The Semiclassical Sine-Gordon Equation and Rational Solutions of Painlevé-II

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The Sine-Gordon Equation

Semiclassical limit for pure-impulse initial data.

Consider the following Cauchy problem for $u^\epsilon = u^\epsilon(x,t)$:

$$\epsilon^2 u^\epsilon_{tt} - \epsilon^2 u^\epsilon_{xx} + \sin(u^\epsilon) = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

$$u^\epsilon(x,0) = F(x), \qquad \epsilon u^\epsilon_t(x,0) = G(x).$$

Here $\epsilon > 0$ is a parameter, and F and G are independent of ϵ.

- Suppose that this Cauchy problem has a unique solution $u^\epsilon(x,t)$ for all sufficiently small $\epsilon > 0$. What can be said about the asymptotic behavior of $u^\epsilon(x,t)$ in the limit $\epsilon \downarrow 0$?
- Analyzing this kind of asymptotic question is what it means to study the *semiclassical limit for the sine-Gordon Cauchy problem* in laboratory coordinates.
- For convenience, we consider only *pure impulse* initial data, *i.e.* $F(\cdot) \equiv 0$.

The Sine-Gordon Equation

Impulse threshold for rotation.

The sine-Gordon equation as a perturbed simple pendulum:

$$u^{\epsilon}_{TT} + \sin(u^{\epsilon}) = \epsilon^2 u^{\epsilon}_{xx}, \quad u^{\epsilon}(x,0) = F(x), \quad u^{\epsilon}_T(x,0) = G(x),$$

where $t = \epsilon T$. The unperturbed problem conserves total energy

$$E = \frac{1}{2}(u^{\epsilon}_T)^2 + (1 - \cos(u^{\epsilon})) = \frac{1}{2}G(x)^2, \quad \text{if } F \equiv 0.$$

For $T = \mathcal{O}(1)$, the pendulum at x undergoes approximate

- librational motion ($|u^{\epsilon}| < \pi$) if $E = E(x) < 2$
- rotational motion (u^{ϵ} growing without bound) if $E = E(x) > 2$.

Therefore, a sufficiently strong initial impulse profile should produce both types of motion separated by values $x = x_{\text{crit}}$ where $G(x_{\text{crit}}) = \pm 2$.

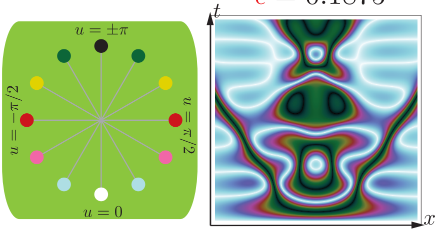
The Sine-Gordon Equation

R. Buckingham and P. D. Miller, *Mem. AMS*, **225**, 1–152, 2013.

This basic picture has been confirmed rigorously for a wide variety of initial impulse profiles G.

$$G(x) = -3\,\mathrm{sech}(x) \qquad\qquad (x,t) \in (-2.5, 2.5) \times (0,5)$$

$$\epsilon = 0.1875$$



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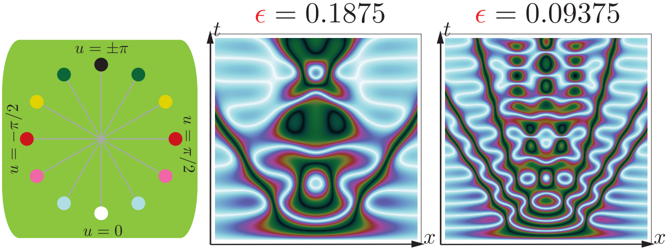

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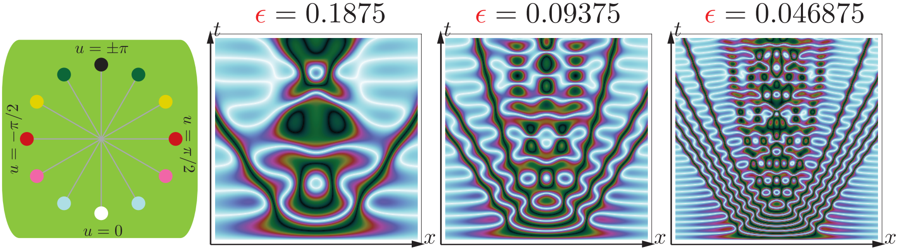

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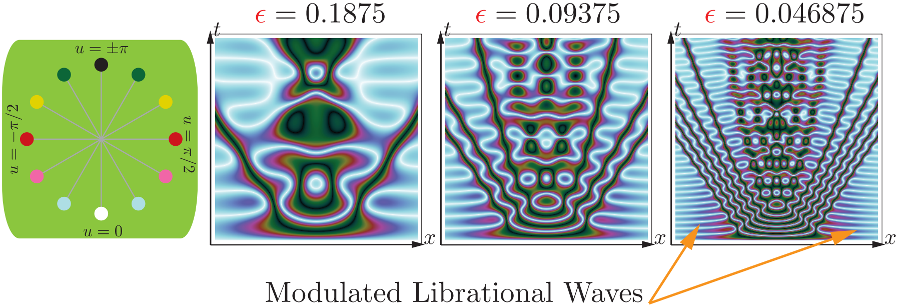

