Topological recursion and moduli spaces

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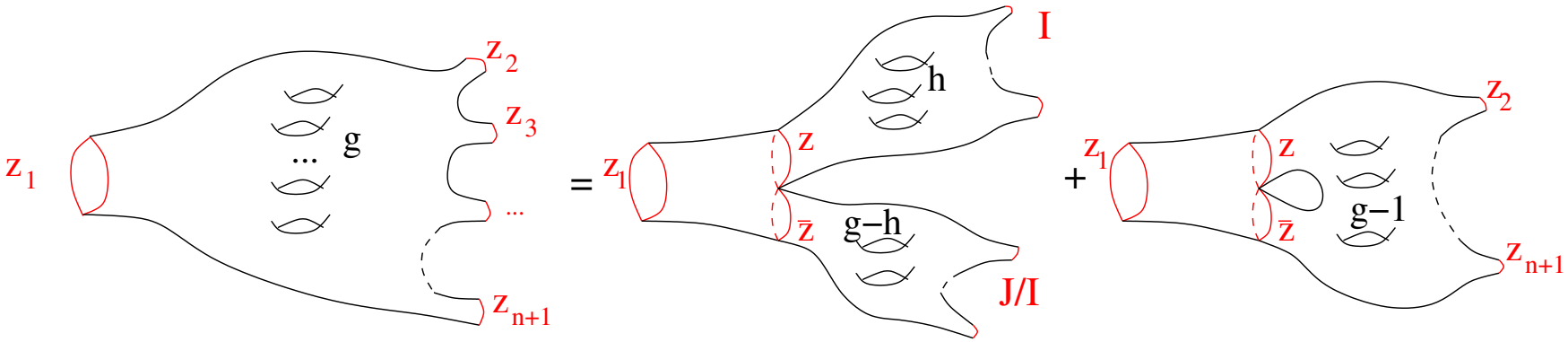

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1. Introduction by examples:

Mirzakhani's recursion
Hurwitz numbers

Weil-Petersson volumes

Fenchel-Nielsen coordinates on the moduli-space of curves of genus g: (l_i, θ_i). Weil-Petersson volumes:

$$\begin{aligned} V_{g,n}(L_1,\dots,L_n) &= \int_{\mathcal{M}_{g,n}(L_1,\dots,L_n)} \prod_i dl_i \wedge d\theta_i \\ &= \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{e}^{2\pi^2\kappa_1 + \frac{1}{2}\sum_i L_i^2 \psi_i} \end{aligned}$$

Laplace transform:

$$W_{g,n}(z_1,\dots,z_n) = \int_0^\infty \dots \int_0^\infty \prod_{i=1}^n L_i dL_i \, \mathrm{e}^{-z_i L_i} V_{g,n}(L_1,\dots,L_n)$$

Examples : $\quad V_{0,3}(L_1,L_2,L_3) = 1 \qquad W_{0,3} = \dfrac{1}{z_1^2\, z_2^2\, z_3^2}$

$$V_{1,1}(L_1) = \frac{1}{24}(2\pi^2 + \frac{L_1^2}{2}) \qquad W_{1,1}(z_1) = \frac{1}{24}(\frac{2\pi^2}{z_1^2} + \frac{3}{z_1^4})$$

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$$V_{0,4} = 2\pi^2 + \frac{L_1^2 + L_2^2 + L_3^2 + L_4^2}{2} \qquad W_{0,4} = \frac{1}{z_1^2\, z_2^2\, z_3^2\, z_4^2}\,(2\pi^2 + \sum_{i=1}^4 \frac{3}{z_i^2})$$

The Mirzakhani Topological recursion

Using hyperbolic geometry, M. Mirzakhani found a recursion among $V_{g,n}$'s. After Laplace transform it implies:

Theorem (Mirzakhani 2004 + Laplace tr. Eynard-Orantin 2007)

$$\begin{aligned} & W_{g,n+1}(z_0, \overbrace{z_1, \dots, z_n}^{J}) \\ = & \operatorname*{Res}_{z\to 0} \frac{\pi\, dz}{(z_0^2 - z^2)\sin 2\pi z} \Big[W_{g-1,n+2}(z, -z, J) \\ & + \sum_{h+h'=g;\, I\uplus I'=J}{}' W_{h,1+|I|}(z, I)\, W_{h',1+|I'|}(-z, I') \Big] \end{aligned}$$

where we have defined $W_{0,2}(z_1, z_2) = \frac{1}{(z_1-z_2)^2}$.

