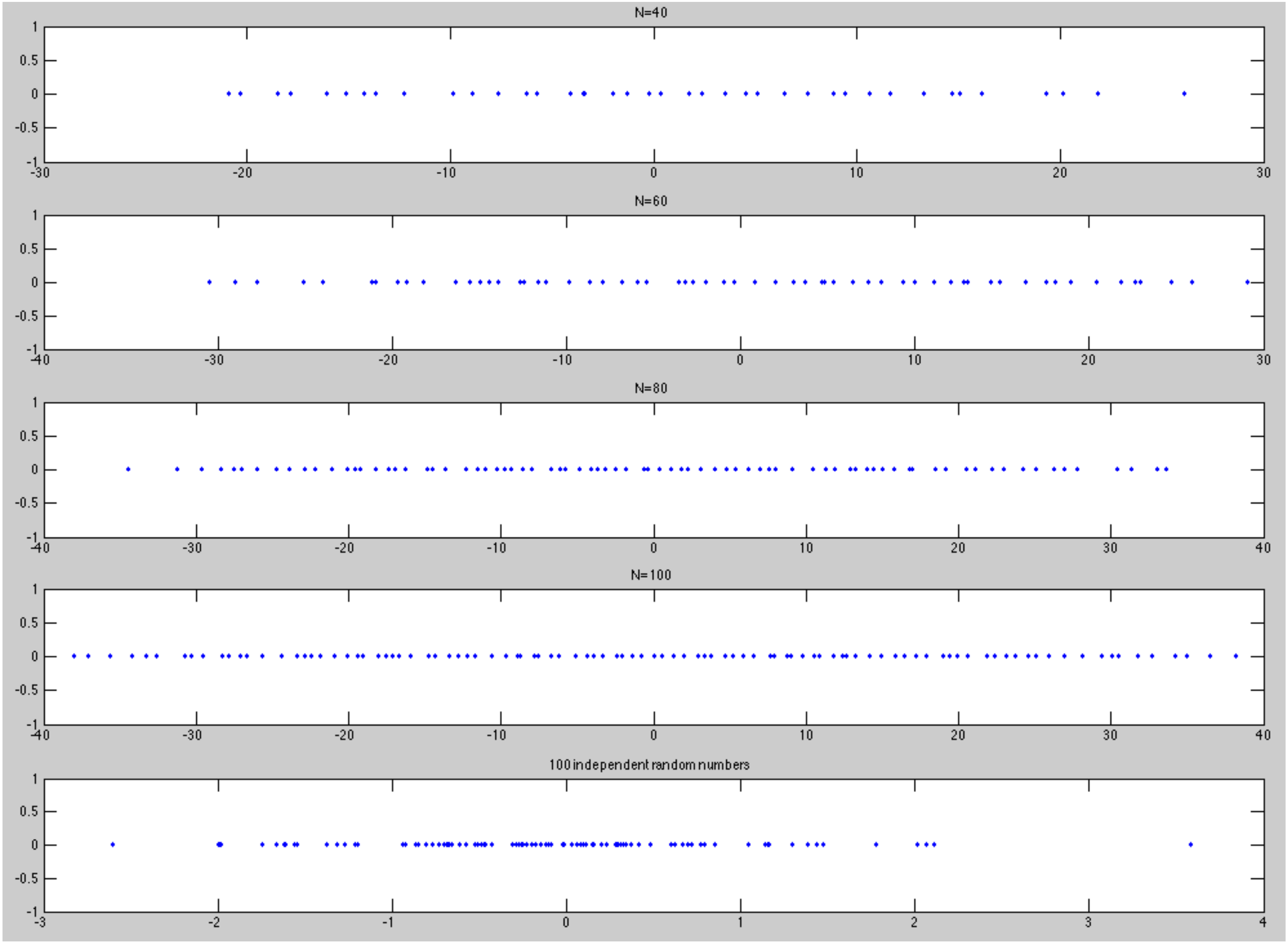
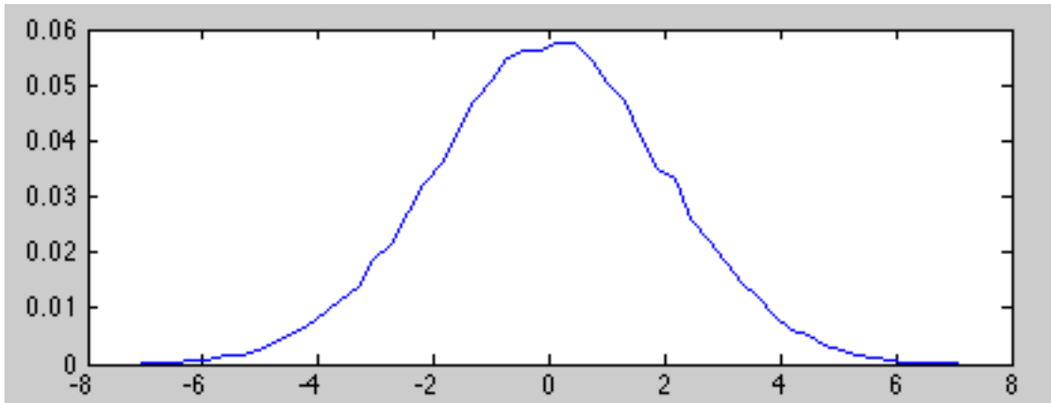
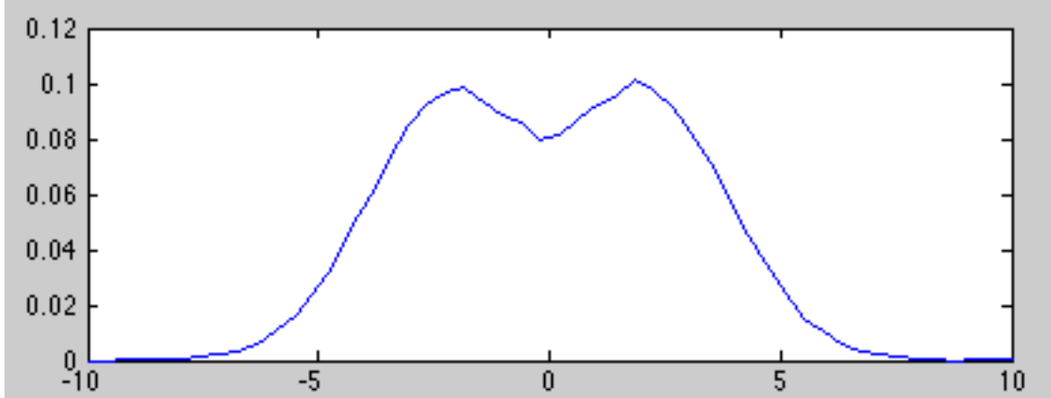


