1. Random matrices: a brief intro

2. Asymptotics of Taylor polynomials and the normal matrix model.

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The basic example of random matrices: $N \times N$ Hermitian matrices:

\[
M_{jj}: \text{Gaussian random variable: } \text{Prob } \{ M_{jj} < x \} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt
\]
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$M_{jj}$: Gaussian random variable: $\text{Prob} \ \{ M_{jj} < x \} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt$

$M_{jk}$: Complex Gaussian random variable, $M_{jk} = M_{jk}^{(R)} + iM_{jk}^{(I)}$, with

$\text{Prob} \ \left\{ M_{jk}^{(R)} < x \right\} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2} dt$

$\text{Prob} \ \left\{ M_{jk}^{(I)} < x \right\} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2} dt$
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\[
\begin{pmatrix}
M_{11} & M_{12}^{(R)} + iM_{12}^{(I)} & \cdots & M_{1N}^{(R)} + iM_{1N}^{(I)} \\
M_{12}^{(R)} - iM_{12}^{(I)} & M_{22} & \cdots & M_{2N}^{(R)} + iM_{2N}^{(I)} \\
\vdots & \cdots & \cdots & \vdots \\
M_{1N}^{(R)} - iM_{1N}^{(I)} & M_{2N}^{(R)} - iM_{2N}^{(I)} & \cdots & M_{NN}
\end{pmatrix}
\]

This is referred to as the Gaussian Unitary Ensemble
Using Matlab, we can generate random matrices and compute their eigenvalues.
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N = 200

N = 400

N = 600

N = 800
The basic example of random matrices: $N \times N$ Hermitian matrices:

$M_{jj}$: Gaussian random variable, $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}M_{jj}^2} dM_{jj}$

$M_{jk}$: Complex Gaussian random variable, $\frac{1}{\pi} e^{-|M_{jk}|^2} dM_{jk}^{(R)} dM_{jk}^{(I)}$

Joint prob. measure: $\frac{1}{\# N} \exp \left\{-\text{Tr} \left[ \frac{1}{2} M^2 \right]\right\} dM,$

$$dM = \prod_{j<k} dM_{jk}^{(R)} dM_{jk}^{(I)} \prod_{j=1}^{N} dM_{jj}$$

This is referred to as the Gaussian Unitary Ensemble
Support seems to grow like $\sqrt{N}$...

Rescale the matrices $M \mapsto N^{1/2}M$:

Joint prob. measure: $\frac{1}{\#N} \exp \left\{ -N \text{ Tr} \left[ \frac{1}{2} M^2 \right] \right\} \, dM,$

$$dM = \prod_{j<k} dM_{jk}^{(R)} \, dM_{jk}^{(I)} \prod_{j=1}^{N} dM_{jj}$$

Still unitarily invariant...
Unitary Ensembles of Random Matrices

Consider the probability measure on $N \times N$ Hermitean matrices given by

$$\frac{1}{\hat{Z}_N} \exp \left\{ -N \text{Tr} \left[ V(M) \right] \right\} dM$$

$$dM = \prod_{j < k} dM^R_{jk} \prod_{j = 1}^N dM_{jj} \prod_{j = 1}^N dM^I_{jk}$$

$$\hat{Z}_N = \int \exp \left\{ -N \text{Tr} \left[ V(M) \right] \right\} dM$$

Historically, the interest has been in the probabilistic description of the eigenvalues, as $N \to \infty$

Gaussian Unitary Ensemble: $V(x) = x^2/2$
There are many different matrix models.

\[
\frac{1}{\hat{Z}_N} \exp \left\{ -N \text{Tr} \left[ V(M) \right] \right\} dM
\]

Example 2: \( N \times N \) Unitary matrices
(Using Haar measure)

Example 3: A coupled Matrix Model

\[
\frac{1}{\hat{Z}_N} \exp \left\{ -N \text{Tr} \left[ \frac{1}{2} A^2 + B^2 + C^2 - it (ABC + ABC) \right] \right\} dA \ dB \ dC
\]
Example 4: Normal Matrix Model:

A weaker symmetry requirement: $[M, M^*] = 0$.

