Topological recursion and moduli spaces

Bertrand Eynard, IPHT CEA Saclay,

Sissa, September 2013
1. Introduction by examples:
Mirzakhani’s recursion
Hurwitz numbers
Weil-Petersson volumes

Fenchel-Nielsen coordinates on the moduli-space of curves of genus $g$: $(l_i, \theta_i)$. Weil-Petersson volumes:

$$V_{g,n}(L_1, \ldots, L_n) = \int_{\mathcal{M}_{g,n}(L_1, \ldots, L_n)} \prod_i dl_i \wedge d\theta_i$$

$$= \int_{\mathcal{M}_{g,n}} e^{2\pi^2 \kappa_1 + \frac{1}{2} \sum_i L_i^2 \psi_i}$$

Laplace transform:

$$W_{g,n}(z_1, \ldots, z_n) = \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^n L_i dL_i \ e^{-z_i L_i} V_{g,n}(L_1, \ldots, L_n)$$

Examples:

$V_{0,3}(L_1, L_2, L_3) = 1$  \quad  $W_{0,3} = \frac{1}{z_1^2 z_2^2 z_3^2}$

$V_{1,1}(L_1) = \frac{1}{24} \left( 2\pi^2 + \frac{L_1^2}{2} \right)$  \quad  $W_{1,1}(z_1) = \frac{1}{24} \left( \frac{2\pi^2}{z_1^2} + \frac{3}{z_1^4} \right)$
Weil-Petersson volumes

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Laplace transform:

$$W_{g,n}(z_1, \ldots, z_n) = \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^n L_i \, dL_i \, e^{-z_i L_i} \, V_{g,n}(L_1, \ldots, L_n)$$

$$V_{1,1}(L_1) = \frac{1}{24} \left( 2\pi^2 + \frac{L_1^2}{2} \right)$$

$$W_{1,1}(z_1) = \frac{1}{24} \left( \frac{2\pi^2}{z_1^2} + \frac{3}{z_1^4} \right)$$

$$V_{0,4} = 2\pi^2 + \frac{L_1^2 + L_2^2 + L_3^2 + L_4^2}{2}$$

$$W_{0,4} = \frac{1}{Z_1^2 Z_2^2 Z_3^2 Z_4^2} \left( 2\pi^2 + \sum_{i=1}^4 \frac{3}{Z_i^2} \right)$$
The Mirzakhani Topological recursion

Using hyperbolic geometry, M. Mirzakhani found a recursion among $V_{g,n}$'s. After Laplace transform it implies:

**Theorem (Mirzakhani 2004 + Laplace tr. Eynard-Orantin 2007)**

\[
W_{g,n+1}(z_0, z_1, \ldots, z_n) = \text{Res}_{z \to 0} \frac{\pi \, dz}{(z_0^2 - z^2) \sin 2\pi z} \left[ W_{g-1,n+2}(z, -z, J) \right. \\
+ \sum_{h + h' = g; I \cup I' = J} W_{h,1+|I|}(z, I) \, W_{h',1+|I'|}(-z, I') \left. \right]
\]

where we have defined $W_{0,2}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2}$.

$\sum'$ means we exclude $(h, I) = (0, \emptyset)$ and $(h', I') = (0, \emptyset)$.
Example of computation

\[ W_{1,1}(z_0) = \operatorname{Res}_{z \to 0} \left( \frac{\pi}{(z_0^2 - z^2) \sin 2\pi z} \right) W_{0,2}(z, -z) \]

with \( W_{0,2}(z, z') = 1/(z - z')^2 \)
Example of computation

\[ W_{1,1}(z_0) = \text{Res}_{z \to 0} \frac{\pi \, dz}{(z_0^2 - z^2) \sin 2\pi z} \frac{1}{(z - (-z))^2} \]

with \( W_{0,2}(z, z') = 1/(z - z')^2 \)
Example of computation

\[ W_{1,1}(z_0) = \text{Res}_{z \to 0} \frac{\pi \, dz}{(z_0^2 - z^2) \sin 2\pi z} \frac{1}{4z^2} \]
Example of computation

\[ W_{1,1}(z_0) = \lim_{z \to 0} \frac{dz}{z^2_0(1 - \frac{z^2}{z_0^2})} \frac{\pi}{2\pi z(1 - \frac{(2\pi)^2 z^2}{6} + \ldots)} \frac{1}{4z^2} \]
Example of computation