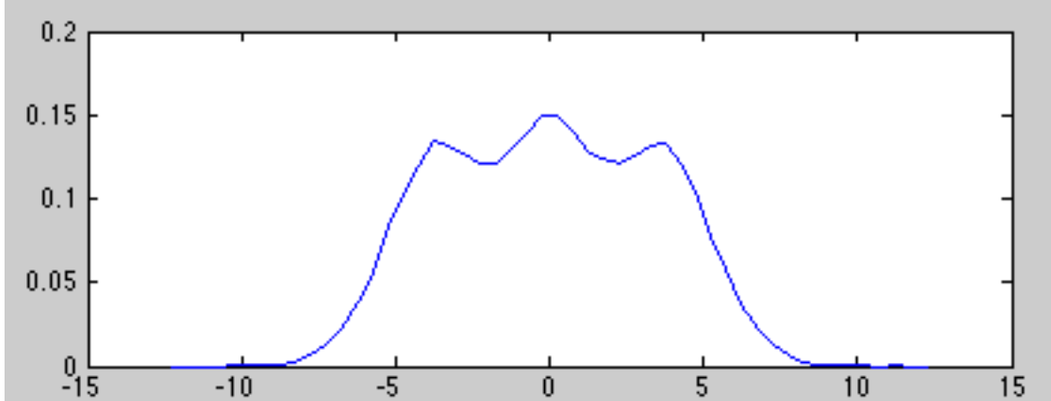
N=1



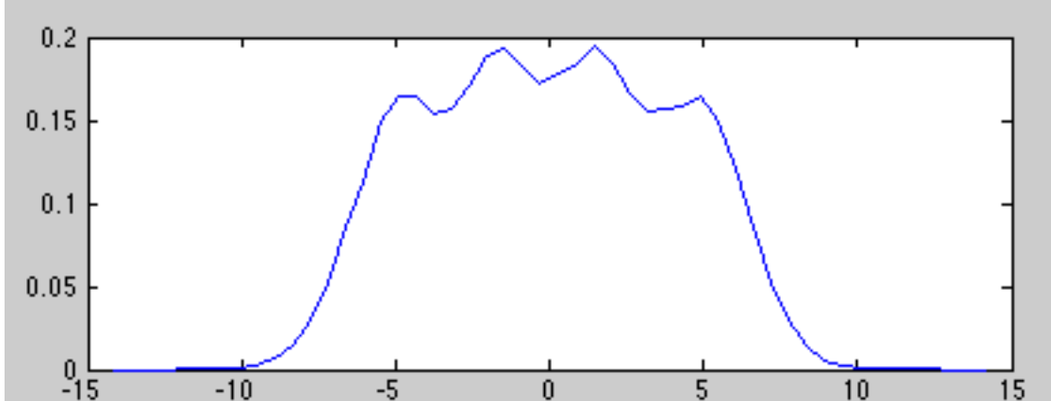
N=2



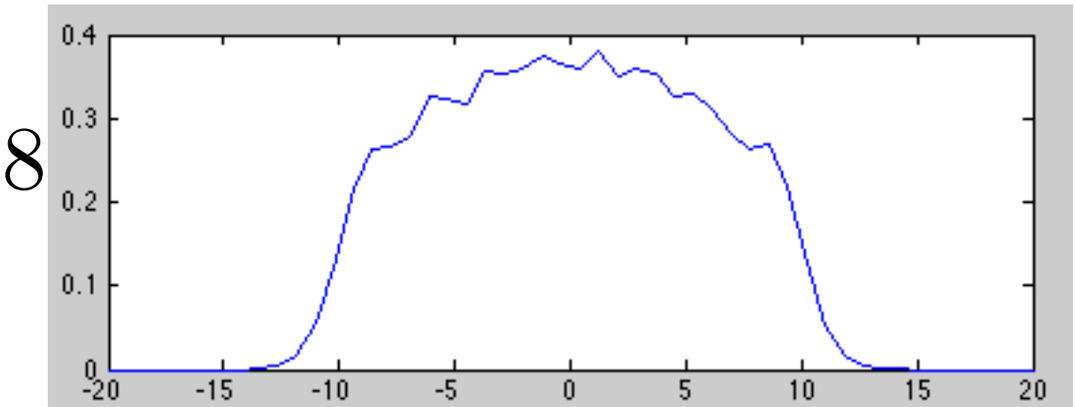
N=3



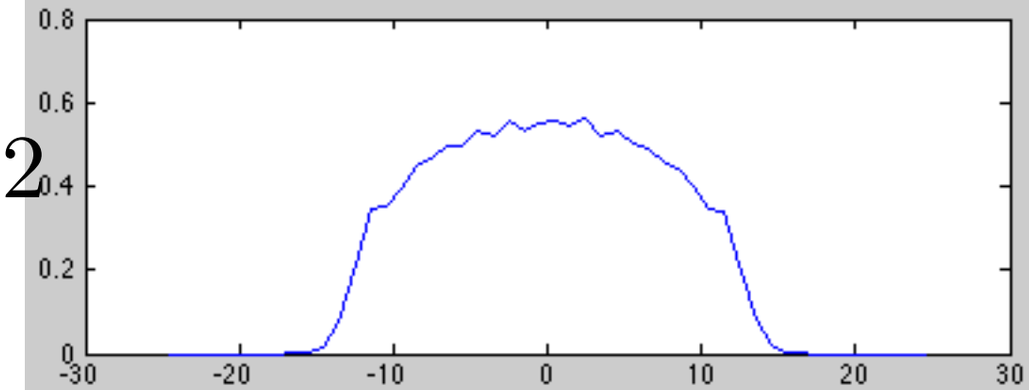
N=4



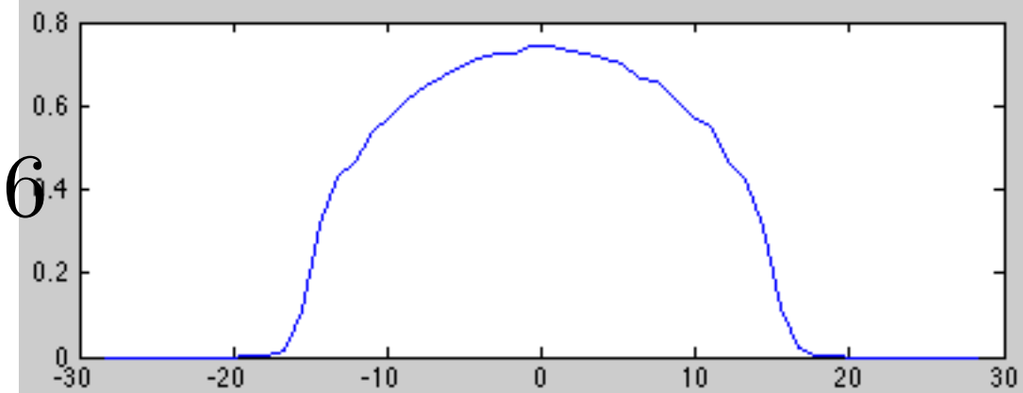
N=8



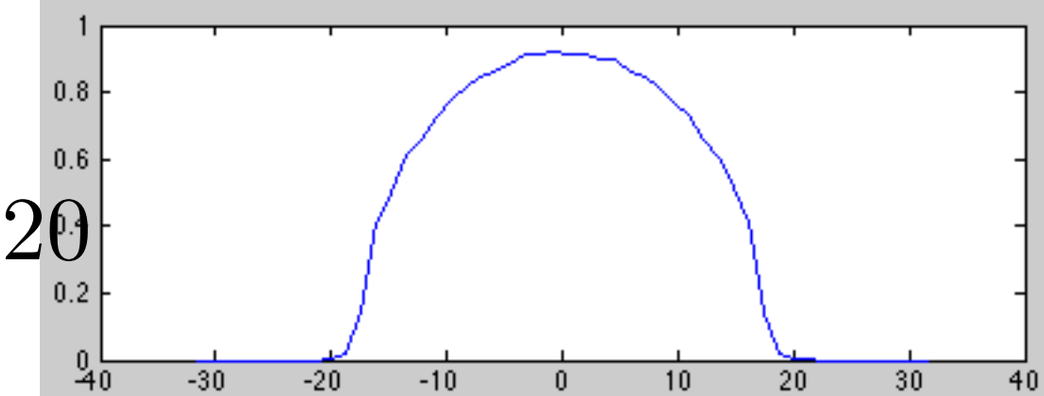
N=12



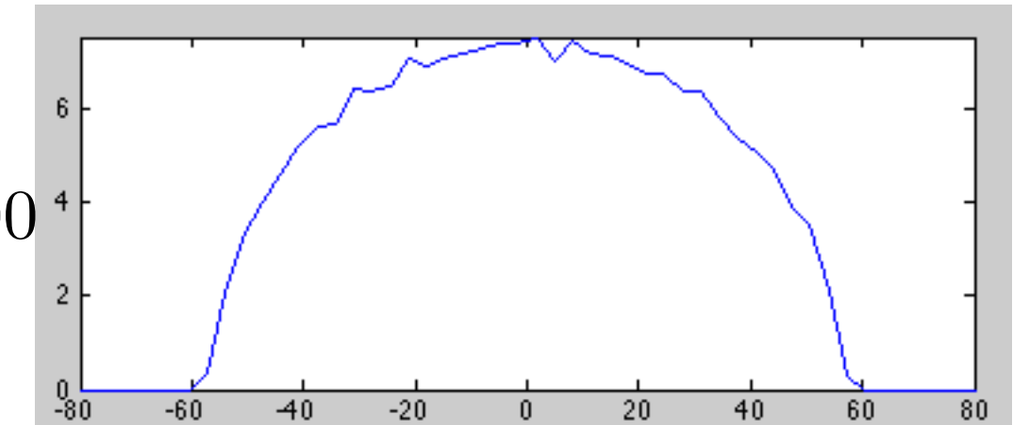
N=16



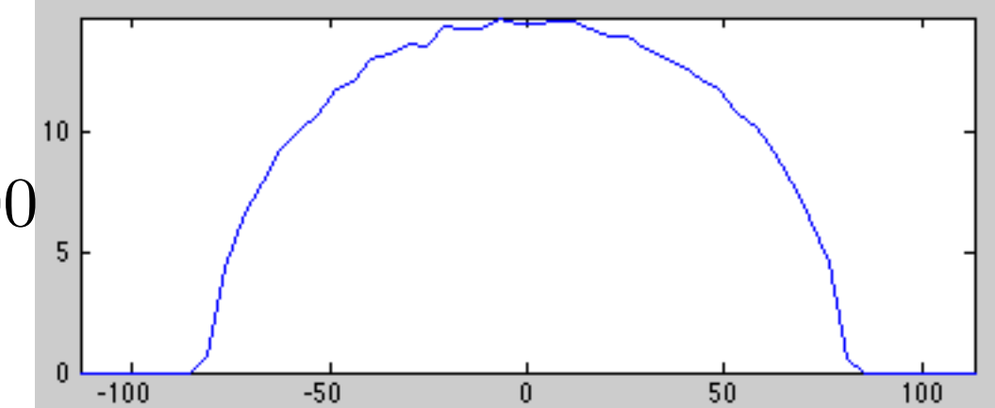
N=20



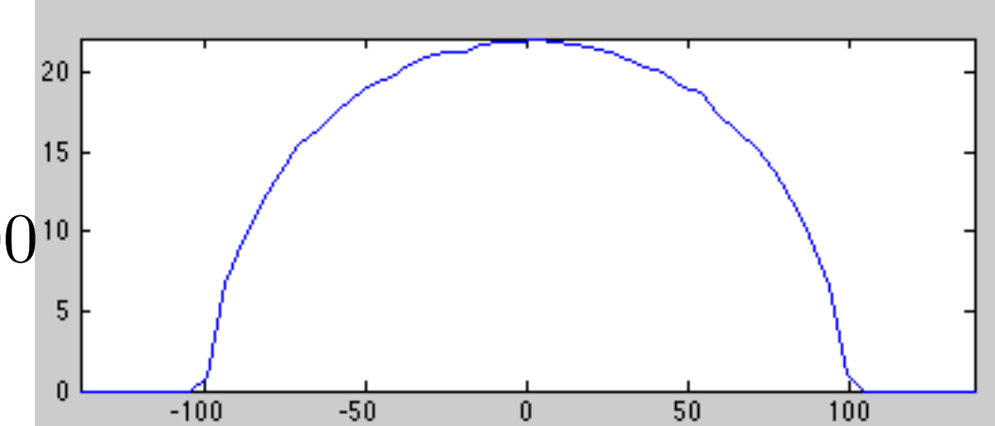
$N = 200$



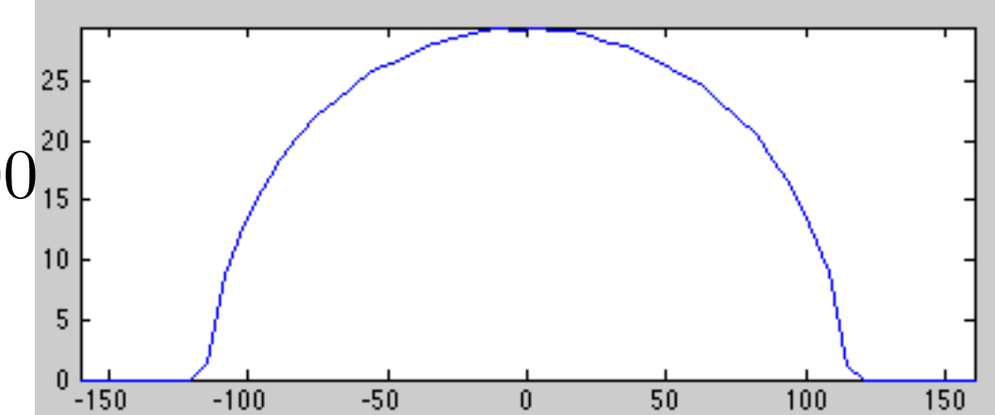
$N = 400$



$N = 600$



$N = 800$



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M_{jk}: Complex Gaussian random variable, $\frac{1}{\pi} e^{-|M_{jk}|^2} dM_{jk}^{(R)} dM_{jk}^{(I)}$

Joint prob. measure: $$\frac{1}{\#_N} \exp\left\{ -\mathrm{Tr}\left[\frac{1}{2} M^2 \right] \right\} dM,$$

$$dM = \prod_{j<k} dM_{jk}^{(R)} dM_{jk}^{(I)} \prod_{j=1}^{N} dM_{jj}$$

This is referred to as the Gaussian Unitary Ensemble

Support seems to grow like $\sqrt{N}$...

Rescale the matrices $M \mapsto N^{1/2} M$:

Joint prob. measure: $$\frac{1}{\#_N} \exp\left\{ -N \ \mathrm{Tr}\left[\frac{1}{2} M^2 \right] \right\} dM,$$