Given $V(z, \bar{z})$, one defines

$$\frac{1}{Z_N} \exp \left[ - \frac{N}{t} \text{Tr} \left( V(M, M^*) \right) \right] \text{“}dM\text{“}$$

Induced measure on eigenvalues:

$$\frac{1}{\hat{Z}_N} e^{-\frac{N}{t} \sum_{j=1}^{N} V(z_j, \bar{z}_j)} \prod_{1 \leq j < k \leq N} |z_j - z_k|^2 \ dA(z, \bar{z})$$
What does one want to calculate

as $N \to \infty$?
Mean density of eigenvalues

Consider the random variable \( \frac{1}{N} \# \{ \lambda_j \in \mathcal{B} \} \).

What is its average behavior?

\[
E \left[ \frac{1}{N} \# \{ \lambda_j \in \mathcal{B} \} \right] = \int_{\mathcal{B}} \rho_1^{(N)} \left\{ dx \right\} dy
\]

\( \rho_1^{(N)} \) is called the mean density of eigenvalues

Question: behavior when \( N \to \infty \)???
Movie of $\rho_1^{(N)}$ for $N = 1$ through $N = 50$. 

\[ \frac{1}{\#N} \exp \left\{ -N \text{ Tr} \left[ \frac{1}{2} M^2 \right] \right\} dM, \]
\[
\frac{1}{\#_N} \exp \left\{ -N \text{Tr} \left[ \frac{1}{2} M^2 \right] \right\} \, dM,
\]

Movie of \( \rho_1^{(N)} \) for \( N = 1 \) through \( N = 50 \).
Movie of $\rho_1^{(N)}$ for $N = 1$ through $N = 50$. 

\[ \frac{1}{\#_N} \exp\{-N \text{ Tr}[\, M M^*]\} \, dM, \]
\[
\frac{1}{\#_N} \exp \left\{ -N \; \text{Tr} \left[ \begin{array}{c} MM^* + \frac{T^2}{2} \left( M^2 + (M^*)^2 \right) \end{array} \right] \right\} \; dM,
\]

Movie of $\rho_1^{(N)}$ for $N = 1$ through $N = 50$. 

Wednesday, September 18, 13
\[
\frac{1}{\#_N} \exp \left\{ -N \operatorname{Tr} \left[ M^2 (M^*)^2 \right] \right\} \, dM,
\]

Movie of \( \rho_1^{(N)} \) for \( N = 1 \) through \( N = 60 \).
Occupation Probabilities

\[ F(B, z) = \sum_{k=0}^{\infty} \text{Prob } \{ \mathcal{B} \text{ has } k \text{ eigenvalues} \} z^k \]

\( F(B, z) \) is called the occupation probability generating function.

Question: behavior when \( N \to \infty \)???
Example 4: Normal Matrix Model:

\( n \times n \) matrices with \([M, M^*] = 0\).

Given \( Q(z, \bar{z}) \), one defines

\[
\frac{1}{Z_N} \exp \left[ -N \text{Tr} (Q(M, M^*)) \right] dM
\]

Induced measure on eigenvalues:

\[
\frac{1}{\hat{Z}_N} e^{-N \sum_{j=1}^{n} Q(z_j, \bar{z}_j)} \prod_{1 \leq j < k \leq n} |z_j - z_k|^2 \ dA(z, \bar{z})
\]
Orthogonal Polynomials

\[ P_{n,N}(z) = z^n + \mathcal{O}(z^{n-1}) \quad (n = 0, 1, \ldots) \]

\[
\int_{\mathbb{C}} P_{k,N}(z) \overline{P_{l,N}(z)} e^{-NQ(z,\bar{z})} dA(x,y) = h_{k,N} \delta_{k,l}
\]

\[
\rho_{1}^{(N)}(z,\bar{z}) = e^{-NQ(z,\bar{z})} \sum_{\ell=0}^{N-1} h_{\ell,N}^{-1} |P_{\ell,N}(z)|^2
\]

\[
K_{N}(z,w) = e^{-\frac{1}{N}(Q(z)+Q(w))} \frac{1}{N} \sum_{k=0}^{n-1} h_{k}^{-1} P_{k,N}(z) \overline{P_{k,N}(w)}
\]

Their asymptotic behavior is wide open.
Small collection of examples for which calculations are explicit