\[ W_{1,1}(z_0) = \frac{1}{8z_0^2} \text{Res}_{z \to 0} \frac{dz}{z^3} \frac{1}{1 - \frac{z^2}{z_0^2}} \frac{1}{1 - \frac{(2\pi)^2 z^2}{6}} + \ldots \]
Example of computation

\[ W_{1,1}(z_0) = \frac{1}{8z_0^2} \lim_{z \to 0} dz \frac{1}{z^3} \left( 1 + \frac{z^2}{z_0^2} + \frac{(2\pi)^2 z^2}{6} + O(z^4) \right) \]
Example of computation

\[ W_{1,1}(z_0) = \frac{1}{8z_0^2} \left( \frac{1}{z_0^2} + \frac{(2\pi)^2}{6} \right) \]
Example of computation

\[ W_{1,1}(z_0) = \frac{1}{8z_0^2} \left( \frac{1}{z_0^2} + \frac{(2\pi)^2}{6} \right) \]

\[ \checkmark \]
Rewriting as the general topological recursion

• $z \in \Sigma = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} = \text{Riemann sphere}$. 
• Turn $W_{g,n}$ to symmetric differential forms on $\Sigma^n$:

$$\omega_{g,n}(z_1, \ldots, z_n) = W_{g,n}(z_1, \ldots, z_n) \, dz_1 \otimes dz_2 \otimes \cdots \otimes dz_n$$

• Introduce $y(z) = \frac{\sin(2\pi z)}{4\pi}$, and $x(z) = z^2$, and $\omega_{0,1} = ydx$.
• Branchpoint $a = 0$, solution of $dx(a) = 0$. Galois involution $s_a(z) = -z$, such that $x(s_a(z)) = x(z)$

\[
W_{g,n+1}(z_0, z_1, \ldots, z_n) = \operatorname{Res}_{z \to 0} \frac{\pi \, dz}{(z_0^2 - z^2) \sin 2\pi z} \left[ W_{g-1,n+2}(z, -z, J) \right.
\left. + \sum_{h + h' = g; I \uplus I' = J} W_{h,1+|I|}(z, I) \, W_{h',1+|I'|}(-z, I') \right]
\]
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\[
W_{g,n+1}(z_0, z_1, \ldots, z_n) \, dz_0 \, dz_1 \cdots \, dz_n
= \text{Res}_{z \to 0} \left( \frac{dz_0 \, \pi \, dz}{(z_0^2 - z^2) \sin 2\pi z} \right) 
\left[ W_{g-1,n+2}(z, -z, J) \, dz_1 \cdots \, dz_n \right]
+ \sum_{h+h'=g; \, I \sqcup I' = J} W_{h,1+|I|}(z, I) \, dz_I \, W_{h',1+|I'|}(-z, I') \, dz_{I'}
\]
Rewriting as the general topological recursion

- \( z \in \Sigma = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} = \text{Riemann sphere.} \)
- Turn \( W_{g,n} \) to symmetric differential forms on \( \Sigma^n \):
  \[
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  \]
- Introduce \( y(z) = \frac{\sin(2\pi z)}{4\pi} \), and \( x(z) = z^2 \), and \( \omega_{0,1} = ydx \).
- Branchpoint \( a = 0 \), solution of \( dx(a) = 0 \). Galois involution \( s_a(z) = -z \), such that \( x(s_a(z)) = x(z) \)

\[
W_{g,n+1}(z_0, \underbrace{z_1, \ldots, z_n}_J) \, dz_0 \, dz_1 \cdots \, dz_n
= \text{Res}_{z \to 0} \frac{dz_0 \, \pi \, \frac{\sin(2\pi z_0)}{(z_0^2 - z^2) \sin 2\pi z} \, (-dz)}{(z_0^2 - z^2) \sin 2\pi z} \left[ W_{g-1,n+2}(z, -z, J) \, dz(-dz) \, dz_1 \cdots \right.
+ \sum_{h+h' = g; \, l+l' = J} W_{h,1+|l|}(z, l) \, dz \, dz_l \, W_{h',1+|l'|}(-z, l')(-dz) \, dz_{l'} \left. \right]
\]
Rewriting as the general topological recursion