$$dM = \prod_{j<k} dM_{jk}^{(R)} dM_{jk}^{(I)} \prod_{j=1}^{N} dM_{jj}$$

Still unitarily invarant...

Unitary Ensembles of Random Matrices

Consider the probability measure on $N \times N$ Hermitean matrices given by

$$\frac{1}{\hat{Z}_N} \exp\left\{-N \operatorname{Tr}\left[\quad V(M) \quad \right]\right\} dM$$

$$dM = \prod_{j<k} dM^R_{jk}\, dM^I_{jk} \prod_{j=1}^{N} dM_{jj}$$

$$\hat{Z}_N = \int \exp\left\{-N \operatorname{Tr}\left[\quad V(M) \quad \right]\right\} dM$$

Historically, the interest has been in the probabilistic description of the eigenvalues, as $N \to \infty$

Gaussian Unitary Ensemble: $V(x)=x^2/2$

There are many different matrix models.

$$\frac{1}{\hat{Z}_N}\exp\left\{-N\mathrm{Tr}\ \left[V(M)\right]\right\}dM$$

Example 2: $N \times N$ Unitary matrices
(Using Haar measure)

Example 3: A coupled Matrix Model

$$\frac{1}{\hat{Z}_N}\exp\left\{-N\mathrm{Tr}\ \left[\frac{1}{2}A^2+B^2+C^2-it\left(ABC+ABC\right)\right]\right\}dA\ dB\ dC$$

Example 4: Normal Matrix Model:

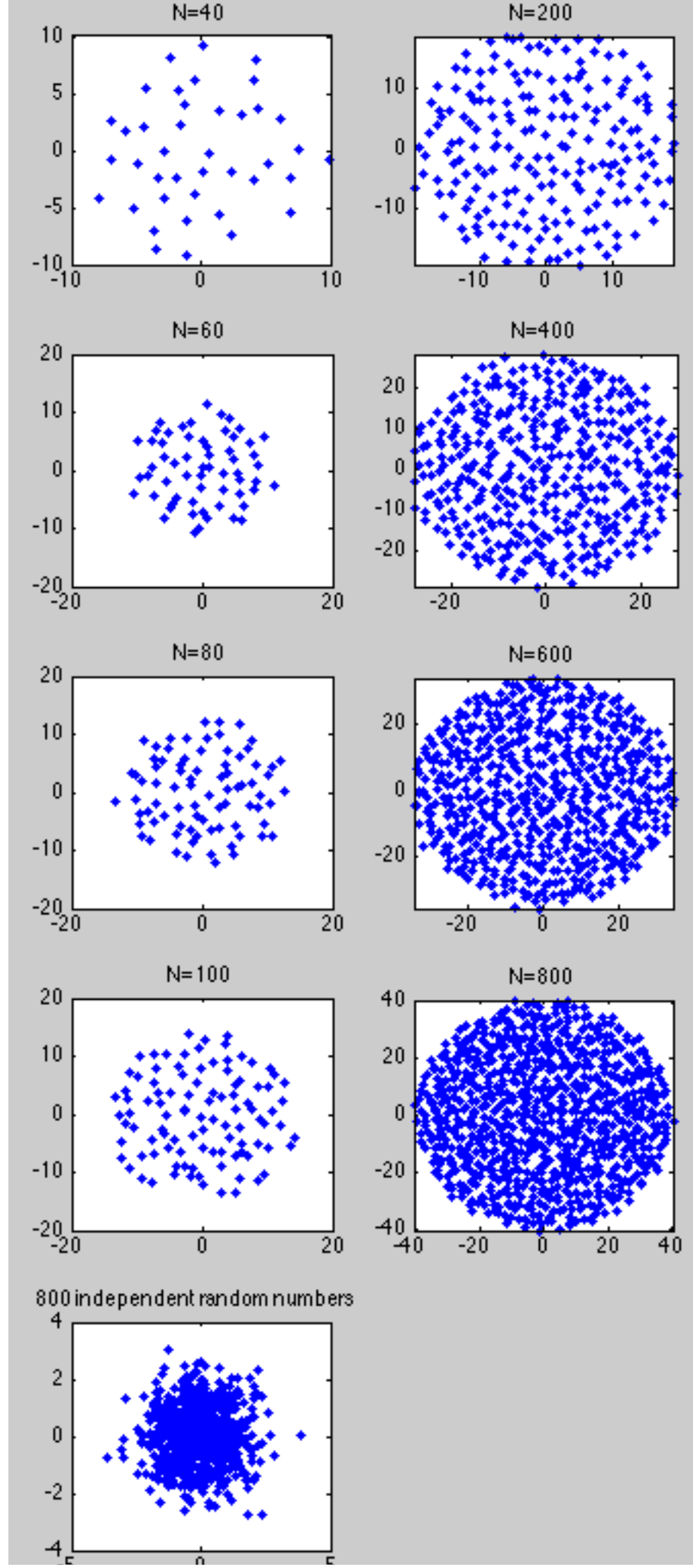
A weaker symmetry requirement: $[M, M^*] = 0$.