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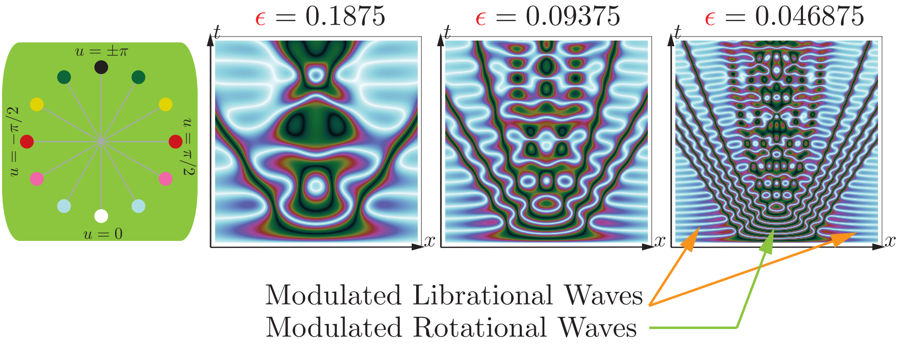
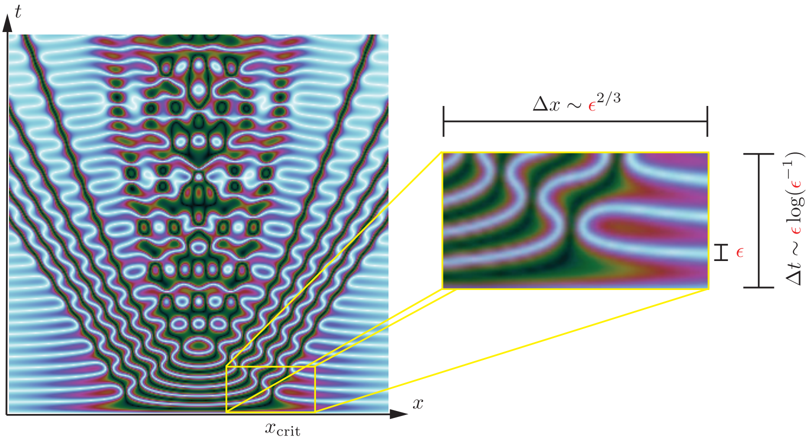


The Sine-Gordon Equation

R. Buckingham and P. D. Miller, *J. d'Anal. Math.*, **118**, 397–492, 2012.

Consider the behavior of $u^{\epsilon}(x,t)$ near a point $x = x_{\text{crit}}$ where the initial data crosses the pendulum separatrix at $t = 0$:



Let $\nu := [12G'(x_{\text{crit}})]^{-1} > 0$ and set $\Delta x := x - x_{\text{crit}}$.

The Sine-Gordon Equation

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Set $\mathcal{U}_0(y) := 1$ and $\mathcal{V}_0(y) := -y/6$. Generate $\{\mathcal{U}_m, \mathcal{V}_m\}_{m\in\mathbb{Z}}$ by the recursions

$$\mathcal{U}_{m+1}(y) := -\frac{1}{6}y\mathcal{U}_m(y) - \frac{\mathcal{U}_m'(y)^2}{\mathcal{U}_m(y)} + \frac{1}{2}\mathcal{U}_m''(y) \quad \text{and} \quad \mathcal{V}_{m+1}(y) := \frac{1}{\mathcal{U}_m(y)}$$

$$\mathcal{U}_{m-1}(y) := \frac{1}{\mathcal{V}_m(y)} \quad \text{and} \quad \mathcal{V}_{m-1}(y) := \frac{1}{2}\mathcal{V}_m''(y) - \frac{\mathcal{V}_m'(y)^2}{\mathcal{V}_m(y)} - \frac{1}{6}y\mathcal{V}_m(y).$$

It turns out that $(\mathcal{U}, \mathcal{V}) = (\mathcal{U}_m, \mathcal{V}_m)$ satisfy for each m the coupled system of second-order Painlevé II-type equations

$$\mathcal{U}''(y) + 2\mathcal{U}(y)^2\mathcal{V}(y) + \frac{1}{3}y\mathcal{U}(y) = 0$$
$$\mathcal{V}''(y) + 2\mathcal{U}(y)\mathcal{V}(y)^2 + \frac{1}{3}y\mathcal{V}(y) = 0.$$

The Sine-Gordon Equation

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Fix an integer m and assume that (x, t) lies in the horizontal strip S_m in the (x, t)-plane given by the inequality

$$\left| t - \frac{2}{3} m\epsilon \log(\epsilon^{-1}) \right| \leq \frac{1}{3}\epsilon \log(\epsilon^{-1}).$$

Suppose also that $\Delta x = \mathcal{O}(\epsilon^{2/3})$. Then as $\epsilon \to 0$,

$$\cos(\tfrac{1}{2}u^{\epsilon}(x, t)) = (-1)^m \operatorname{sgn}(\mathcal{U}_m(y)) \operatorname{sech}(T) + o(1)$$
$$\sin(\tfrac{1}{2}u^{\epsilon}(x, t)) = (-1)^{m+1} \tanh(T) + o(1)$$

where

$$T := \frac{t}{\epsilon} - 2m \log\left(\frac{4\nu^{1/3}}{\epsilon^{1/3}}\right) + \log|\mathcal{U}_m(y)|, \quad \text{and} \quad y := \frac{\Delta x}{2\nu^{1/3}\epsilon^{2/3}}.$$

The Sine-Gordon Equation

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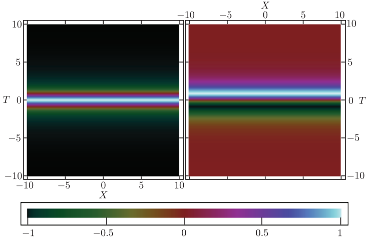
The leading terms determine a limiting function $u(T)$ modulo 2π:

$$\cos(u(T)) := 2\,\mathrm{sech}^2(T) - 1 \text{ and } \sin(u(T)) := -2\sigma\,\mathrm{sech}(T)\tanh(T),$$

and $u(T)$ is an X-independent solution of the unscaled equation

$$u_{TT} - u_{XX} + \sin(u) = 0.$$

This exact solution represents a superluminal (infinite velocity) kink with unit magnitude *topological charge* $\sigma := \mathrm{sgn}(\mathcal{U}_m(y))$.