- \( z \in \Sigma = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} = \text{Riemann sphere.} \)
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\[
\omega_{g,n+1}(z_0, Z_1, \ldots, Z_n) = \text{Res}_{z \to 0} \frac{dz_0}{(z_0^2 - z^2) \sin 2\pi z} \, (-dz) \left[ \omega_{g-1,n+2}(z, -z, J) \right. \\
+ \sum_{h + h' = g; I \cup I' = J} \omega_{h,1+|I|}(z, I) \omega_{h',1+|I'|}( -z, I') \left. \right]
\]
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- $z \in \Sigma = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} = \text{Riemann sphere}.$
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  \[ \omega_{g,n}(z_1, \ldots, z_n) = W_{g,n}(z_1, \ldots, z_n) \, dz_1 \otimes dz_2 \otimes \cdots \otimes dz_n \]
- Introduce $y(z) = \frac{\sin(2\pi z)}{4\pi}$, and $x(z) = z^2$, and $\omega_{0,1} = ydx$.
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\[
\omega_{g,n+1}(z_0, \overbrace{z_1, \ldots, z_n}^J) = \operatorname{Res}_{z \to 0} \frac{dz_0}{(z_0^2 - z^2) \sin 2\pi z} \left( -dz \right) \left[ \omega_{g-1,n+2}(z, -z, J) \right] \\
+ \sum_{h+h'=g; \; l \cup l' = J} \omega_{h,1+|l|}(z, l) \omega_{h',1+|l'|}(z, l') \\
\]

Use that \[ \frac{2zdz_0}{z_0^2 - z^2} = \int_{-z}^z \frac{dz_0 dz'}{(z_0 - z')^2} = \int_{-z}^z \omega_{0,2}(z_0, z') \]
Rewriting as the general topological recursion

• $z \in \Sigma = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} =$ Riemann sphere.
• Turn $W_{g,n}$ to symmetric differential forms on $\Sigma^n$:

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• Branchpoint $a = 0$, solution of $dx(a) = 0$. Galois involution $s_a(z) = -z$, such that $x(s_a(z)) = x(z)$

$$\omega_{g,n+1}(z_0, \overbrace{Z_1, \ldots, Z_n}^{J}) = \mathrm{Res}_{z \to 0} \left( \frac{\omega_{0,2}(z_0, Z')}{-\frac{\sin 2\pi z}{\pi} (2zdz)} \right) \left[ \omega_{g-1,n+2}(Z, -Z, J) \right.$$

$$+ \sum_{h+h'=g; \; l \uplus l'=J} \omega_{h,1+|l|}(Z, l) \omega_{h',1+|l'|}(-Z, l') \left. \right]$$

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\[
\omega_{g,n+1}(Z_0, Z_1, \ldots, Z_n) = \text{Res}_{z \to 0} \frac{\int_{-z}^{z} \omega_{0,2}(z_0, z')}{-2(y(z) - y(-z))} \, dx(z) \left[ \omega_{g-1,n+2}(z, -z, J) \right]
\]
\[
+ \sum_{h+h' = g; I \oplus I' = J} \omega_{h,1+|I|}(z, I) \omega_{h',1+|I'|}(-z, I')
\]
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$$
\omega_{g,n+1}(Z_0, Z_1, \ldots, Z_n) = - \lim_{z \to 0} \frac{\int_z^Z \omega_{0,2}(Z_0, Z')}{2(\omega_{0,1}(Z) - \omega_{0,1}(-Z))} \left[ \omega_{g-1,n+2}(Z, -Z, J) \omega_{h,1+|l|}(Z, l) \omega_{h',1+|l'|}(-Z, l') \right] \\
+ \sum_{h+h'=g; \, l+l'=J} \omega_{h,1+|l|}(Z, l) \omega_{h',1+|l'|}(-Z, l')
$$
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- Branchpoint $a = 0$, solution of $dx(a) = 0$. Galois involution $s_a(z) = -z$, such that $x(s_a(z)) = x(z)$

$$\omega_{g,n+1}(z_0, \overline{Z_1}, \ldots, Z_n)$$

$$= - \text{Res}_{z \rightarrow a} \frac{\int_{s_a(z)}^Z \omega_{0,2}(z_0, z')}{2(\omega_{0,1}(Z) - \omega_{0,1}(s_a(Z)))} \left[ \omega_{g-1,n+2}(Z, s_a(Z), J) \right]$$

$$+ \sum_{h+h' = g; \, l \sqcup l' = J} \omega_{h,1+|l|}(Z, l) \omega_{h',1+|l'|}(s_a(Z), l')$$
General topological recursion