$\sum'$ *means we exclude* $(h, I) = (0, \emptyset)$ *and* $(h', I') = (0, \emptyset)$.

Example of computation

$$W_{1,1}(z_0) = \operatorname*{Res}_{z\to 0} \frac{\pi\, dz}{(z_0^2 - z^2)\sin 2\pi z}\; W_{0,2}(z,-z)$$

with $W_{0,2}(z,z') = 1/(z-z')^2$

Example of computation

$$W_{1,1}(z_0) = \operatorname*{Res}_{z\to 0} \frac{\pi\, dz}{(z_0^2 - z^2)\sin 2\pi z} \frac{1}{(z-(-z))^2}$$

with $W_{0,2}(z,z') = 1/(z-z')^2$

Example of computation

$$W_{1,1}(z_0) = \operatorname*{Res}_{z\to 0} \frac{\pi\, dz}{(z_0^2 - z^2)\sin 2\pi z} \frac{1}{4z^2}$$

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$$W_{1,1}(z_0) = \operatorname*{Res}_{z\to 0} \frac{dz}{z_0^2(1-\frac{z^2}{z_0^2})} \frac{\pi}{2\pi z(1-\frac{(2\pi)^2 z^2}{6}+\dots)} \frac{1}{4z^2}$$

Example of computation

$$W_{1,1}(z_0) = \frac{1}{8z_0^2} \operatorname*{Res}_{z\to 0} \frac{dz}{z^3} \frac{1}{1-\frac{z^2}{z_0^2}} \frac{1}{1-\frac{(2\pi)^2 z^2}{6}+\dots}$$

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$$W_{1,1}(z_0) = \frac{1}{8z_0^2} \operatorname*{Res}_{z\to 0} \frac{dz}{z^3} \left(1 + \frac{z^2}{z_0^2} + \frac{(2\pi)^2 z^2}{6} + O(z^4)\right)$$

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Rewriting as the general topological recursion

- $z \in \Sigma = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} =$ Riemann sphere.
- Turn $W_{g,n}$ to symmetric differential forms on Σ^n:

$$\omega_{g,n}(z_1,\dots,z_n) = W_{g,n}(z_1,\dots,z_n)\, dz_1 \otimes dz_2 \otimes \cdots \otimes dz_n$$

- Introduce $y(z) = \frac{\sin(2\pi z)}{4\pi}$, and $x(z) = z^2$, and $\omega_{0,1} = y dx$.
- Branchpoint $a = 0$, soultion of $dx(a) = 0$. Galois involution $s_a(z) = -z$, such that $x(s_a(z)) = x(z)$

$$\begin{aligned} & W_{g,n+1}(z_0, \overbrace{z_1,\dots,z_n}^{J}) \\ = & \operatorname*{Res}_{z\to 0} \frac{\pi\, dz}{(z_0^2 - z^2)\sin 2\pi z} \Big[W_{g-1,n+2}(z,-z,J) \\ & + \sum_{h+h'=g;\, I\uplus I'=J}^{\prime} W_{h,1+|I|}(z,I)\, W_{h',1+|I'|}(-z,I') \Big] \end{aligned}$$

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\end{aligned}$$

Use that $\frac{2zdz_0}{z_0^2-z^2} = \int_{-z}^{z} \frac{dz_0 dz'}{(z_0-z')^2} = \int_{-z}^{z} \omega_{0,2}(z_0,z')$

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= \quad & \operatorname*{Res}_{z \to 0} \frac{\int_{-z}^{z} \omega_{0,2}(z_0, z')}{-\frac{\sin 2\pi z}{\pi}\,(2zdz)} \Big[\omega_{g-1,n+2}(z, -z, J) \\
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= \quad & -\operatorname*{Res}_{z \to 0} \frac{\int_{-z}^{z} \omega_{0,2}(z_0, z')}{2(\omega_{0,1}(z) - \omega_{0,1}(-z))} \Big[\omega_{g-1,n+2}(z, -z, J) \\
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\end{aligned}$$

General topological recursion

Data: Spectral curve $\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$

• Σ a Riemann surface (not necessarily compact or connected)

• $x : \Sigma \to \mathbb{C}P^1$ analytical on Σ, branchpoints $\{a\}$ such that $dx(a) = 0$, local Galois involutions s_a such that $x \circ s_a = x$, $s_a \neq \mathrm{Id}$, $s_a(a) = a$.