$\cos(u(T))$ $\sin(u(T))$

The Sine-Gordon Equation

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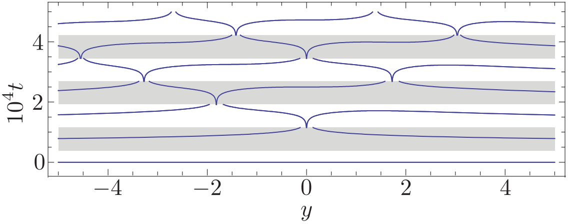
The kink is slowly modulated in the direction parallel to the wavefront; the center ($T = 0$) is located along a vertical translate of the graph of $-\log|\mathcal{U}_m(y)|$:



The strips $S_0, \ldots, S_6$ in the (y,t)-plane for $\epsilon = 10^{-5}$ and $4\nu^{1/3} = 1$, and the curve $T = 0$ in each strip.

Note the left-right asymmetry of the pattern.

Rational Painlevé-II Functions

Relation to the inhomogeneous Painlevé-II equation.

The logarithmic derivatives

$$\mathcal{P}_m(y) := \frac{\mathcal{U}_m'(y)}{\mathcal{U}_m(y)} \quad \text{and} \quad \mathcal{Q}_m(y) := \frac{\mathcal{V}_m'(y)}{\mathcal{V}_m(y)}$$

satisfy uncoupled equations:

$$\begin{aligned}\mathcal{P}_m''(y) &= 2\mathcal{P}_m(y)^3 + \frac{2}{3}y\mathcal{P}_m(y) - \frac{2}{3}m \\ \mathcal{Q}_m''(y) &= 2\mathcal{Q}_m(y)^3 + \frac{2}{3}y\mathcal{Q}_m(y) + \frac{2}{3}(m-1).\end{aligned}$$

There exists a unique rational solution to the inhomogeneous Painlevé-II equation (PII-α)

$$\mathcal{P}''(y) = 2\mathcal{P}(y)^3 + \frac{2}{3}y\mathcal{P}(y) + \frac{2}{3}\alpha, \quad \alpha \in \mathbb{C}$$

iff $\alpha \in \mathbb{Z}$ (Y. Murata, 1985). Thus *all* such solutions arise in this way.

Rational Painlevé-II Functions

History & applications.

The functions $\mathcal{P}_m(y)$ appear to have been discovered as solutions of PII-α for $\alpha = m \in \mathbb{Z}$ via Bäcklund transformations by Airault (1979).

The functions $\mathcal{P}_m(y)$ are known to be important in several applications:

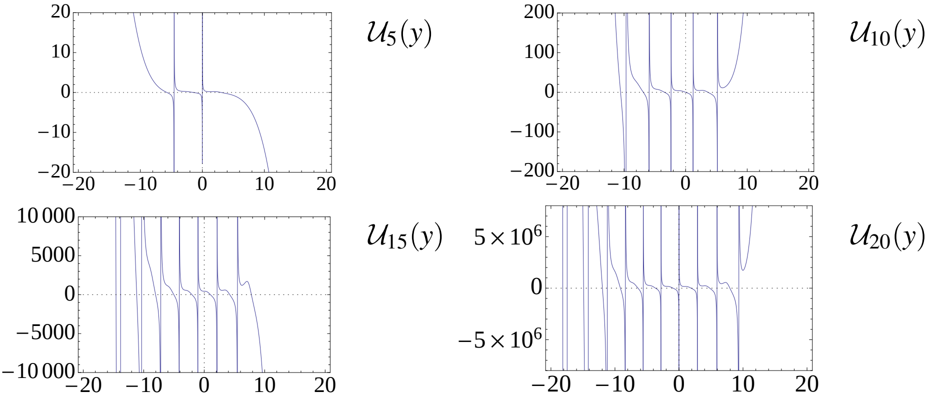
- Their singularities describe equilibrium configurations of interacting fluid vortices in the plane. (P. Clarkson, 2009)
- They appear in string theory. (C. Johnson, 2006).
- The related functions $\mathcal{U}_m(y)$ describe the universal wave pattern near a simple crossing of the pendulum separatrix in the semiclassical sine-Gordon equation.

Key point: in the latter application the question of the large-$|m|$ asymptotic behavior of the rational Painlevé-II functions is natural. It is associated with understanding how the universal wave pattern near the critical point matches onto the larger-time dynamics of sine-Gordon.

Rational Painlevé-II Functions

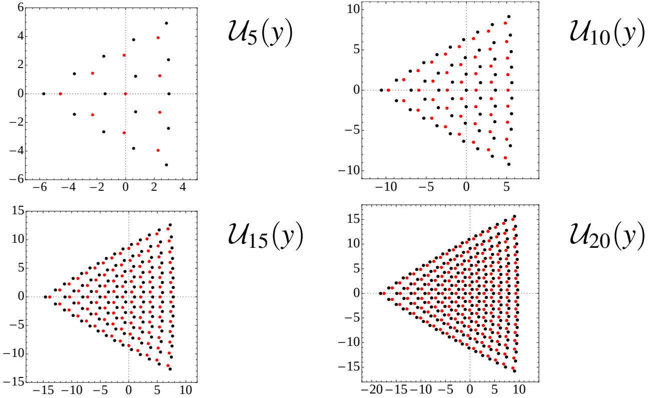
Poles and zeros.

The real poles and zeros of $\mathcal{U}_m(y)$ are important in part because they locate the grazing collisions of the kinks.



More insight can be gained by looking in the complex y-plane...

The functions $\mathcal{U}_m(y)$ are ratios of consecutive *Yablonskii-Vorob'ev polynomials*. The complex zeros of these polynomials were studied numerically by Clarkson and Mansfield (2003).



Red/black dots: poles/zeros. These pictures suggest an explanation for the left-right asymmetry of the universal wave pattern.