Data: Spectral curve $S = (\Sigma, x, y, \omega_{0,2})$

- $\Sigma$ a Riemann surface (not necessarily compact or connected)
- $x : \Sigma \to \mathbb{C}P^1$ analytical on $\Sigma$, branchpoints $\{a\}$ such that $dx(a) = 0$, local Galois involutions $s_a$ such that $x \circ s_a = x$, $s_a \neq \text{Id}$, $s_a(a) = a$.
- $y : \Sigma \to \mathbb{C}P^1$, $\omega_{0,1} = ydx$ (in fact $y$=germ of analytical function).
- $\omega_{0,2}$ symmetric 2-form on $\Sigma \times \Sigma$, double pole on diagonal.

Then define $\omega_{g,n} =$ symmetric n-form, by the recursion:

$$
\omega_{g,n+1}(Z_0, Z_1, \ldots, Z_n) = - \sum_a \text{Res}_{z \to a} \int_z^Z \omega_{0,2}(Z_0, Z') \frac{\omega_{0,1}(Z) - \omega_{0,1}(s_a(Z))}{2}\left[\omega_{g-1,n+2}(Z, s_a(Z), J) + \sum_{h+h'=g; l+l'=J} \omega_{h,1+l}(Z, l) \omega_{h',1+l'}(s_a(Z), l')\right]
$$
Other example: Hurwitz numbers

Data:

• $\Sigma = \mathbb{C} \setminus \mathbb{R}_-$
• $x(z) = -z + \ln z$. $dx(z) = (-1 + 1/z)dz$, $a = 1$, $x(a) = -1$, $s_a(z) = 1 - (z - 1) + \frac{2}{3}(z - 1)^2 - \frac{4}{3}(z - 1)^3 + O(z - 1)^4$.
• $y(z) = z$. $\omega_{0,1} = ydx = (1 - z)dz$
• $\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$.

\[
\omega_{g,n+1}(z_0, z_1, \ldots, z_n) = - \sum_a \text{Res}_{z \to a} \frac{\int_{s_a(z)}^z \omega_{0,2}(z_0, z')}{2(\omega_{0,1}(z) - \omega_{0,1}(s_a(z)))}
\]

\[
\left[ \omega_{g-1,n+2}(z, s_a(z), J) + \sum_{h+h'=g; l+l'=J} \omega_{h,1+|l|}(z, l) \omega_{h',1+|l'|}(s_a(z), l') \right]
\]

Bouchard-Mariño conjecture (proved BEMS,EMS,E,DOSSZ...)

\[
\omega_{g,n}(z_1, \ldots, z_n) = \sum_{\mu=(\mu_1, \ldots, \mu_n)} H_g(\mu) \prod_{i=1}^n \mu_i e^{\mu_i x(z_i)} dx(z_i)
\]
Other example: Hurwitz numbers

Examples of computations:

\[
\omega_{1,1}(z_0) = \sum_{\mu=0}^{\infty} H_1(\{\mu\}) \mu e^{\mu x(z_0)} dx(z_0)
\]

recursion

\[
dz_0 \equiv \frac{dz_0}{24} \left( \frac{1 - 2z_0}{(1 - z_0)^4} - \frac{1}{(1 - z_0)^2} \right)
\]

\[
= \frac{1}{24} \left( d\tilde{\xi}_1(z_0) - d\tilde{\xi}_0(z_0) \right) = \text{ELSV}
\]

Define:

\[
\tilde{\xi}_0(z) = \frac{1}{1 - z}, \quad \tilde{\xi}_1(z) = \frac{z}{(1 - z)^3}, \quad \tilde{\xi}_{n+1}(z) = \frac{d\tilde{\xi}_n(z)}{dx(z)}
\]

Remark: \(\tilde{\xi}_n(z) = \sum_{\mu} \frac{\mu^{\mu+n}}{\mu!} e^{\mu x(z)}\) (where \(x(z) = -z + \ln z\)).
Other example: Hurwitz numbers