• $y : \Sigma \to \mathbb{C}P^1$, $\omega_{0,1} = ydx$ (in fact y=germ of analytical function).

• $\omega_{0,2}$ symmetric 2-form on $\Sigma \times \Sigma$, double pole on diagonal.

Then define $\omega_{g,n}$ = symmetric n-form, by the recursion:

$$\begin{aligned}
&\omega_{g,n+1}(z_0, \overbrace{z_1, \dots, z_n}^{J}) \\
= \quad & -\sum_a \operatorname*{Res}_{z \to a} \frac{\int_{s_a(z)}^{z} \omega_{0,2}(z_0, z')}{2(\omega_{0,1}(z) - \omega_{0,1}(s_a(z)))} \Big[\omega_{g-1,n+2}(z, s_a(z), J) \\
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\end{aligned}$$

Other example: Hurwitz numbers

Data:

- $\Sigma = \mathbb{C} \setminus \mathbb{R}_-$
- $x(z) = -z + \ln z$. $dx(z) = (-1 + 1/z)dz$, $a = 1$, $x(a) = -1$, $s_a(z) = 1 - (z-1) + \frac{2}{3}(z-1)^2 - \frac{4}{3}(z-1)^3 + O(z-1)^4)$.
- $y(z) = z$. $\omega_{0,1} = ydx = (1-z)dz$
- $\omega_{0,2}(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$.

$$\omega_{g,n+1}(z_0, \overbrace{z_1, \dots, z_n}^{J}) = -\sum_a \operatorname*{Res}_{z \to a} \frac{\int_{s_a(z)}^{z} \omega_{0,2}(z_0, z')}{2(\omega_{0,1}(z) - \omega_{0,1}(s_a(z)))}$$

$$\left[\omega_{g-1,n+2}(z, s_a(z), J) + \sum_{h+h'=g;\, I \uplus I' = J}{}' \omega_{h,1+|I|}(z, I)\, \omega_{h',1+|I'|}(s_a(z), I')\right]$$

Bouchard-Mariño conjecture (proved BEMS,EMS,E,DOSSZ...)

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{\mu = (\mu_1, \dots, \mu_n)} H_g(\mu) \prod_{i=1}^{n} \mu_i\, \mathrm{e}^{\mu_i x(z_i)}\, dx(z_i)$$

Other example: Hurwitz numbers

Examples of computations:

$$\begin{aligned}\omega_{1,1}(z_0) &= \sum_{\mu=0}^{\infty} H_1(\{\mu\})\mu\, \mathrm{e}^{\mu x(z_0)} dx(z_0) \\ &\overset{\text{recursion}}{=} \frac{dz_0}{24}\left(\frac{1-2z_0}{(1-z_0)^4} - \frac{1}{(1-z_0)^2}\right) \\ &= \frac{1}{24}\left(d\tilde{\xi}_1(z_0) - d\tilde{\xi}_0(z_0)\right) \qquad = \mathrm{ELSV}\end{aligned}$$

Define:

$$\tilde{\xi}_0(z) = \frac{1}{(1-z)},\ \tilde{\xi}_1(z) = \frac{z}{(1-z)^3},\ \tilde{\xi}_{n+1}(z) = \frac{d\tilde{\xi}_n(z)}{dx(z)}$$

Remark: $\tilde{\xi}_n(z) = \sum_\mu \frac{\mu^{\mu+n}}{\mu!}\, \mathrm{e}^{\mu x(z)}$ (where $x(z) = -z + \ln z$).

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General remarks: polynomiality

Recursion involves residues at $a \to$ Taylor expand everything:

- $y(z) \sim_{z\to a} \sum_k \tilde{t}_{a,k}(x(z)-x(a))^{k/2}$ i.e.

$$t_{a,k} = \frac{(2k+1)!!\,\tilde{t}_{a,2k+1}}{2^{k-1}} = \frac{(2k+1)!!}{2^k} \operatorname*{Res}_{z\to a} \frac{y(z)dx(z)}{(x(z)-x(a))^{k+3/2}}$$

- $$\omega_{0,2}(z,z') \underset{z'\to a}{\sim} -\sum_n \frac{2^n}{2n+1!!}(x(z')-x(a))^{n-1/2}\, d\xi_n(z)$$

i.e. $d\xi_n(z) = -\frac{(2n-1)!!}{2^n} \operatorname{Res}_{z'\to a} \frac{\omega_{0,2}(z,z')}{(x(z')-x(a))^{n+\frac{1}{2}}}$ and also define

$$B_{a,k;b,l} = \frac{(2k-1)!!\,(2l-1)!!}{2^{k+l}} \operatorname*{Res}_{z\to a} \operatorname*{Res}_{z'\to b} \frac{\omega_{0,2}(z,z')}{(x(z)-x(a))^{k+\frac{1}{2}}\,(x(z')-x(b)}$$

The results of residues computations are necessarily (universal) polynomials of the $t_{a,k}$'s, $B_{a,k;b,l}$, and $d\xi_{a,k}(z_i)$'s.