- $Q(x, y) = x^2 + y^2$: Polynomials are $P_n(z) = z^n$, and

$$
\rho_N = \frac{1}{\pi} e^{-N|z|^2} \sum_{j=0}^{n-1} \frac{(N|z|^2)^j}{j!}
$$

as $N \to \infty$, support is a disc of radius 1, and $\lim_{N \to \infty} \rho_N = 1$. 
Small collection of examples for which calculations are explicit

- \( Q(x, y) = x^2 + y^2 \): Polynomials are \( P_n(z) = z^n \), and

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\]

as \( N \to \infty \), support is a disc of radius 1, and \( \lim_{N \to \infty} \rho_N = 1 \).

\[
K_N(z, w) = \frac{1}{\pi} e^{-N(|z|^2+|w|^2)/2} \sum_{j=0}^{n-1} \frac{(Nzw)^j}{j!}
\]
Small collection of examples for which calculations are explicit

- $Q(x, y) = x^2 + y^2$: Polynomials are $P_n(z) = z^n$, and

$$
\rho_N = \frac{1}{\pi} e^{-N|z|^2} \sum_{j=0}^{n-1} \frac{(N|z|^2)^j}{j!}
$$

as $N \to \infty$, support is a disc of radius 1, and $\lim_{N \to \infty} \rho_N = 1$.

- $Q = Q(|z|)$: $P_n = z^n$, limiting support is again a disc,

$$
\rho_N = \frac{e^{-NQ(|z|)}}{N} \sum_{j=0}^{n-1} \frac{|z|^{2j}}{2\pi \int_0^\infty r^{2j+1} e^{-NQ} \, dr} \to \frac{1}{4} \Delta V(|z|).
$$

$$
K_N(z, w) = \frac{e^{-N(Q(|z|)+Q(|w|))/2}}{N} \sum_{j=0}^{n-1} \frac{(zw)^j}{2\pi \int_0^\infty r^{2j+1} e^{-NQ} \, dr}
$$

• Analysis of kernel via potential theory, Bergmann kernel: Work of Ameur, Hedenmalm, Makarov

• Approach to general asymptotic analysis of 2D orthogonal polynomials: $\overline{\partial}$ problem, described by Its and Takhtajan, and Balogh and Harnad.
We have recently considered \( Q = |z|^2 + 2c \log \frac{1}{|z - a|} \)

\[
\frac{1}{Z_N} e^{-N \text{Tr} \left( M M^* \right)} |\det(M - a)|^{2Nc} \ dM
\]

arXiv:1209.6366
(With F. Balogh (SISSA), M. Bertola (Concordia Univ.), S. Y. Lee (Caltech)
With remaining time:

Asymptotic behavior of the zeros of \( \sum_{j=0}^{N-1} \frac{z^j}{j!} \)

Related questions

Application to NMM (for fun)
\[ \sum_{j=0}^{N-1} \frac{z^j}{j!} \]
\[
\sum_{j=0}^{N-1} \frac{(Nz)^j}{j!}
\]
\[
\sum_{j=0}^{N-1} \frac{(Nz)^j}{j!} = e^{Nz} \left(1 + O\left(\frac{1}{N}\right)\right)
\]
\[ f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(s)}{s - z} \, ds . \]
\[ f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(s)}{s-z} \, ds. \]

\[ = \sum_{j=0}^{N-1} \frac{f^j(0)}{j!} z^j + \frac{z^N}{2\pi i} \oint_{\gamma} \frac{f(s)}{s^n} \frac{ds}{s-z}. \]
\[ f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(s)}{s - z} ds. \]

\[ = \sum_{j=0}^{N-1} \frac{f^j(0)}{j!} z^j + \frac{z^N}{2\pi i} \int_{\gamma} \frac{f(s)}{s^n} \frac{ds}{s - z}. \]

If \( f(s) = e^s \), \( z \mapsto Nz \), \( s \mapsto Ns \):

\[ e^{Nz} = \sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j + \frac{z^N}{2\pi i} \int_{\tilde{\gamma}} e^{N(s - \ln(s))} \frac{ds}{s - z}. \]
\[
\sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j = e^{Nz} - \frac{z^N}{2\pi i} \int_{\tilde{\gamma}} e^{N(s - \ln(s))} \frac{ds}{s - z}.
\]

Steepest descent method for integrals:

\[
\sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j = e^{Nz} - \frac{z^N e^N}{1 - z} \frac{1}{\sqrt{2\pi N}} \left( 1 + O \left( \frac{1}{N} \right) \right)
\]