Examples of computations:

\[ \omega_{1,1}(z_0) = \sum_{\mu=0}^{\infty} H_1(\{\mu\}) \mu e^{\mu x(z_0)} \, dx(z_0) \]

\[ \text{recursion} \quad dz_0 \overset{!}{=} \frac{dz_0}{24} \left( \frac{1 - 2z_0}{(1 - z_0)^4} - \frac{1}{(1 - z_0)^2} \right) \]

\[ = \frac{1}{24} \left( d\tilde{\xi}_1(z_0) - d\tilde{\xi}_0(z_0) \right) = \text{ELSV} \]

Define:

\[ \tilde{\xi}_0(z) = \frac{1}{1 - z}, \quad \tilde{\xi}_1(z) = \frac{z}{(1 - z)^3}, \quad \tilde{\xi}_{n+1}(z) = \frac{d\tilde{\xi}_n(z)}{dx(z)} \]

Remark: \[ \tilde{\xi}_n(z) = \sum_{\mu} \frac{\mu^{\mu+n}}{\mu!} e^{\mu x(z)} \] (where \( x(z) = -z + \ln z \)).
Other example: Hurwitz numbers

Examples of computations:

\[ \omega_{1,1}(z_0) = \sum_{\mu=0}^{\infty} H_1(\{\mu\}) \mu e^{\mu z_0} dx(z_0) \]

\[ = \frac{dz_0}{24} \left( \frac{1 - 2z_0}{(1 - z_0)^4} - \frac{1}{(1 - z_0)^2} \right) \]

Define:

\[ \tilde{\xi}_0(z) = \frac{1}{1 - z} , \quad \tilde{\xi}_1(z) = \frac{z}{(1 - z)^3} , \quad \tilde{\xi}_{n+1}(z) = \frac{d\tilde{\xi}_n(z)}{dx(z)} \]

Remark: \[ \tilde{\xi}_n(z) = \sum_{\mu} \frac{\mu^{\mu+n}}{\mu!} e^{\mu z} \] (where \( x(z) = -z + \ln z \)).
Other example: Hurwitz numbers

Examples of computations:

\[
\omega_{1,1}(z_0) = \sum_{\mu=0}^{\infty} H_1(\{\mu\}) \mu \mu \mu (z_0) e^{\mu x(z_0)} dx(z_0)
\]

recursion

\[
\frac{dz_0}{24} \left( \frac{1 - 2z_0}{(1 - z_0)^4} - \frac{1}{(1 - z_0)^2} \right)
\]

\[
= \frac{1}{24} \left( d\tilde{\xi}_1(z_0) - d\tilde{\xi}_0(z_0) \right) = ELSV
\]

Define:

\[
\tilde{\xi}_0(z) = \frac{1}{(1 - z)}, \quad \tilde{\xi}_1(z) = \frac{z}{(1 - z)^3}, \quad \tilde{\xi}_{n+1}(z) = \frac{d\tilde{\xi}_n(z)}{dx(z)}
\]

Remark: \( \tilde{\xi}_n(z) = \sum_{\mu} \frac{\mu^{\mu+n}}{\mu!} e^{\mu x(z)} \) (where \( x(z) = -z + \ln z \)).
General remarks: polynomiality

Recursion involves residues at $a \to$ Taylor expand everything:

- $y(z) \sim_{z \to a} \sum_k \tilde{t}_{a,k}(x(z) - x(a))^{k/2}$ i.e.
  $$t_{a,k} = \frac{(2k + 1)!! \tilde{t}_{a,2k+1}}{2^{k-1}} = \frac{(2k + 1)!!}{2^k} \text{Res}_{z \to a} \frac{y(z)dx(z)}{(x(z) - x(a))^{k+3/2}}$$

- $\omega_{0,2}(z, z') \sim_{z' \to a} - \sum_n \frac{2^n}{2n + 1!!} (x(z') - x(a))^{n-1/2} d_{\xi n}(z)$
  i.e. $d_{\xi n}(z) = - \frac{(2n-1)!!}{2^n} \text{Res}_{z' \to a} \frac{\omega_{0,2}(z,z')}{(x(z') - x(a))^{n+1/2}}$ and also define
  $$B_{a,k;b,l} = \frac{(2k - 1)!! (2l - 1)!!}{2^{k+l}} \text{Res}_{z \to a} \text{Res}_{z' \to b} \frac{\omega_{0,2}(z,z')}{(x(z) - x(a))^{k+1/2} (x(z') - x(b))}$$