General remarks: polynomiality

Theorem (trivial)

There exists some universal polynomial coefficients $\tilde{C}_{g,n}(\{t_{a,k}\},\{B_{a,k;b,l}\},a_1,d_1,\dots,a_n,d_n)$, such that the $\omega_{g,n}$'s defined by the recursion are

$$\omega_{g,n}(z_1,\dots,z_n) = \sum_{a_1,\dots,a_n}\sum_{d_1,\dots,d_n} \tilde{C}_{g,n}(\{t\},\{B\},a_1,d_1,\dots,a_n,d_n) \prod_{i=1}^{n} d\xi_{a_i,d_i}(z_i)$$

Notation:

$$\tilde{C}_{g,n}(\{t_{a,k}\},\{B_{a,k;b,l}\},a_1,d_1,\dots,a_n,d_n) =< \Lambda(S) \prod_{i=1}^{n} \tau_{a_i,d_i} >_g$$

S =spectral curve =data of $\{t_{a,k}\},\{B_{a,k;b,l}\}$.

General remarks: polynomiality

The recursion satisfies some properties (proved by recursion).
Linearity in $\omega_{0,2}$ of the recursion kernel implies:

$$\frac{\partial \omega_{g,n}(z_1,\dots,z_n)}{\partial B_{a,k;b,l}} = \frac{\partial^2 \omega_{g,n}(z_1,\dots,z_n)}{\partial t_{a,k}\,\partial t_{b,l}}$$

This implies that: $\{B\}$ dependence = Wick's theorem,

$$\tilde{C}_{g,n}(\{t\},\{B\},a_i,d_i) = \sum_{\text{graphs}} \prod_{v=\text{vertices}} C_{g_v,n_v}(\{t_{a_v,k}\},\{d_{v,e}\}_{e\mapsto v}) \frac{1}{\#\text{Aut}} \prod_{e=\text{edges}=(v_+,v_-)} B_{a_{v_+},d_{v_+,e};a_{v_-},d_{v_-,e}}$$

where $C_{g,n}(\{t_k\},d_1,\dots,d_k)$ is a universal polynomial of its variables t_k's.

General remarks: polynomiality

Form-cycle duality (=special geometry, SW) proved by recursion:

$$\frac{\partial \omega_{g,n}(z_1,\dots,z_n)}{\partial t_{a,k}} = \oint_{z\in t^*_{a,k}} \omega_{g,n+1}(z_1,\dots,z_n,z)$$

implies linear 1st order linear ODE for $C_{g,n}$, i.e.= exponential

$$C_{g,n}(\{t_k\},d_1,\dots,d_n) = \sum_m \frac{1}{m!}\sum_{j_1,\dots,j_m}\prod_i t_{j_i} C_{g,n+m}(\{0\},d_1,\dots,d_n,j_1+1,\dots,j_m+1)$$

i.e. symbolically

$$C_{g,n}(\{t_k\},d_1,\dots,d_n) =< \tau_{d_1}\dots\tau_{d_n}\ \mathrm{e}^{\sum_j t_j\tau_{j+1}} >_g$$

The universal coefficients $< \tau_{d_1}\dots\tau_{d_n} >_g$ can be computed on an example, Airy curve $x=z^2, y=z$, i.e. Kontsevich integral:

$$< \tau_{d_1}\dots\tau_{d_n} >_g = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n c_1(\mathcal{L}_i)^{d_i} = \mathrm{Witten-Kontsecich\ intersection\ nu}$$

Theorem

Theorem (Eynard 2011)