Flaschka and Newell (1980) considered the Lax pair:

$$\frac{\partial \mathbf{v}}{\partial \zeta} = \begin{bmatrix} -i(4\zeta^2 + x + 2q^2) & 4\zeta q - 2ir + \nu\zeta^{-1} \\ 4\zeta q + 2ir + \nu\zeta^{-1} & i(4\zeta^2 + x + 2q^2) \end{bmatrix} \mathbf{v}, \quad \frac{\partial \mathbf{v}}{\partial x} = \begin{bmatrix} -i\zeta & q \\ q & i\zeta \end{bmatrix} \mathbf{v}$$

for which the compatibility condition is $q''(x) = 2q^3 + xq - \nu$. Note that

$$\mathcal{P} = (\tfrac{2}{3})^{1/3} q, \quad y = (\tfrac{3}{2})^{1/3} x, \quad \alpha = -\nu \quad \Longrightarrow \quad \text{PII-}\alpha \text{ for } \mathcal{P}(y).$$

In this setting, the rational solutions $\mathcal{P}_m(y)$ for $\alpha = m \in \mathbb{Z}$ correspond to the case that *all Stokes multipliers are zero*. The inverse monodromy problem amounts to the construction of a meromorphic matrix function of ζ with only one pole of order $|m|$ at $\zeta = 0$ with given exponential behavior at $\zeta = \infty$, i.e. a Riemann-Hilbert problem *without jumps* solvable by determinants. The rational functions $\mathcal{P}_m(y)$ are the *solitons* of PII.

Encoding the rational functions $\mathcal{P}_m(y)$ in terms of a Riemann-Hilbert problem with jumps instead of poles is preferable for asymptotic analysis in the limit $\alpha = m \to \infty$.

Luckily, such a representation is exactly what comes out of sine-Gordon: a Riemann-Hilbert problem with jumps and no poles characterizing $\mathcal{P}_m(y)$ (as well as $\mathcal{U}_m(y)$) related to the Lax pair found by Jimbo and Miwa (1981):

$$\frac{\partial \mathbf{v}}{\partial \zeta} = \frac{1}{2}\begin{bmatrix} -3\zeta^2 - 6\mathcal{U}\mathcal{V} - y & 6\mathcal{U}\zeta + 2\mathcal{W} \\ -6\mathcal{V}\zeta - 2\mathcal{Z} & 3\zeta^2 + 6\mathcal{U}\mathcal{V} + y \end{bmatrix}\mathbf{v}, \quad \frac{\partial \mathbf{v}}{\partial y} = \frac{1}{2}\begin{bmatrix} -\zeta & 2\mathcal{U} \\ -2\mathcal{V} & \zeta \end{bmatrix}\mathbf{v}$$

with compatibility conditions

$$\mathcal{W}(y) = -3\mathcal{U}'(y), \qquad \mathcal{W}'(y) = 6\mathcal{U}(y)^2\mathcal{V}(y) + y\mathcal{U}(y),$$
$$\mathcal{Z}(y) = 3\mathcal{V}'(y), \qquad \mathcal{Z}'(y) = -6\mathcal{U}(y)\mathcal{V}(y)^2 - y\mathcal{V}(y).$$

The sine-Gordon inverse-scattering problem can be written as a matrix Riemann-Hilbert problem: given a contour $\Sigma \subset \mathbb{C}$ and a matrix function $V : \Sigma \to SL(2, \mathbb{C})$, find $\mathbf{M} : \mathbb{C} \setminus \Sigma \to SL(2, \mathbb{C})$ such that:

- $\mathbf{M}_+(w) = \mathbf{M}_-(w)\mathbf{V}(w)$ holds at each point of Σ and
- $\mathbf{M}(w) \to \mathbb{I}$ as $w \to \infty$.

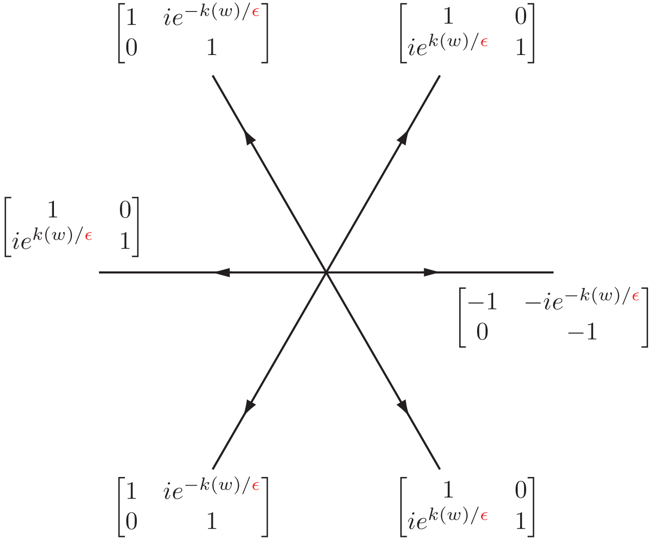
After some systematic preparations (introduction of g-function, Deift-Zhou steepest descent method) this problem is converted into another equivalent one in which:

- As $\epsilon \to 0$, $\mathbf{V}(w)$ converges uniformly to something simple except near one point, $w = w_* \approx -1$.
- Away from w_* there is an obvious approximate solution indexed by an arbitrary parameter $m \in \mathbb{Z}$.

Techniques of Asymptotic Analysis

Semiclassical sine-Gordon origin of Jimbo-Miwa problem.