Given $V(z, \overline{z})$, one defines

$$\frac{1}{Z_N} \exp\left[-\frac{N}{t} \mathrm{Tr}\ \left(V\left(M, M^*\right)\right)\right] \text{“}dM\text{“}$$

Induced measure on eigenvalues:

$$\frac{1}{\hat{Z}_N} e^{-\frac{N}{t}\sum_{j=1}^{N} V(z_j, \overline{z_j})} \prod_{1 \leq j < k \leq N} |z_j - z_k|^2\ dA(z, \overline{z})$$


N=40
N=200
N=60
N=400
N=80
N=600
N=100
N=800
800 independent random numbers

What does one want to calculate

as $N \to \infty$?

Mean density of eigenvalues

Consider the random variable $\frac{1}{N}\#\{\ \lambda_j \in \mathcal{B}\}$.

What is its average behavior?

$$\mathbb{E}\left[\frac{1}{N}\#\{\ \lambda_j \in \mathcal{B}\}\right] = \int_{\mathcal{B}} \rho_1^{(N)} \left\{ \begin{array}{c} dx \\ dxdy \end{array} \right\}$$

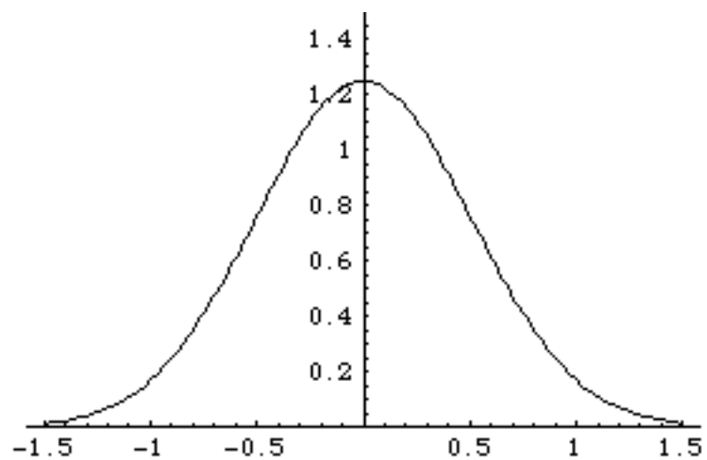
$\rho_1^{(N)}$ is called the mean density of eigenvalues

Question: behavior when $N \to \infty$???

$$\frac{1}{\#_N} \exp\left\{-N \operatorname{Tr}\left[\frac{1}{2}M^2\right]\right\} dM,$$

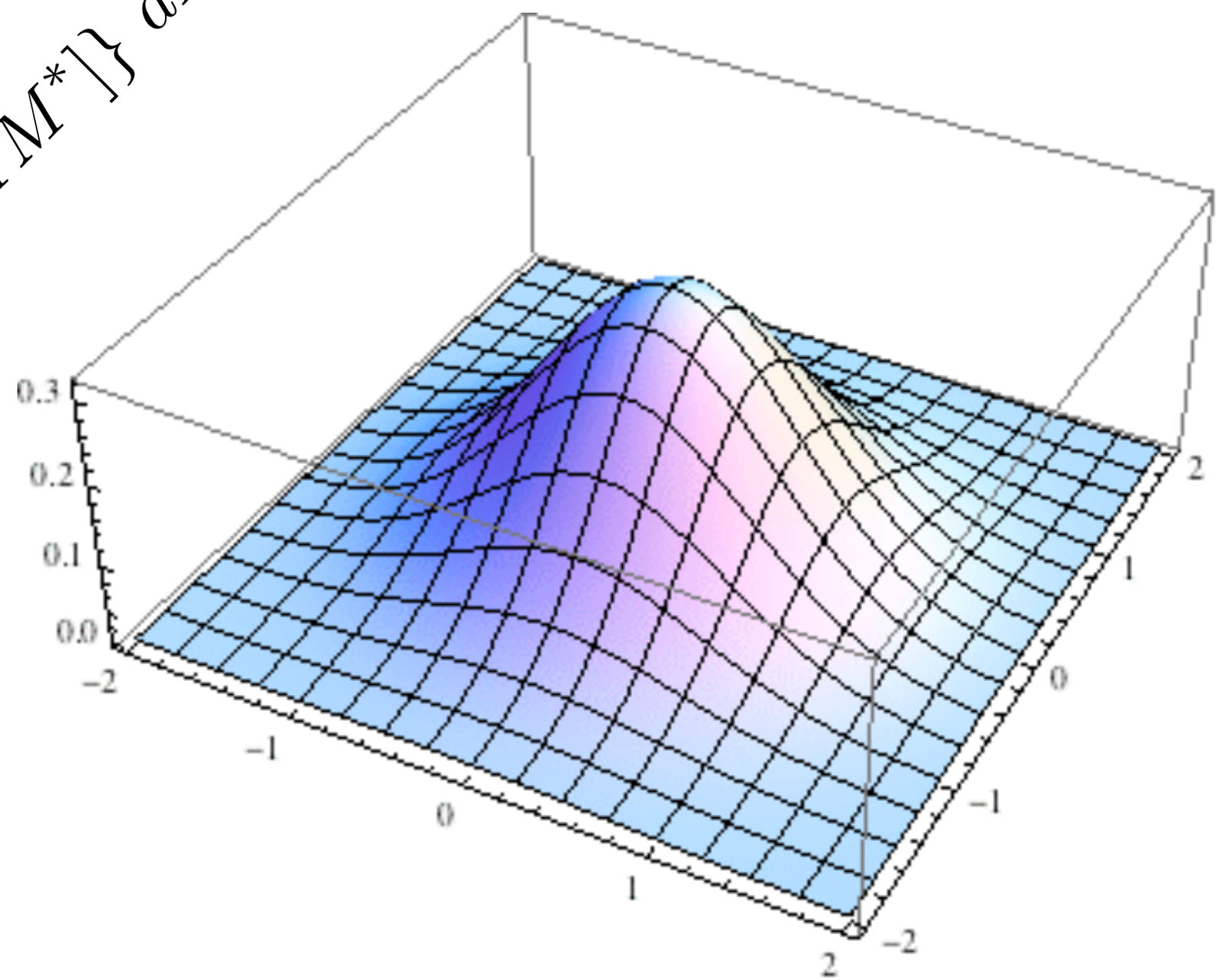
Movie of $\rho_1^{(N)}$ for $N = 1$ through $N = 50$.

$$\frac{1}{\#_N} \exp\left\{-N \operatorname{Tr}\left[\frac{1}{2}M^2\right]\right\} dM,$$