Looking for zeros:

\[
e^{-Nz} \sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j = 1 - \frac{(ze^{1-z})^N}{1 - z} \frac{1}{\sqrt{2\pi N}} \left( 1 + O \left( \frac{1}{N} \right) \right)
\]
Looking for zeros:

\[ e^{-Nz} \sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j = 1 - \frac{(ze^{1-z})^N}{1 - z} \frac{1}{\sqrt{2\pi N}} \left( 1 + O\left( \frac{1}{N} \right) \right) \]

\( \{ z : \left| ze^{1-z} \right| = 1 \} \)
And the density of eigenvalues

\[ e^{-Nz} \sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j = 1 - \frac{(ze^{1-z})^N}{1-z} \frac{1}{\sqrt{2\pi N}} \left( 1 + O \left( \frac{1}{N} \right) \right) \]

\{ z : |ze^{1-z}| = 1 \}

\[
\rho_1^{(N)}(z) = e^{-N|z|^2} \sum_{j=0}^{N-1} \frac{1}{j!} (N|z|^2)^j = 1 - \frac{(|z|^2e^{1-|z|^2})^N}{1-|z|^2} \frac{1}{\sqrt{2\pi N}} \left( 1 + O \left( \frac{1}{N} \right) \right)
\]
And the kernel

\[ e^{-Nz} \sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j = 1 - \frac{(ze^{1-z})^N}{1 - z} \frac{1}{\sqrt{2\pi N}} \left( 1 + \mathcal{O}\left( \frac{1}{N} \right) \right) \]

\{ z : |ze^{1-z}| = 1 \}

\[ K_N(z, w) = e^{-N(|z|^2 + |w|^2)/2} \sum_{j=0}^{N-1} \frac{1}{j!} (Nzw)^j = \]

\[ e^{-N(|z|^2 + |w|^2)/2} e^{Nzw} \left( 1 - \frac{(z\overline{w}e^{1-z\overline{w}})^N}{1 - z\overline{w}} \frac{1}{\sqrt{2\pi N}} \left( 1 + \mathcal{O}\left( \frac{1}{N} \right) \right) \right) \]
\[
\sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j = e^{Nz} - \frac{z^N}{2\pi i} \int_{\tilde{\gamma}} e^{N(s-\ln(s))} \frac{ds}{s-z}.
\]

Steepest descent method for integrals:
\[
\sum_{j=0}^{N-1} \frac{1}{j!} (Nz)^j = e^{Nz} - \frac{z^N e^N}{1 - z} \frac{1}{\sqrt{2\pi N}} \left( 1 + \mathcal{O} \left( \frac{1}{N} \right) \right)
\]

Trouble for \(z\) near the stationary phase point 1.
\[ Q = |z|^{2n} \]

\[
K_N = nN^{1/n} e^{-\left(|z|^{2n} + |w|^{2n}\right)/2} \sum_{j=0}^{N-1} \frac{(N^{1/n} z \bar{w})^j}{\pi \Gamma \left( \frac{j+1}{n} \right)}
\]

Taylor polynomial of degree \( N - 1 \) for

\[
\sum_{j=0}^{\infty} \frac{x^j}{\pi \Gamma \left( \frac{j+1}{n} \right)}
\]

Which, it turns out (thanks Dario e Matteo!) is a special function:

\[
\frac{n}{\pi} \sum_{j=0}^{\infty} \frac{x^j}{\Gamma \left( \frac{j+1}{n} \right)} = \frac{1}{\pi} x^{n-1} e^{x^n} \left[ n^2 + \sum_{\ell=1}^{n-1} \frac{\ell}{\Gamma \left( \frac{n-\ell}{n} \right)} \int_0^{\infty} t^{1-\frac{\ell}{n}} e^{-t} \, dt \right]
\]
Thank You!!!
Other fundamental quantities: gap probabilities

\[ \text{Prob} \{ \text{no evals in } (a, b) \} \]

\[ F_N(\mu) = \text{Prob} \{ \lambda_{max} < \mu \} \]

These are "local" or "microscopic" statistical quantities...
GUE: Density of $\left( \lambda_{\text{max}} - \sqrt{2N} \right) N^{1/6}$