For $w \approx w_*$, the jump matrices $\mathbf{V}(w)$ don't converge uniformly:



The exponent function $k(w) = k(w; x, t)$ is globally complicated, but is approximately cubic near w_* when $x \approx x_{\text{crit}}$ and $t \approx 0$. By Chester, Friedman, and Ursell (1957), there are analytic spacetime coordinates $r(x, t) \approx \Delta x/(2\nu^{1/3})$ and $s(x, t) \approx t$ such that $k(w) = k(w; x, t)$ *is exactly a cubic polynomial* in a new variable ζ:

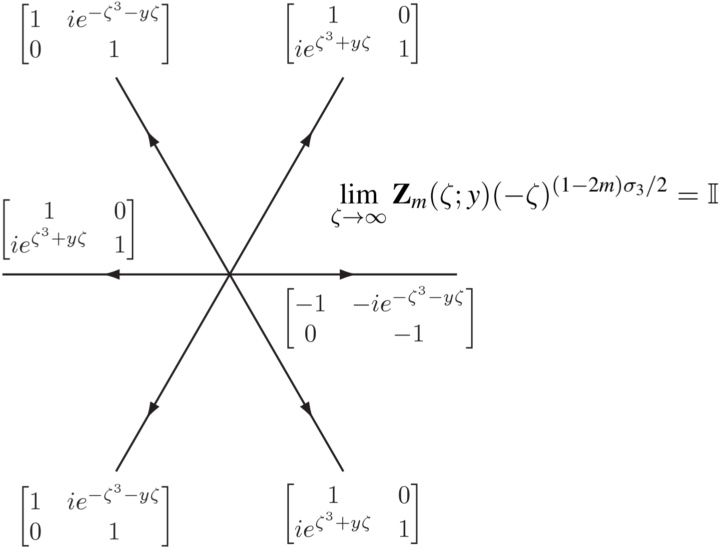
$$\frac{k(w)}{\epsilon} = \zeta^3 + y\zeta - \frac{s}{\epsilon}, \quad \zeta := \frac{W(w)}{\epsilon^{1/3}}, \quad y := \frac{r}{\epsilon^{2/3}}.$$

The large constant term in the cubic can be removed by an explicit transformation. The value of m indexing the "outer" approximation has to be matched to the value of s; this leads to the necessity of introducing the coordinate strips S_m.

Techniques of Asymptotic Analysis

Semiclassical sine-Gordon origin of Jimbo-Miwa problem.

The problem reduces to the following *parametrix* RHP for $\mathbf{Z}_m(\zeta; y)$:



Let $\mathbf{L}_m := \mathbf{Z}_m e^{-(\zeta^3+y\zeta)\sigma_3/2}$.

- This transformation removes all exponential factors from the jump matrices. It follows that

$$\mathbf{U}_m := \frac{\partial \mathbf{L}_m}{\partial y}\mathbf{L}_m^{-1} \quad \text{and} \quad \mathbf{V}_m := \frac{\partial \mathbf{L}_m}{\partial \zeta}\mathbf{L}_m^{-1}$$

  are both entire functions of ζ.
- Prescribed behavior of $\mathbf{Z}_m$ as $\zeta \to \infty$ implies that $\mathbf{U}_m$ and $\mathbf{V}_m$ are polynomials of degree 1 and 2 respectively. The coefficients in these polynomials come from the large-ζ expansion of $\mathbf{Z}_m$.
- Therefore, $\mathbf{L}_m$ satisfies the overdetermined linear system

$$\frac{\partial \mathbf{L}_m}{\partial y} = \mathbf{U}_m\mathbf{L}_m \quad \text{and} \quad \frac{\partial \mathbf{L}_m}{\partial \zeta} = \mathbf{V}_m\mathbf{L}_m.$$

This is precisely the Jimbo-Miwa Lax pair.

We analyze the parametrix RHP for $\mathbf{Z}_m(\zeta;y)$ in the limit $m\to\infty$ using the *Deift-Zhou steepest descent method*. Note that for each fixed $m\in\mathbb{Z}$, we have the exact formulae:

$$\mathcal{U}_m(y)=A_{m,12}(y)\quad\text{and}\quad\mathcal{P}_m(y)=A_{m,22}(y)-\frac{B_{m,12}(y)}{A_{m,12}(y)}$$

where the matrices $\mathbf{A}_m(y)$ and $\mathbf{B}_m(y)$ are obtained from the expansion:

$$\mathbf{Z}_m(\zeta;y)(-\zeta)^{(1-2m)\sigma_3/2}=\mathbb{I}+\mathbf{A}_m(y)\zeta^{-1}+\mathbf{B}_m(y)\zeta^{-2}+\mathcal{O}(\zeta^{-3}),\quad\zeta\to\infty.$$

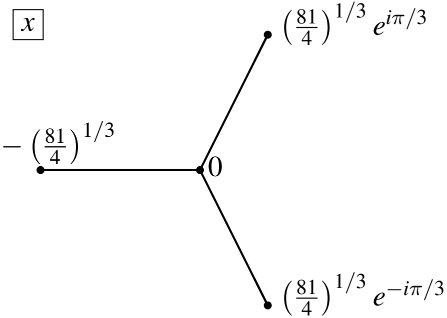
We scale: $z:=(m-\frac{1}{2})^{-1/3}\zeta$ and $x:=(m-\frac{1}{2})^{-2/3}y$.

Some asymptotic features of the functions $\mathcal{U}_m(y)$ and $\mathcal{P}_m(y)$ resemble those of more general solutions of PII-α in the limit of large α obtained by Kapaev (1997) by means of the *isomonodromy method*, an important precursor of the steepest descent method.

Asymptotic Description of $\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m

Formulae valid for sufficiently large $|x|$. One-cut/genus-zero analysis.

The cubic equation $3S^3 + 4xS + 8 = 0$ has a unique solution $S = S(x)$ that is analytic for $x \in \mathbb{C} \setminus \Sigma_S$ where Σ_S is the contour



Note that $S(x) = -2x^{-1} + \mathcal{O}(x^{-4})$ as $x \to \infty$ and $S(x)$ is real for $x \in \mathbb{R}$.

Asymptotic Description of $\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m

Formulae valid for sufficiently large $|x|$. One-cut/genus-zero analysis.