The results of residues computations are necessarily (universal) polynomials of the $t_{a,k}$'s, $B_{a,k;b,l}$, and $d_{\xi a,k}(z_i)$'s.
General remarks: polynomiality

Theorem (trivial)

There exists some universal polynomial coefficients $\tilde{C}_{g,n}(\{t_{a,k}\}, \{B_{a,k;b,l}\}, a_1, d_1, \ldots, a_n, d_n)$, such that the $\omega_{g,n}$'s defined by the recursion are

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{a_1,\ldots,a_n} \sum_{d_1,\ldots,d_n} \tilde{C}_{g,n}(\{t\}, \{B\}, a_1, d_1, \ldots, a_n, d_n) \prod_{i=1}^{n} d_{\xi_{a_i},d_i}(z_i)$$

Notation:

$$\tilde{C}_{g,n}(\{t_{a,k}\}, \{B_{a,k;b,l}\}, a_1, d_1, \ldots, a_n, d_n) = \langle \wedge(S) \prod_{i=1}^{n} \tau_{a_i,d_i} >_g$$

$S =$ spectral curve = data of $\{t_{a,k}\}, \{B_{a,k;b,l}\}$. 
General remarks: polynomiality

The recursion satisfies some properties (proved by recursion). Linearity in $\omega_{0,2}$ of the recursion kernel implies:

$$\frac{\partial \omega_{g,n}(z_1, \ldots, z_n)}{\partial B_{a,k;b,l}} = \frac{\partial^2 \omega_{g,n}(z_1, \ldots, z_n)}{\partial t_{a,k} \partial t_{b,l}}$$

This implies that: $\{B\}$ dependence = Wick’s theorem,

$$\tilde{C}_{g,n}(\{t\}, \{B\}, a_i, d_i) = \sum_{\text{graphs } \nu=\text{vertices}} \prod_{\nu} C_{g_{\nu},n_{\nu}}(\{t_{a_{\nu},k}\}, \{d_{\nu,e}\}_{e \rightarrow \nu}) \times \frac{1}{\# \text{Aut}} \prod_{e=\text{edges} = (\nu_+, \nu_-)} B_{a_{\nu_+},d_{\nu+},e;a_{\nu_-},d_{\nu_-},e}$$

where $C_{g,n}(\{t_k\}, d_1, \ldots, d_k)$ is a universal polynomial of its variables $t_k$’s.
General remarks: polynomiality

Form-cycle duality (=special geometry, SW) proved by recursion:

$$\frac{\partial \omega_{g,n}(z_1, \ldots, z_n)}{\partial t_{a,k}} = \oint_{z \in t^*_a, k} \omega_{g,n+1}(z_1, \ldots, z_n, z)$$

implies linear 1st order linear ODE for $C_{g,n}$, i.e. exponential

$$C_{g,n}(\{t_k\}, d_1, \ldots, d_n) = \sum_{m} \frac{1}{m!} \sum_{j_1, \ldots, j_m} \prod_{i} t_{j_i} C_{g,n+m}(\{0\}, d_1, \ldots, d_n, j_1 + 1, \ldots, j_m + 1)$$

i.e. symbolically

$$C_{g,n}(\{t_k\}, d_1, \ldots, d_n) = <\tau_{d_1} \ldots \tau_{d_n} e^{\sum_j t_j \tau_{j+1}} >_g$$

The universal coefficients $<\tau_{d_1} \ldots \tau_{d_n}>_g$ can be computed on an example, Airy curve $x = z^2, y = z$, i.e. Kontsevich integral:

$$<\tau_{d_1} \ldots \tau_{d_n}>_g = \int_{\mathcal{M}_{g,n}} \prod_{i=1}^{n} c_1(\mathcal{L}_i)^{d_i} = \text{Witten - Kontsevich intersection number}$$
Theorem (Eynard 2011)

The $\omega_{g,n}$'s defined by the recursion are

$$
\omega_{g,n}(z_1, \ldots, z_n) = \int_{\overline{M}_{g,n}(\#b.p.)} \frac{1}{n!} \sum_{\delta=(p_+,p_-)} \sum_{k,l} B_{\sigma(p_+),k;\sigma(p_-),l} \ell_\delta t_{\pi^*\tau_{k+1}}
$$

$$
e \sum_{k,l} B_{\sigma(p_+),k;\sigma(p_-),l} \ell_\delta \tau_{k+1} \prod_{i=1}^n \tau_{d_i} d_{\xi_\sigma(p_i),d_i}(z_i)
$$

where $\overline{M}_{g,n}(r) = \{(C, p_1, \ldots, p_n, \sigma)\}$=moduli space of nodal stable surfaces $C$ of genus $g$ with $n$ marked points $p_i$, and with a map $\sigma : C \to \{1, \ldots, r\}$ constant in each component.