The $\omega_{g,n}$'s defined by the recursion are

$$\omega_{g,n}(z_1,\dots,z_n) = \int_{\overline{\mathcal{M}}_{g,n}(\#\mathrm{b.p.})} \mathrm{e}^{\sum_k t_{\sigma,k}\pi_*\tau_{k+1}} \mathrm{e}^{\frac{1}{2}\sum_{\delta=(p_+,p_-)}\sum_{k,l} B_{\sigma(p_+),k;\sigma(p_-),l}\ell_\delta{}_*\tau_k\tau_l} \prod_{i=1}^{n} \tau_{d_i}\ d\xi_{\sigma(p_i),d_i}(z_i)$$

where $\overline{\mathcal{M}}_{g,n}(r) = \{(C,p_1,\dots,p_n,\sigma\}$=moduli space of nodal stable surfaces C of genus g with n marked points p_i, and with a map $\sigma : C \to \{1,\dots,r\}$ constant in each component.
δ =boundary divisors = introduction of a nodal point i.e. 2 new marked points p_+, p_-. ℓ_δ = inclusion of the boundary into the stratum of the moduli space with 2 more marked points

Laplace transforms

The $(2k+1)!!$ can be get rid off by Laplace transform:

$$\sum_k t_{a,k} u^{-k} = \mathrm{e}^{-\sum_k \hat{t}_{a,k} u^{-k}} = \frac{u^{2/3}}{2\sqrt{\pi}} \int_{\gamma_a} y dx \ \mathrm{e}^{-u(x-x(a))}$$

$$\sum_{k,l} B_{a,k;b,l} u^{-k} v^{-l} = \frac{\sqrt{uv}}{2\pi} \int_{z\in\gamma_a} \int_{z'\in\gamma_b} \omega_{0,2}(z,z') \ \mathrm{e}^{-u(x(z)-x(a))} \ \mathrm{e}^{-v(x(z$$

Remark: Arbarello Cornalba define κ classes as forgetful projections of τ classes as:

$$\mathrm{e}^{\sum_k t_{a,k} \pi_* \tau_{k+1}} = \mathrm{e}^{\sum_k \hat{t}_{a,k} \kappa_k}$$

Example of application

$y(z) = \frac{\sin 2\pi z}{4\pi}$, $x(z) = z^2$, $\omega_{0,2}(z,z') = \frac{dzdz'}{(z-z')^2}$.
Only one branchpoint $a = 0$. We have $B_{a,k;a,l} = 0$.
Laplace transform of ydx:

$$\begin{aligned}
\int_{\mathbb{R}} ydx\ \mathrm{e}^{-ux} &= \frac{1}{2\mathrm{i}\pi}\int \mathrm{e}^{2\mathrm{i}\pi z}\ \mathrm{e}^{-uz^2}\ zdz \\
&= \frac{1}{2\mathrm{i}\pi}\mathrm{e}^{-\pi^2/u}\int \mathrm{e}^{-u(z-\mathrm{i}\pi/u)^2}\,(z-\mathrm{i}\pi/u+\mathrm{i}\pi/u)\,dz \\
&= \frac{1}{2u}\mathrm{e}^{-\pi^2/u}\int \mathrm{e}^{-u(z-\mathrm{i}\pi/u)^2}\ dz \\
&= \frac{\sqrt{\pi}}{2u\sqrt{u}}\mathrm{e}^{-\pi^2/u}
\end{aligned}$$

it implies $\hat{t}_k = \pi^2\delta_{k,1}$, i.e.

$$\Lambda(S) = \mathrm{e}^{\pi^2\kappa_1} = \text{Weil Petersson form}$$

Example of application

$y(z) = z$, $x(z) = -z + \ln z$. Only one branchpoint $a = 1$.
Laplace transform of ydx:

$$\begin{aligned}
\int_{\mathbb{R}} ydx\ \mathrm{e}^{-ux} &= \int_{\gamma} \mathrm{e}^{uz} z^{-u} (1-z) dz \\
&= \int_0^{\infty} \mathrm{e}^{-uz} z^{-u} (1+z) dz \\
&= u^{u-1}\Gamma(1-u) + u^{u-2}\Gamma(2-u) \\
&= u^{u-1}\Gamma(-u) \\
&= \frac{\mathrm{e}^{u}\sqrt{2\pi}}{u^{3/2}}\ \mathrm{e}^{-\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} u^{1-2k}}
\end{aligned}$$

This leads to the "Gamma" class=Hodge class

$$\Lambda(S) = \mathrm{e}^{\sum_k \frac{B_{2k}}{2k(2k-1)} (\kappa_{2k-1} + \frac{1}{2}\sum_l (-1)^l \sum_\delta \ell_{\delta *} \tau_{2k-l} \tau_l)} \overset{\text{Mumford}}{=} \Lambda_{\text{Hodge}}$$