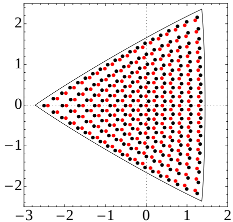
Theorem

There exists a piecewise-analytic simple closed curve ∂T such that uniformly for $x = y/(m-\frac{1}{2})^{2/3}$ bounded outside ∂T (and also for x as close as $\log(m)/m$ from an edge — but not a corner), as $m \to +\infty$,

$$m^{-2m/3}e^{-m\lambda(x)}\mathcal{U}_m = \dot{\mathcal{U}}(x) + \mathcal{O}(m^{-1}), \quad \dot{\mathcal{U}}(x) := e^{xS(x)/6},$$

$$m^{-1/3}\mathcal{P}_m = \dot{\mathcal{P}}(x) + \mathcal{O}(m^{-1}), \quad \dot{\mathcal{P}}(x) := -\frac{1}{2}S(x),$$

where the normalizing exponent for $\mathcal{U}$ is $\lambda(x) := \frac{1}{4}S(x)^3 - \log(3S(x))$.



Poles and zeros of $\mathcal{U}_m(y)$ in the x-plane for $m = 20$ and the curve ∂T. The opening angle of ∂T at each corner is exactly $2\pi/5$. ∂T is (part of) the zero locus of an explicit (in S) harmonic function.

Asymptotic Description of $\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m

Formulae valid in the elliptic region. Two-cut/genus-one analysis.

Boutroux ansatz method: Let x_0 be a fixed complex number and set

$$y = (m - \tfrac{1}{2})^{2/3}(x_0 + (m - \tfrac{1}{2})^{-1}w).$$

Writing $\mathcal{P}_m(y) = (m - \frac{1}{2})^{1/3}q(w)$ converts the exact equation

$$\mathcal{P}_m''(y) = 2\mathcal{P}_m(y)^3 + \frac{2}{3}y\mathcal{P}_m(y) - \frac{2}{3}m$$

into the form

$$q''(w) = 2q(w)^3 + \frac{2}{3}x_0q(w) - \frac{2}{3} + (m - \tfrac{1}{2})^{-1}\left[\frac{2}{3}wq(w) - \frac{1}{3}\right].$$

Neglecting the formally small final term results in an equation solved by elliptic functions with modulus depending on x_0. This turns out to be a valid approximation, as long as x_0 lies within the interior of T, the "elliptic region".

Asymptotic Description of $\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m

Formulae valid in the elliptic region. Two-cut/genus-one analysis.

Theorem

There exists a smooth but non-analytic function $\Lambda : T \to \mathbb{C}$ such that with $y = (m - \frac{1}{2})^{2/3}x$ and $x = x_0 + (m - \frac{1}{2})^{-1}w$, as $m \to +\infty$,

$$m^{-2m/3}e^{-m\Lambda(x)}\mathcal{U}_m = \frac{\dot{\mathcal{U}}_m(w; x_0)}{1 + \mathcal{O}(m^{-1}\dot{\mathcal{U}}_m(w; x_0))},$$

$$m^{-1/3}\mathcal{P}_m = \frac{\dot{\mathcal{P}}_m(w; x_0)}{1 + \mathcal{O}(m^{-1}\dot{\mathcal{P}}_m(w; x_0))},$$

both hold uniformly for x_0 in compact subsets of T and w bounded, where $\dot{\mathcal{U}}_m(w; x_0)$ and $\dot{\mathcal{P}}_m(w; x_0)$ are explicitly constructed in terms of the Riemann theta function associated with a uniquely determined elliptic curve $\Gamma(x_0)$.

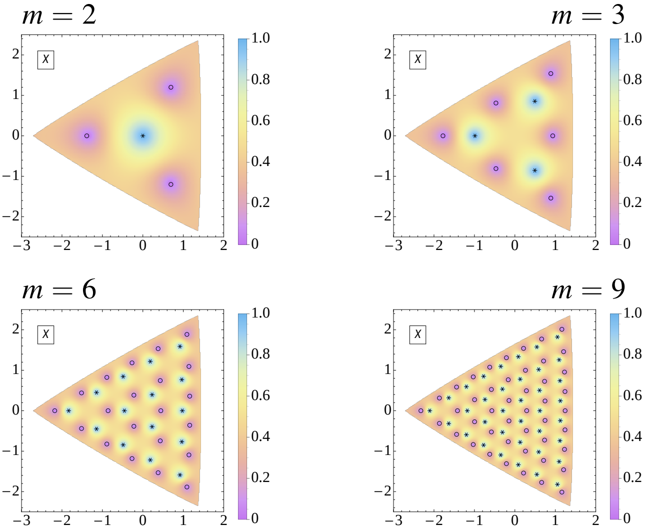
Some notes:

- For each $x_0 \in T$, $\dot{\mathcal{P}}_m(w; x_0)$ is an elliptic function of w that solves the Boutroux ansatz differential equation.
- Accuracy even near poles is obtained using Bäcklund transformations.
- Pole/zero locations accurate to $\mathcal{O}(m^{-2})$ in x; spacing scales as m^{-1}.
- Interpretation of two-variable approximations:
  - x_0 is a coordinate on the base manifold T. Setting $w = 0$ gives a *uniform approximation* that is not meromorphic in $x_0 = x$.
  - w is a coordinate on the tangent space to T at x_0. Fixing x_0 and varying w gives a *tangent approximation* that is meromorphic in w but only locally accurate.

Asymptotic Description of $\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m

Formulae valid in the elliptic region. Two-cut/genus-one analysis.

Plots of $\frac{2}{\pi}\arctan(|\dot{\mathcal{U}}_m(0;x)|)$ with zeros ($\circ$) and poles ($*$) of $\mathcal{U}_m$.