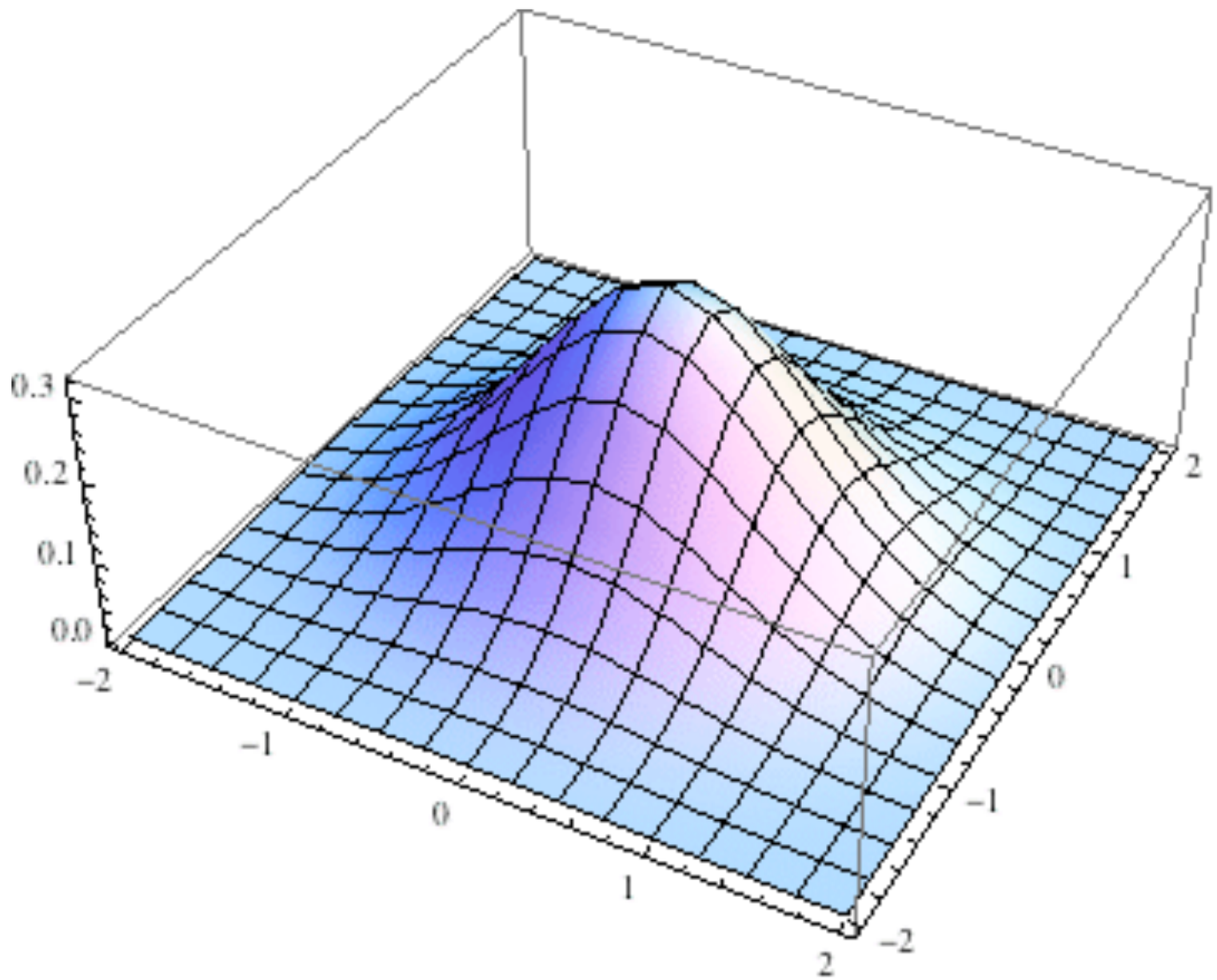
Movie of $\rho_1^{(N)}$ for $N = 1$ through $N = 50$.

$$\frac{1}{\#_N} \exp\{-N \operatorname{Tr}[MM^*]\}\, dM,$$



Movie of $\rho_1^{(N)}$ for $N = 1$ through $N = 50$.

$$\frac{1}{\#_N} \exp\left\{ -N \ \mathrm{Tr}\left[\ MM^* + \frac{T^2}{2}\left(M^2 + (M^*)^2\right)\right]\right\} dM,$$



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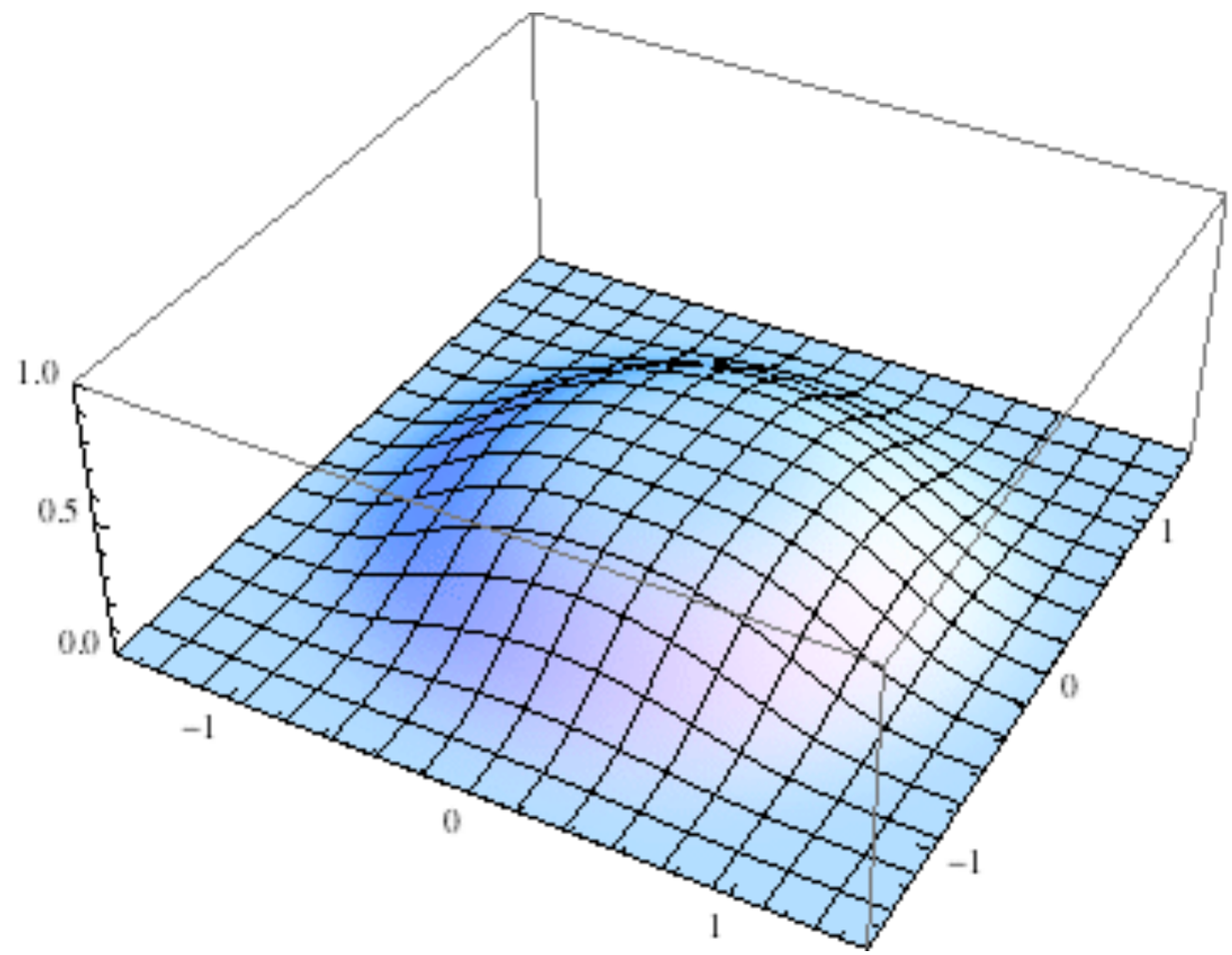

Movie of $\rho_1^{(N)}$ for $N = 1$ through $N = 60$.

Occupation Probabilities

$$\text{Prob}\left\{\mathcal{B} \text{ has } k \text{ eigenvalues}\right\}$$

$$F\left(\mathcal{B}, z\right) = \sum_{k=0}^{\infty} \text{Prob}\left\{\mathcal{B} \text{ has } k \text{ eigenvalues}\right\} z^k$$

$F(\mathcal{B}, z)$ is called the occupation probability generating function.