$\delta =$boundary divisors = introduction of a nodal point i.e. 2 new marked points $p_+, p_-$. $\ell_\delta =$ inclusion of the boundary into the stratum of the moduli space with 2 more marked points.
Laplace transforms

The $(2k + 1)!!$ can be get rid off by Laplace transform:

$$
\sum_k t_{a,k} u^{-k} = e^{-\sum_k \hat{t}_{a,k} u^{-k}} = \frac{u^{2/3}}{2\sqrt{\pi}} \int_{\gamma_a} ydx \ e^{-u(x-x(a))}
$$

$$
\sum_{k,l} B_{a,k;b,l} u^{-k} v^{-l} = \frac{\sqrt{uv}}{2\pi} \int_{z \in \gamma_a} \int_{z' \in \gamma_b} \omega_{0,2}(z, z') \ e^{-u(x(z)-x(a))} \ e^{-v(x(z'))}
$$

Remark: Arbarello Cornalba define $\kappa$ classes as forgetful projections of $\tau$ classes as:

$$
e \sum_k t_{a,k} \pi^* \tau_{k+1} = e \sum_k \hat{t}_{a,k} \kappa_{k}
$$
Example of application

\[ y(z) = \frac{\sin 2\pi z}{4\pi}, \quad x(z) = z^2, \quad \omega_{0,2}(z, z') = \frac{dzdz'}{(z-z')^2}. \]

Only one branchpoint \( a = 0 \). We have \( B_{a,k;a,l} = 0 \).

Laplace transform of \( ydx \):

\[
\int_{\mathbb{R}} ydx \ e^{-ux} = \frac{1}{2i\pi} \int e^{2i\pi z} e^{-uz^2} \ dz
\]

\[
= \frac{1}{2i\pi} e^{-\pi^2/u} \int e^{-u(z-i\pi/u)^2} (z - i\pi/u + i\pi/u) \ dz
\]

\[
= \frac{1}{2u} e^{-\pi^2/u} \int e^{-u(z-i\pi/u)^2} \ dz
\]

\[
= \frac{\sqrt{\pi}}{2u\sqrt{u}} e^{-\pi^2/u}
\]

It implies \( \hat{t}_k = \pi^2 \delta_{k,1} \), i.e.

\[
\Lambda(S) = e^{\pi^2 \kappa_1} = \text{Weil Petersson form}
\]
Example of application

\( y(z) = z, \ x(z) = -z + \ln z. \) Only one branchpoint \( a = 1. \)

Laplace transform of \( ydx: \)

\[
\int_\mathbb{R} ydx \ e^{-ux} = \int_{\gamma} e^{uz} z^{-u} (1 - z)dz = \int_0^\infty e^{-uz} z^{-u} (1 + z)dz = u^{u-1} \Gamma(1 - u) + u^{u-2} \Gamma(2 - u) = u^{u-1} \Gamma(-u)
\]

\[
= \frac{e^u \sqrt{2\pi}}{u^{3/2}} e - \sum_{k=1}^\infty \frac{B_{2k}}{2k(2k-1)} u^{1-2k}
\]

This leads to the "Gamma" class=Hodge class

\[
\Lambda(S) = e^{\sum_k \frac{B_{2k}}{2k(2k-1)} (\kappa_{2k-1} + \frac{1}{2} \sum_l (-1)^l \sum_\delta \ell_\delta \tau_{2k-l} \tau_l)} \quad \text{Mumford} \quad \Lambda_{\text{Hodge}}
\]

Gives ELSV formula:

\[
\omega_{g,n}(\{Z_i\}) = \sum_{\mathcal{M}_{g,n}} \int \Lambda_{\text{Hodge}} \prod_{i=1}^n \tau_{d_i} d\xi_{d_i}(Z_i)
\]
Example of application: Mariño Vafa