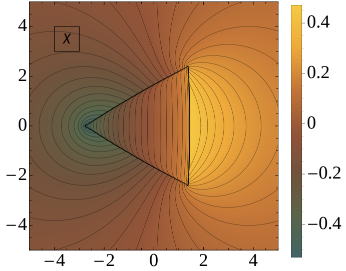
Asymptotic Description of $\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m

Formulae valid in the elliptic region. Two-cut/genus-one analysis.

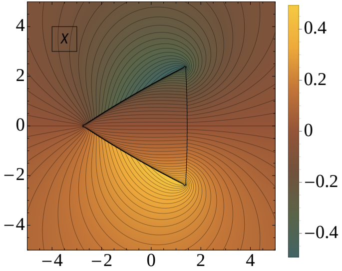
From our approximate formulae:

- We prove that the function $m^{-1/3}\mathcal{P}_m((m-\frac{1}{2})^{2/3}x)$ converges to a continuous limit $\dot{\mathcal{P}}_{\text{macro}}(x)$ in the distributional topology of $\mathscr{D}'(\mathbb{C}\setminus\partial T)$, as well as in the (suitably PV-regularized) distributional topology of $\mathscr{D}'(\mathbb{R}\setminus\{x_c,x_e\})$, where $(x_c,x_e)=T\cap\mathbb{R}$.

$\mathrm{Re}(\dot{\mathcal{P}}_{\text{macro}}(x))$



$\mathrm{Im}(\dot{\mathcal{P}}_{\text{macro}}(x))$

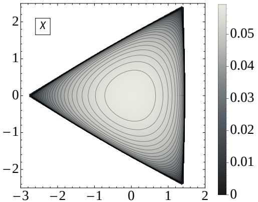


Note $\overline{\partial}\dot{\mathcal{P}}_{\text{macro}}(x)\neq 0$ for $x\in T$.

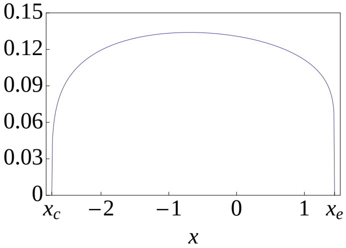
From our approximate formulae:

- We calculate the asymptotic planar density (at $x \in T$) and linear density (at $x \in T \cap \mathbb{R}$) of poles of $\mathcal{U}_m$. Taking out a factor of m^2, these are:

Planar Density $\sigma_{\mathrm{P}}(x)$



Linear Density $\sigma_{\mathrm{L}}(x)$

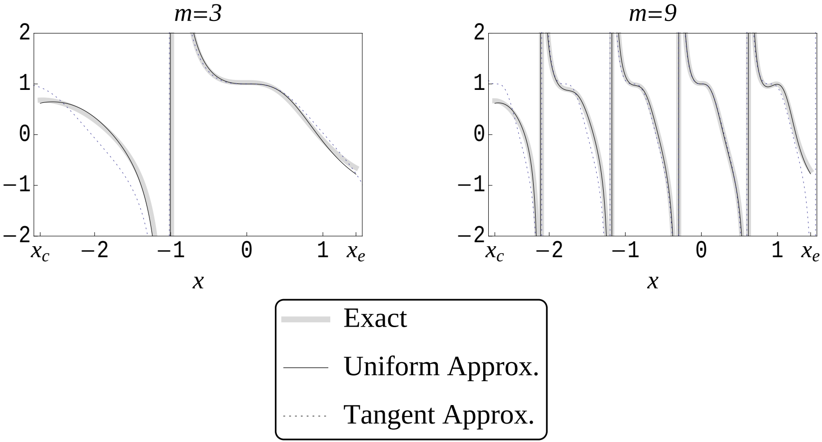


$\sigma_{\mathrm{P}}(x)$ is inversely proportional to the real area of the Jacobian of $\Gamma(x)$. $\sigma_{\mathrm{L}}(x)$ is inversely proportional to the real period of the elliptic function $\dot{\mathcal{P}}_m$ as a function of w.

Asymptotic Description of $\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m

Formulae valid in the elliptic region. Two-cut/genus-one analysis.

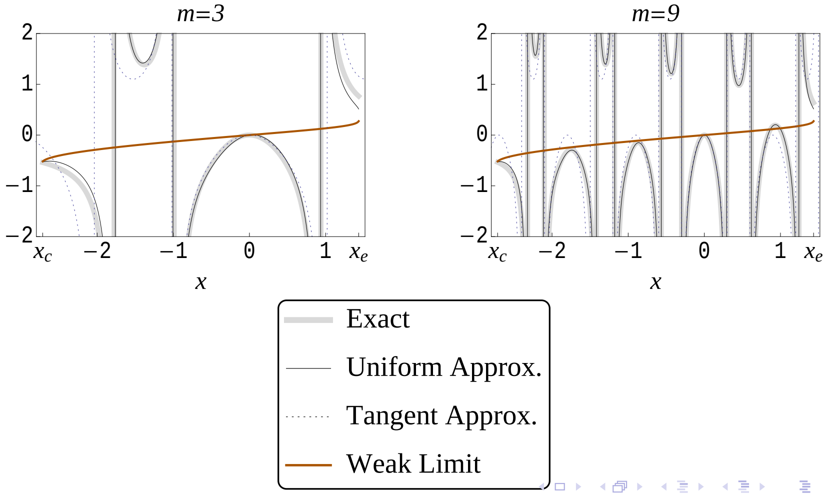
Quantitative comparison for $x \in T \cap \mathbb{R}$ of $m^{-2m/3} e^{-m\Lambda(x)} \mathcal{U}_m((m - \frac{1}{2})^{2/3} x)$, the uniform approximation $\dot{\mathcal{U}}_m(0; x)$, and the tangent approximation based at the origin $\dot{\mathcal{U}}_m((m - \frac{1}{2})x; 0)$:



Asymptotic Description of $\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m

Formulae valid in the elliptic region. Two-cut/genus-one analysis.