Gives ELSV formula:

$$\omega_{g,n}(\{z_i\}) = \sum_{d} \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{\text{Hodge}} \prod_{i=1}^{n} \tau_{d_i} d\tilde{\xi}_{d_i}(z_i)$$

Example of application: Mariño Vafa

Example: $y(z) = \ln z$, $x(z) = \ln(1-z) - f \ln z$, one branchpoint $a = \frac{f}{f-1}$. (satisfies $\mathrm{e}^{x+fy} + \mathrm{e}^{y} - 1 = 0$, mirror of $\mathbf{C}^3$, framing f).
Laplace transform of ydx:

$$\begin{aligned}\int_{\mathbb{R}} ydx\ \mathrm{e}^{-ux} &= \frac{1}{u}\int_0^1 z^{fu}(1-z)^{-u}\frac{dz}{z} = \frac{\Gamma(fu)\Gamma(u)}{\Gamma((f+1)u+1)} \\ &= \frac{\mathrm{e}^{-ux(a)}\sqrt{2\pi}}{u^{3/2}}\ \mathrm{e}^{\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}u^{1-2k}(1+f^{1-2k}-(1+f)^{1-2k})}\end{aligned}$$

This leads to the "Gamma" class=Hodge class

$$\Lambda(S) = \Lambda_{\mathrm{Hodge}}(-1)\Lambda_{\mathrm{Hodge}}(-f)\Lambda_{\mathrm{Hodge}}(1+f)$$

Gives Mariño-Vafa formula:

$$\omega_{g,n}(\{z_i\}) = \sum_{d_1,\dots,d_n}\int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{\mathrm{Hodge}}(-1)\Lambda_{\mathrm{Hodge}}(-f)\Lambda_{\mathrm{Hodge}}(1+f)\prod_{i=1}^{n}\tau_{d_i}d\tilde{\xi}_{d_i}(z$$

Example of application: BKMP conjecture

$\mathfrak{X}$=toric Calabi–Yau 3 fold. Mirror symmetry: $\hat{\mathfrak{X}}$ =curve $P(\mathrm{e}^x, \mathrm{e}^y) = 0$.
Example: $\mathfrak{X}$ =local $\mathbb{P}^2$, $\mathrm{e}^{x+fy} + \mathrm{e}^y - 1 + Q\mathrm{e}^{-x-(f+1)y} = 0$.
(genus 1 curve, i.e. ln x and ln y=elliptic functions).
The previous theorem implies (after some combinatorics)

$$\omega_{g,n}(\{z_i\}) = \sum_{\text{localization graphs}} \frac{1}{\#\mathrm{Aut}} \prod_{e=(v_+,v_-)=\text{edges}} Q_e^{d_e/f_e} \prod_{v=\text{vertices}} (\#) \int_{\overline{\mathcal{M}}_{g_v,n_v}} \Lambda_H(f_v)\Lambda_H(f'_v)\Lambda_H(f''_v) \prod_{e\mapsto v} \tau_{d_e} \prod_{i=1}^{n} d\xi_{d_i}(z_i)$$

it proves (BKMP) that $\omega_{g,n}$= open Gromov-Witten invariants:

$$\omega_{g,n}(\{z_i\}) = \sum_{d\in H_2(\mathfrak{X},\{L_i\},\mathbb{Z}),\, d_i\in H_1(L_i,\mathbb{Z})} N_{g,d,\{d_i\}}(\mathfrak{X},\{L_i\}) \prod_{i=1}^{n} \mathrm{e}^{\frac{-d_i}{f(L_i)}x(z_i)}\, dx(z_i$$

5. Conclusion

Conclusion:

- The TR is a universal recursion relation satisfied by many enumeration problems (also appears in random matrices, enumeration of partitions, 3d partitions, knots, statistical physics, Liouville CFT,...)
- Once we know $W_{0,1}$ and $W_{0,2}$, the TR is very efficient at explicitly computing higher genus generating functions.
- Gives new proofs of ELSV, Mariño-Vafa, Mirzakhani, BKMP...

Further prospects:

- $\exists$ recent new conjecture: topological recursion with $\mathcal{S}$ =A-polynomial, computes asymptotics of Jones polynomial of knots ? [Dijkgraaf-Fuji-Manabe 2010, Borot-E 2012, Gukov, Sulkowski, Mulase,...] = generalization of volume conjecture. Find a proof ???
- categorification refinement, AGT

The end

Thank you for your attention

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