Example: \( y(z) = \ln z, \ x(z) = \ln (1 - z) - f \ln z, \) one branchpoint \( a = \frac{f}{f-1}. \) (satisfies \( e^{x+fy} + e^y - 1 = 0, \) mirror of \( \mathbb{C}^3, \) framing \( f). \)

Laplace transform of \( ydx: \)

\[
\int_{\mathbb{R}} ydx \ e^{-ux} = \frac{1}{u} \int_0^1 z^{fu} (1 - z)^{-u} \frac{dz}{z} = \frac{\Gamma(fu)\Gamma(u)}{\Gamma((f + 1)u + 1)}
\]

\[
= \frac{e^{-ux(a)} \sqrt{2\pi}}{u^{3/2}} \ e^{\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \ u^{1-2k}(1+f^{1-2k}-(1+f)^{1-2k})}
\]

This leads to the "Gamma" class=Hodge class

\[
\Lambda(S) = \Lambda_{\text{Hodge}}(-1)\Lambda_{\text{Hodge}}(-f)\Lambda_{\text{Hodge}}(1+f)
\]

Gives Mariño-Vafa formula:

\[
\omega_{g,n}\left(\{z_i\}\right) = \sum_{d_1,...,d_n} \int_{\mathcal{M}_{g,n}} \Lambda_{\text{Hodge}}(-1)\Lambda_{\text{Hodge}}(-f)\Lambda_{\text{Hodge}}(1+f) \prod_{i=1}^{n} \tau_{d_i} d\xi_{d_i}(z)
\]
Example of application: BKMP conjecture

\( \mathfrak{x} = \text{toric Calabi–Yau 3 fold. Mirror symmetry: } \hat{\mathfrak{x}} \) = curve

\[ P( e^x, e^y) = 0. \]

Example: \( \mathfrak{x} = \text{local } \mathbb{P}^2, \) \( e^{x+fy} + e^y - 1 + Q e^{-x-(f+1)y} = 0. \)

(genus 1 curve, i.e. \( \ln x \) and \( \ln y = \) elliptic functions).

The previous theorem implies (after some combinatorics)

\[
\omega_{g,n}(\{z_i\}) = \sum_{\text{localization graphs}} \frac{1}{\# \text{Aut}} \prod_{e=(v_+,v_-)=\text{edges}} Q_e^{d_{e}/f_e} \prod_{\nu=\text{vertices}} (\#) \int_{\mathcal{M}_{g,v},n_v} \Lambda_{H(f_\nu)} \Lambda_{H(f'_\nu)} \Lambda_{H(f''_\nu)} \prod_{e \mapsto v} \tau_{d_e} \prod_{i=1}^n d \xi_{d_i}(z_i)
\]

it proves (BKMP) that \( \omega_{g,n} = \) open Gromov-Witten invariants:

\[
\omega_{g,n}(\{z_i\}) = \sum_{d \in H_2(\mathfrak{x},\{L_i\},\mathbb{Z}), \ d_i \in H_1(L_i,\mathbb{Z})} N_{g,d,\{d_i\}}(\mathfrak{x},\{L_i\}) \prod_{i=1}^n e^{\frac{-d_i}{f(L_i)} x(z_i)} d(x(z_i))
\]
5. Conclusion
Conclusion:

- The TR is a universal recursion relation satisfied by many enumeration problems (also appears in random matrices, enumeration of partitions, 3d partitions, knots, statistical physics, Liouville CFT,...)
- Once we know $W_{0,1}$ and $W_{0,2}$, the TR is very efficient at explicitly computing higher genus generating functions.
- Gives new proofs of ELSV, Mariño-Vafa, Mirzakhani, BKMP...

Further prospects:

- \exists recent new conjecture: topological recursion with $S = A$-polynomial, computes asymptotics of Jones polynomial of knots? [Dijkgraaf-Fuji-Manabe 2010, Borot-E 2012, Gukov, Sulkowski, Mulase,...] = generalization of volume conjecture. Find a proof ???
- categorification refinement, AGT
The end

Thank you for your attention

Acknowledgments:
This work was supported:

by the CRM Montréal, by CERN, by the Quebec government with the FQRNT.