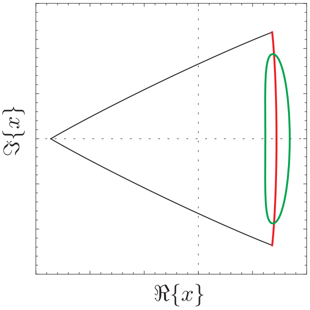
Quantitative comparison for $x \in T \cap \mathbb{R}$ of $m^{-1/3}\mathcal{P}_m((m-\frac{1}{2})^{2/3}x)$, the uniform approximation $\dot{\mathcal{P}}_m(0;x)$, the tangent approximation based at the origin $\dot{\mathcal{P}}_m((m-\frac{1}{2})x;0)$, and the weak limit $\dot{\mathcal{P}}_{\text{macro}}(x)$:

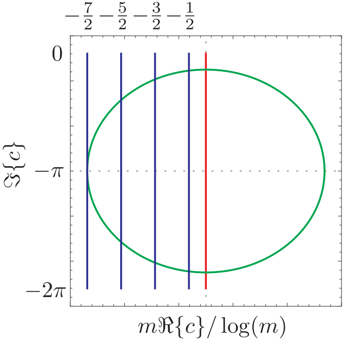


Asymptotic Description of $\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m

Formulae valid near an edge. Hermite parametrix plus “improvement”.

Each of the three smooth arcs of ∂T has a representation of the form $\Re\{f_k(x)\} = 0$, $k = 0, \pm 1$, where $f_{\pm 1}(x) = f_0(e^{\pm 2\pi i/3}x)$, and where $f_0(x)$ is analytic for $|\arg(x)| < \pi/3$. The relation $c = f_0(x)$ defines a conformal map taking the arc of ∂T with $|\arg(x)| < \pi/3$ to the segment of the imaginary axis in the c-plane with $-2\pi < \Im\{c\} < 0$:





Theorem

Assume that $\Re\{c(x)\} = \mathcal{O}(m^{-1}\log(m))$ *and that* $-2\pi < \alpha \le \Im\{c(x)\} \le \beta < 0$. *Then* $\mathcal{U}_m$ *and* $\mathcal{P}_m$ *are approximated by explicit trigonometric formulae, and there exists a family of functions* $d_K(x)$ *such that the poles of* $\mathcal{U}_m$ *are approximated by*

$$c(x) = -(K+\tfrac{1}{2})m^{-1}\log(m) + m^{-1}d_K(x) + 2\pi i j m^{-1}, \quad j \in \mathbb{Z}, \quad K = 0, 1, 2, 3, \ldots$$

Thus, the poles lie approximately along the vertical lines $\Re\{c\} = -(K+\frac{1}{2})m^{-1}\log(m)$. Moreover, the poles are "staggered" in both K and m, and the staggering effect weakens as $\Im\{c\}$ approaches -2π or 0, that is, as x approaches a corner of T.

Asymptotic Description of $\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m

Formulae valid near a corner. Painlevé-I tritronquée parametrix.

Here are results are formulated in terms of the famous *tritronquée* solution $y_{\mathrm{TT}}(v)$ of the Painlevé-I equation (PI):

$$y''(v) = 6y(v)^2 - v.$$

The solution $y_{\mathrm{TT}}(v)$ is distinguished by its asymptotic behavior:

$$y_{\mathrm{TT}}(v) = \left(\frac{v}{6}\right)^{1/2} + \mathcal{O}(v^{-2}), \quad v \to \infty, \quad |\arg(v)| < \frac{4\pi}{5}.$$

All solutions of PI have double poles as their only singularities. The *Hamiltonian* associated to each solution $y(v)$ and defined by

$$H(v) := \frac{1}{2}y'(v)^2 + vy(v) - 2y(v)^3$$

Has only simple poles. Let $H_{\mathrm{TT}}(v)$ denote the Hamiltonian of the tritronquée solution $y_{\mathrm{TT}}(v)$.

Asymptotic Description of $\mathcal{U}_m$ and $\mathcal{P}_m$ for Large m

Formulae valid near a corner. Painlevé-I tritronquée parametrix.

The negative real corner of T is the point $x_c = -(9/2)^{2/3}$.

Theorem

There exists a function $\hat{\lambda}(x)$ analytic at x_c, such that with $v = -2^{1/15}3^{-1/3}m^{4/5}(x - x_c)$,

$$\begin{aligned}\mathcal{U}_m &= m^{2m/3}e^{m\hat{\lambda}(x)}6^{-1/3}e^{-\hat{\lambda}(x_c)/2-1/3} \\ &\qquad\cdot\left[1 + 2^{2/5}H_{\mathrm{TT}}(v)m^{-1/5} + \mathcal{O}(m^{-2/5})\right] \\ \mathcal{P}_m &= -m^{1/3}6^{-1/3} \\ &\qquad\cdot\left[1 + 2^{12/5}H_{\mathrm{TT}}(v)m^{-1/5} + \mathcal{O}(m^{-2/5})\right]\end{aligned}$$

both hold uniformly for $v = O(1)$ such that v is also bounded away from all poles of $H_{\mathrm{TT}}(v)$.

Conclusion

- In a semiclassical multi-scaling limit solutions to the sine-Gordon equation with initial data crossing the pendulum separatrix exhibit a universal structure near the crossing points. Superluminal kinks are centered along the real graphs of the rational functions $\mathcal{U}_m$ associated with the Painlevé-II-α equation.
- The rational PII-α functions are "solitons" from the point of view of the Flaschka-Newell theory, but they can be obtained from the Jimbo-Miwa theory by means of a Riemann-Hilbert problem with jumps along contours rather than poles.
- The latter formulation makes possible the extraction of detailed asymptotic formulae (also effective for numerical computation) for $\mathcal{U}_m$ and $\mathcal{P}_m$ in the limit of large m by the steepest descent method.

Thank You!

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