Question: behavior when $N \to \infty$???

Example 4: Normal Matrix Model:

$n \times n$ matrices with $[M, M^*] = 0$.

Given $Q(z, \overline{z})$, one defines

$$\frac{1}{Z_N} \exp\left[-N \mathrm{Tr}\ \left(Q\left(M, M^*\right)\right)\right] dM$$

Induced measure on eigenvalues:

$$\frac{1}{\hat{Z}_N} e^{-N \sum_{j=1}^{n} Q(z_j, \overline{z_j})} \prod_{1 \leq j < k \leq n} \left|z_j - z_k\right|^2\ dA(z, \overline{z})$$

Orthogonal Polynomials

$$P_{n,N}(z) = z^n + \mathcal{O}\left(z^{n-1}\right) \qquad (n = 0, 1, \dots)$$

$$\int_{\mathbb{C}} P_{k,N}(z)\overline{P_{l,N}(z)}e^{-NQ(z,\overline{z})}dA(x,y) = h_{k,N}\delta_{kl}$$

$$\rho_1^{(N)}(z,\overline{z}) = e^{-NQ(z,\overline{z})}\sum_{\ell=0}^{N-1} h_{\ell,N}^{-1}\left|P_{\ell,N}(z)\right|^2$$

$$K_N(z,w) = e^{-\frac{1}{N}(Q(z)+Q(w))}\frac{1}{N}\sum_{k=0}^{n-1} h_k^{-1}P_{k,N}(z)\overline{P_{k,N}(w)}$$

Their asymptotic behavior is wide open.

Small collection of examples for which calculations are explicit

- $Q(x,y) = x^2 + y^2$: Polynomials are $P_n(z) = z^n$, and

$$\rho_N = \frac{1}{\pi} e^{-N|z|^2} \sum_{j=0}^{n-1} \frac{\left(N|z|^2\right)^j}{j!}$$

  as $N \to \infty$, support is a disc of radius 1, and $\lim_{N\to\infty} \rho_N = 1$.

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$$K_N(z,w) = \frac{1}{\pi} e^{-N\left(|z|^2+|w|^2\right)/2} \sum_{j=0}^{n-1} \frac{\left(Nz\overline{w}\right)^j}{j!}$$

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- $Q = Q\left(|z|\right)$: $P_n = z^n$, limiting support is again a disc,

$$\rho_N = \frac{e^{-NQ(|z|)}}{N} \sum_{j=0}^{n-1} \frac{|z|^{2j}}{2\pi \int_0^\infty r^{2j+1} e^{-NQ} dr} \quad \to \quad \frac{1}{4}\Delta V(|z|) \ .$$

$$K_N(z,w) = \frac{e^{-N(Q(|z|)+Q(|w|))/2}}{N} \sum_{j=0}^{n-1} \frac{(z\overline{w})^j}{2\pi \int_0^\infty r^{2j+1} e^{-NQ} dr}$$

- General existence of support set via potential theory: straightforward. Support properties: more complicated.

  New examples : Balogh and Merzi.

- Analysis of kernel via potential theory, Bergmann kernel: Work of Ameur, Hedenmalm, Makarov

- Approach to general asymptotic analysis of 2D orthogonal polynomials: $\overline{\partial}$ problem, described by Its and Takhtajan, and Balogh and Harnad.

We have recently considered $Q = |z|^2 + 2c \log \frac{1}{|z-a|}$

$$\frac{1}{Z_N} e^{[-N \mathrm{Tr}\ (MM^*)]} |\det(M-a)|^{2Nc} dM$$

arXiv:1209.6366

(With F. Balogh (SISSA), M. Bertola (Concordia Univ.), S. Y. Lee (Caltech)

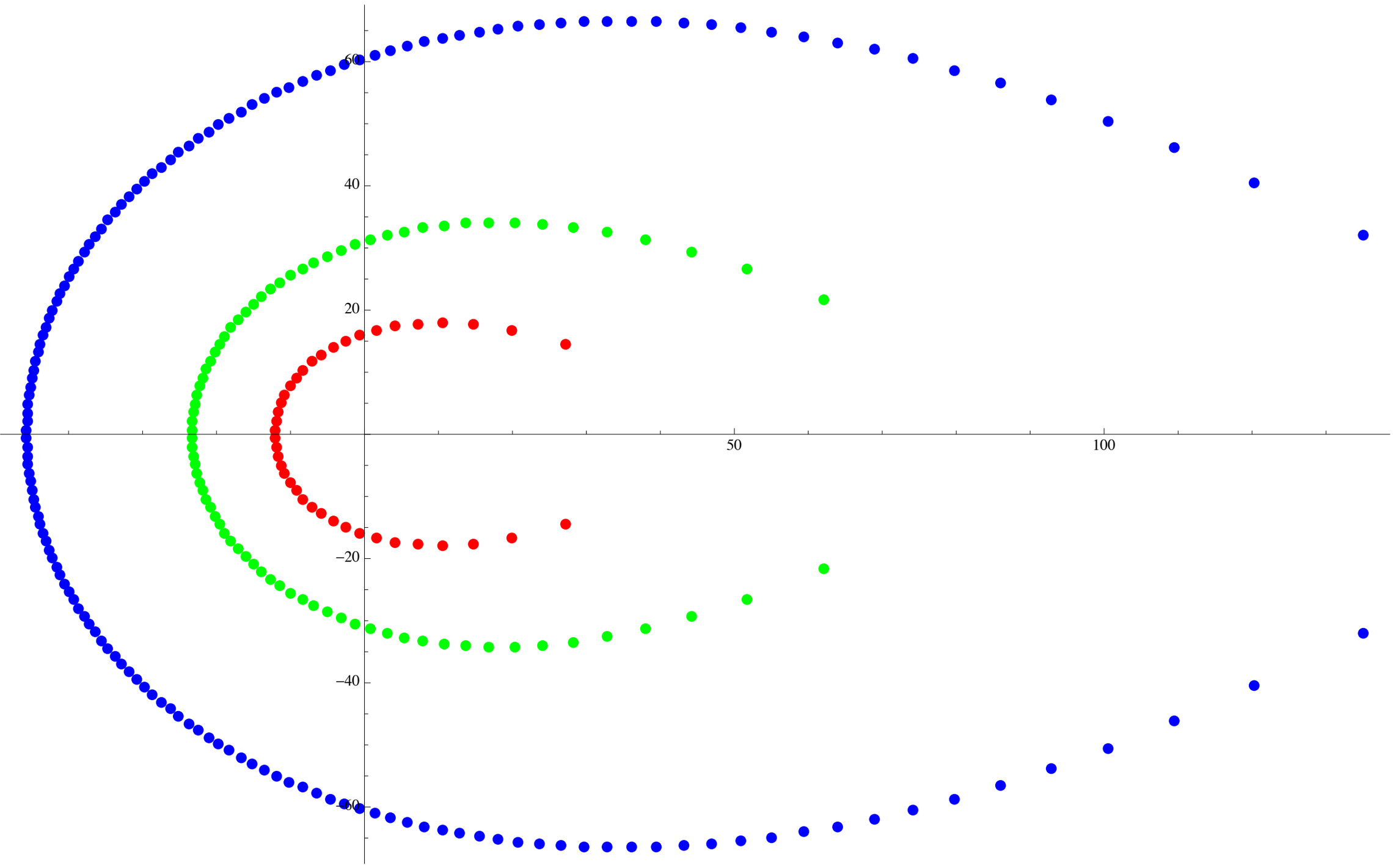
With remaining time:

Asymptotic behavior of the zeros of $\sum_{j=0}^{N-1} \frac{z^j}{j!}$

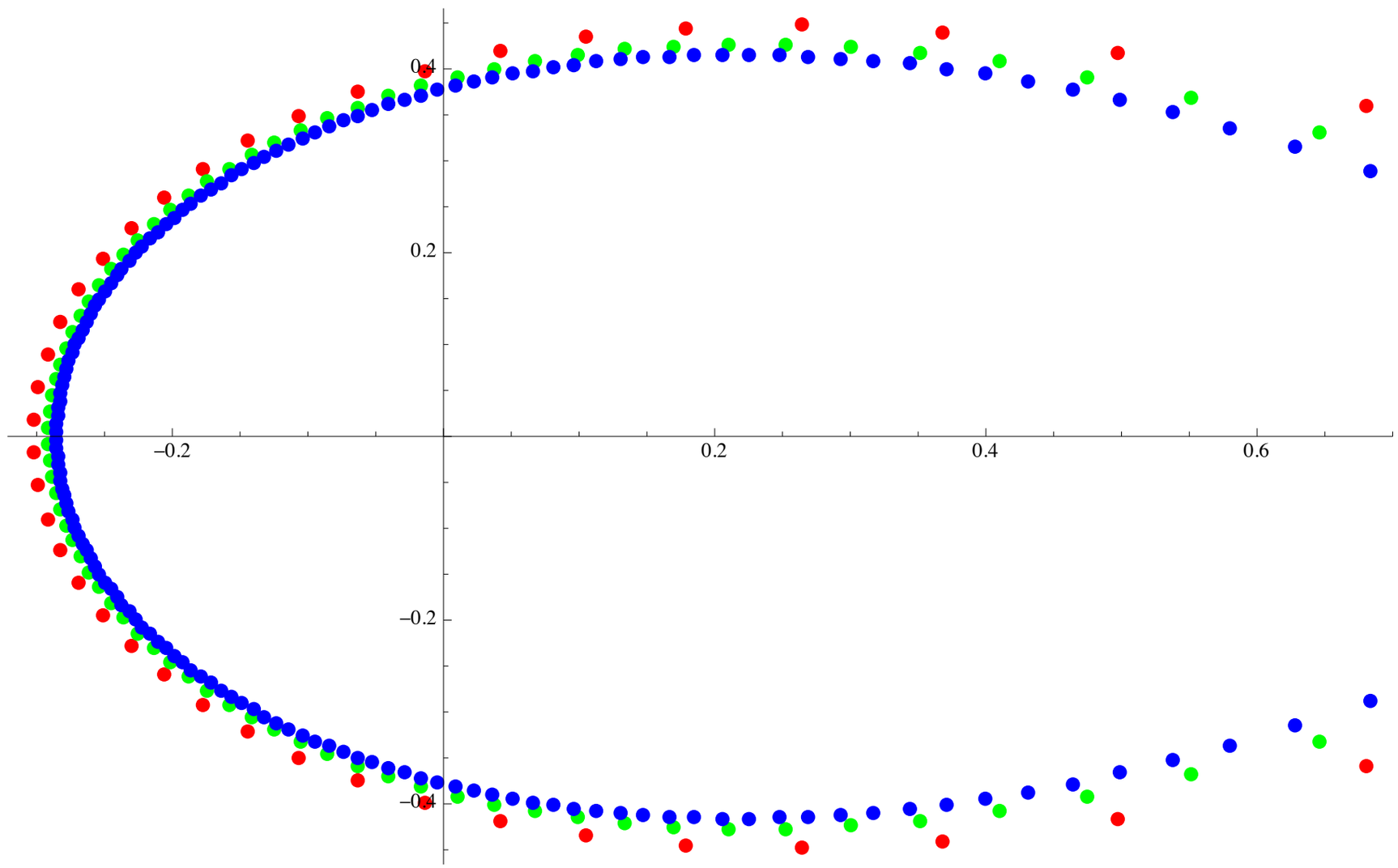
Related questions

Application to NMM (for fun)

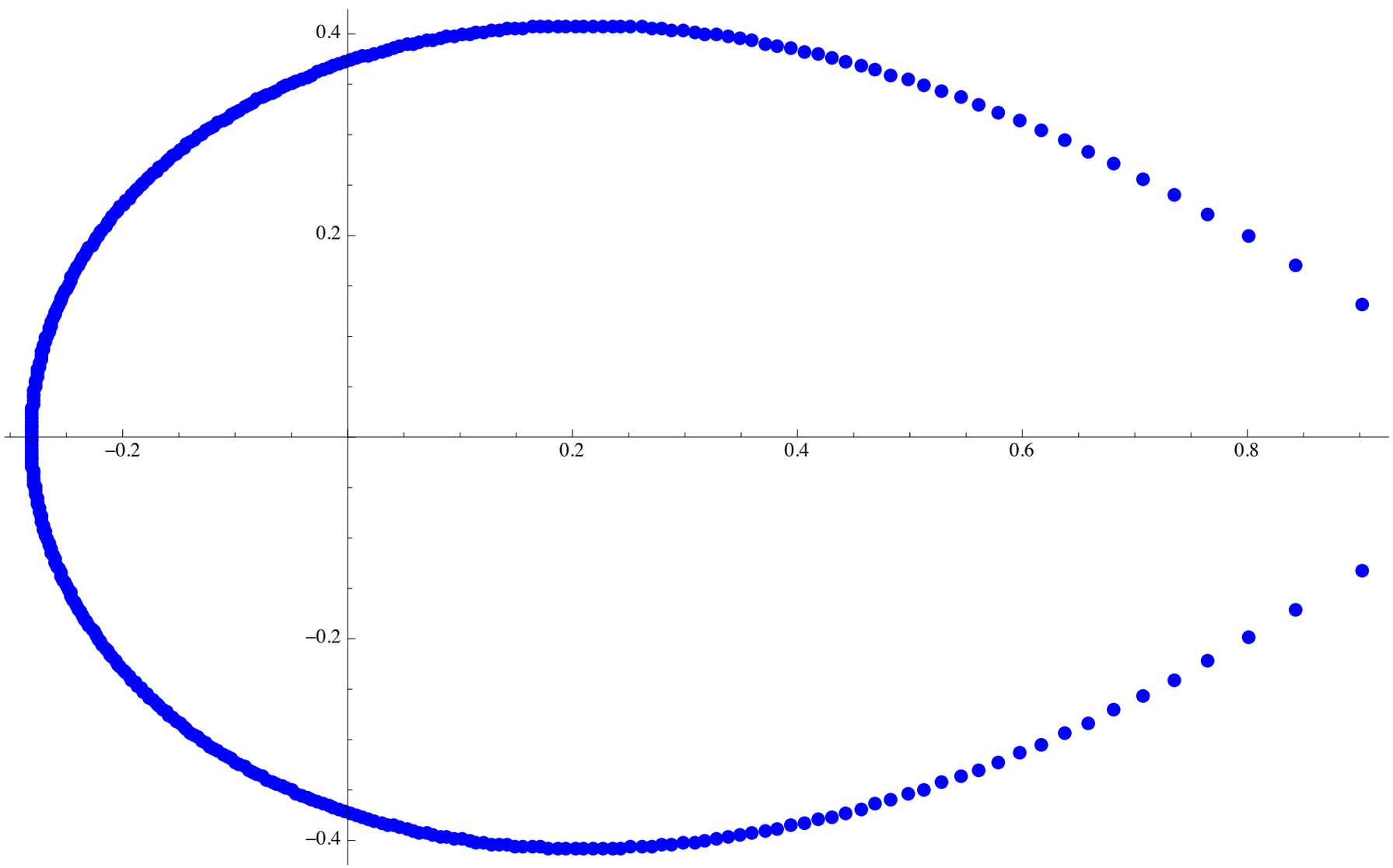
$$\sum_{j=0}^{N-1} \frac{z^j}{j!}$$



$$\sum_{j=0}^{N-1} \frac{(Nz)^j}{j!}$$



$$\sum_{j=0}^{N-1} \frac{(Nz)^j}{j!} = e^{Nz}\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right)$$



$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{s - z} ds \ .$$

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$$= \sum_{j=0}^{N-1} \frac{f^j(0)}{j!}z^j + \frac{z^N}{2\pi i}\int_\gamma \frac{f(s)}{s^n}\frac{ds}{s-z} \ .$$

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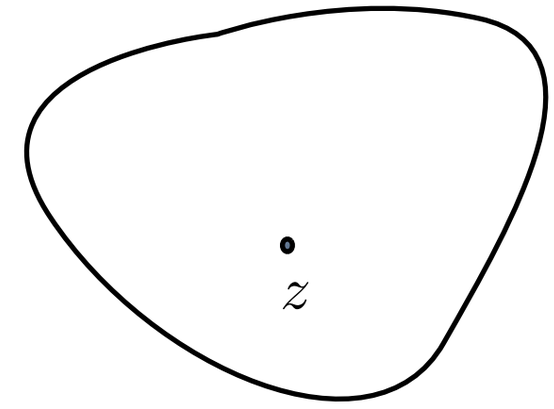

$$= \sum_{j=0}^{N-1} \frac{f^j(0)}{j!} z^j + \frac{z^N}{2\pi i}\int_\gamma \frac{f(s)}{s^n}\frac{ds}{s-z} \ .$$

If $f(s) = e^s$, $z \mapsto Nz$, $s \mapsto Ns$:

$$e^{Nz} = \sum_{j=0}^{N-1} \frac{1}{j!}(Nz)^j + \frac{z^N}{2\pi i}\int_{\tilde\gamma} e^{N(s-\ln(s))}\frac{ds}{s-z} \ .$$

$$\sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j = e^{Nz} - \frac{z^N}{2\pi i} \int_{\tilde{\gamma}} e^{N(s - \ln(s))} \frac{ds}{s-z} \ .$$

Steepest descent method for integrals:

$$\sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j = e^{Nz} - \frac{z^N e^N}{1-z} \frac{1}{\sqrt{2\pi N}} \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right)$$

Looking for zeros:

$$e^{-Nz} \sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j = 1 - \frac{\left(ze^{1-z}\right)^N}{1-z} \frac{1}{\sqrt{2\pi N}} \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right)$$

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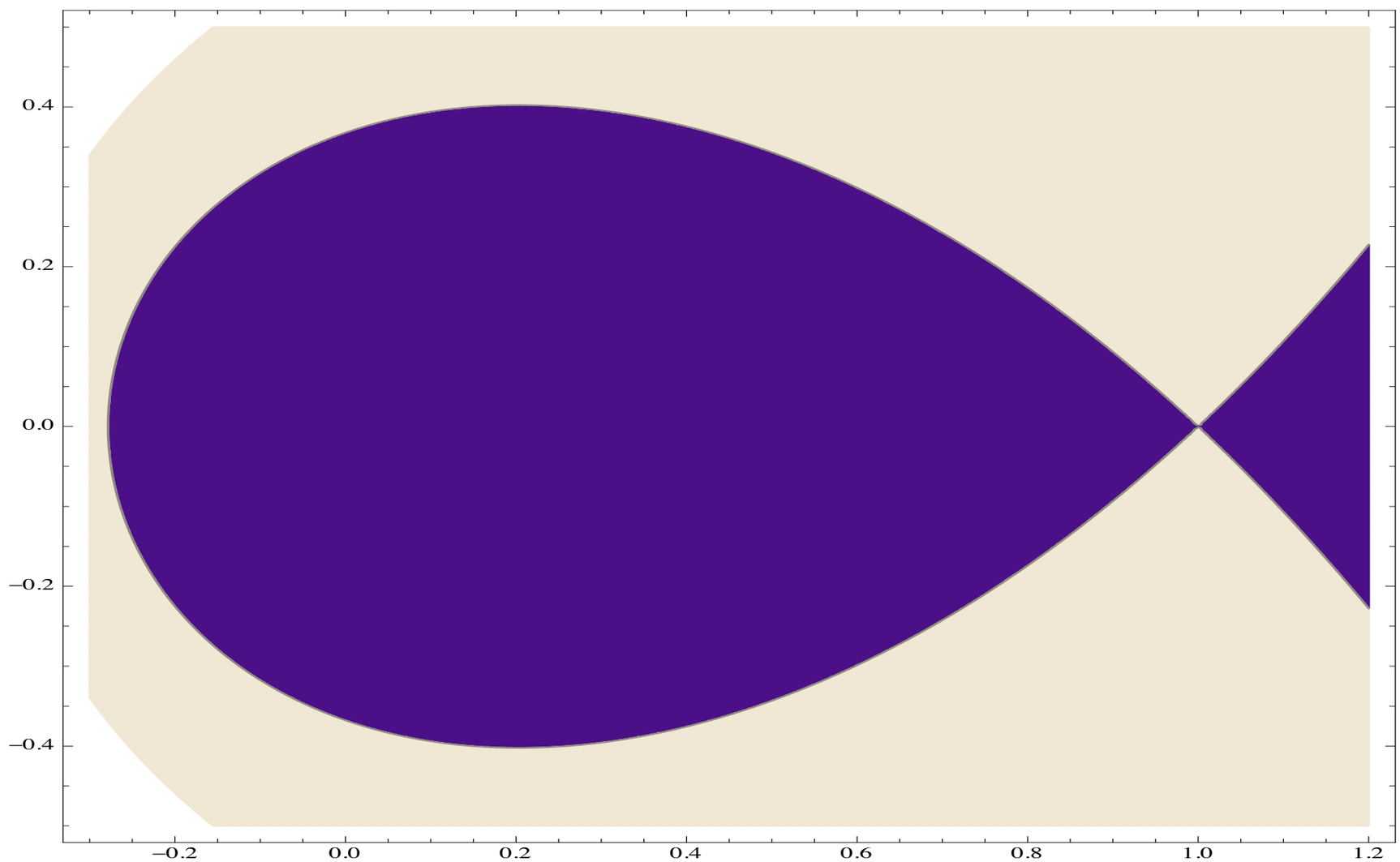
$$\left\{z : \left|ze^{1-z}\right| = 1\right\}$$



And the density of eigenvalues

$$e^{-Nz}\sum_{j=0}^{N-1}\frac{1}{j!}\left(Nz\right)^j = 1 - \frac{\left(ze^{1-z}\right)^N}{1-z}\frac{1}{\sqrt{2\pi N}}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right)$$

$$\left\{z : \left|ze^{1-z}\right| = 1\right\}$$

$$\rho_1^{(N)}(z) = e^{-N|z|^2}\sum_{j=0}^{N-1}\frac{1}{j!}\left(N|z|^2\right)^j = 1 - \frac{\left(|z|^2 e^{1-|z|^2}\right)^N}{1-|z|^2}\frac{1}{\sqrt{2\pi N}}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right)$$

And the kernel

$$e^{-Nz}\sum_{j=0}^{N-1}\frac{1}{j!}(Nz)^j = 1 - \frac{\left(ze^{1-z}\right)^N}{1-z}\frac{1}{\sqrt{2\pi N}}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right)$$

$$\left\{z : \left|ze^{1-z}\right| = 1\right\}$$

$$K_N(z,w) = e^{-N(|z|^2+|w|^2)/2}\sum_{j=0}^{N-1}\frac{1}{j!}(Nz\overline{w})^j =$$

$$e^{-N(|z|^2+|w|^2)/2}e^{Nz\overline{w}}\left(1 - \frac{\left(z\overline{w}e^{1-z\overline{w}}\right)^N}{1-z\overline{w}}\frac{1}{\sqrt{2\pi N}}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right)\right)$$

$$\sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j = e^{Nz} - \frac{z^N}{2\pi i} \int_{\tilde{\gamma}} e^{N(s - \ln(s))} \frac{ds}{s-z} \ .$$

Steepest descent method for integrals:

$$\sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j = e^{Nz} - \frac{z^N e^N}{1-z} \frac{1}{\sqrt{2\pi N}} \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right)$$

Trouble for z near the stationary phase point 1.

$$Q = |z|^{2n}$$

$$K_N = nN^{1/n} e^{-\left(|z|^{2n}+|w|^{2n}\right)/2} \sum_{j=0}^{N-1} \frac{(N^{1/n} z\overline{w})^j}{\pi\Gamma\left(\frac{j+1}{n}\right)}$$

Taylor polynomial of degree $N-1$ for $\displaystyle\sum_{j=0}^{\infty} \frac{x^j}{\pi\Gamma\left(\frac{j+1}{n}\right)}$

Which, it turns out (thanks Dario e Matteo!) is a special function:

$$\frac{n}{\pi} \sum_{j=0}^{\infty} \frac{x^j}{\Gamma\left(\frac{j+1}{n}\right)} = \frac{1}{\pi} x^{n-1} e^{x^n} \left[n^2 + \sum_{\ell=1}^{n-1} \frac{\ell}{\Gamma\left(\frac{n-\ell}{n}\right)} \int_{x^n}^{\infty} t^{-1-\frac{\ell}{n}} e^{-t} dt \right]$$

Thank You!!!

Other fundamental quantities: gap probabilities

$$\text{Prob } \{ \text{ no evals in } (a, b)\}$$

$$F_N(\mu) = \text{Prob } \{\lambda_{max} < \mu\}$$

These are "local" or "microscopic" statistical quantities...

GUE: Density of $\left(\lambda_{\max} - \sqrt{2N}\right) N^{1/6}$