Painlevé functions and conformal blocks

Oleg Lisovyy

LMPT, Tours, France

with O. Gamayun and N. Iorgov

arXiv:1207.0787, 1302.1832

Painlevé equations:

- classification of ODEs $w'' = F(w, w', t)$ without movable critical points
- non-autonomous hamiltonian systems
- confluence cascade

$$
\begin{tikzcd}
 & & PIV \arrow[dr] & & \\
PVI \arrow[r] & PV \arrow[ur] \arrow[dr] & & PII \arrow[r] & PI \\
 & & PIII \arrow[ur] & &
\end{tikzcd}
$$

Solutions:

- classical special functions

$$
\begin{tikzcd}
 & & Hermite \arrow[dr] & & \\
{}_2F_1 \arrow[r] & Whittaker \arrow[ur] \arrow[dr] & & Airy \arrow[r] & \varnothing \\
 & & Bessel \arrow[ur] & &
\end{tikzcd}
$$

- elliptic (PVI)
- algebraic
- transcendental (almost all solutions!)

Example

Toeplitz determinant

$$D_N^{(z,z')}(t) = \det \left[f_{j-k} \right]_{j,k=1,\ldots,N},$$

with the symbol

$$\sum_{\ell \in \mathbb{Z}} f_\ell \zeta^\ell = \left(1 - \sqrt{t}\zeta\right)^z \left(1 - \sqrt{t}\zeta^{-1}\right)^{z'}$$

is a Painlevé VI tau function:

$$D_N^{(z,z')}(t) = (1-t)^{\frac{N(N+z+z')}{2}}\, \tau(t)$$

Namely, $\sigma(t) = t(t-1)\frac{d}{dt} \ln \tau(t)$ satisfies

$$-\frac{1}{2}\Big(t(t-1)\sigma''\Big)^2 =$$

$$= \det \begin{pmatrix} 2\theta_0^2 & t\sigma' - \sigma & \sigma' + \theta_0^2 + \theta_t^2 + \theta_1^2 - \theta_\infty^2 \\ t\sigma' - \sigma & 2\theta_t^2 & (t-1)\sigma' - \sigma \\ \sigma' + \theta_0^2 + \theta_t^2 + \theta_1^2 - \theta_\infty^2 & (t-1)\sigma' - \sigma & 2\theta_1^2 \end{pmatrix}$$

with $(\theta_0, \theta_t, \theta_1, \theta_\infty) = \left(0, \frac{N}{2}, -\frac{N+z+z'}{2}, \frac{z-z'}{2}\right)$.

[Gessel, '90; Borodin, '01]:

$$D_N^{(z,z')}(t) = (1-t)^{-zz'} \sum_{\lambda\in\mathbb{Y},\lambda_1\le N} P^{(z,z',t)}(\lambda)\,,$$

where $P^{(z,z',t)}(\lambda)$ is the z-measure

$$P^{(z,z',t)}(\lambda) = (1-t)^{zz'} \prod_{(i,j)\in\lambda} \frac{(i-j+z)(i-j+z')}{h_\lambda^2(i,j)}\, t^{|\lambda|}$$

- $\mathbb{Y}$ is the set of all partitions $\lambda = (\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_\ell > 0)$, identified with Young diagrams
- $|\lambda| = \lambda_1 + \ldots + \lambda_\ell$ is the total number of boxes in λ
- $h_\lambda(i,j)$ denotes hook length of the box $(i,j)\in\lambda$
- $\sum_{\lambda\in\mathbb{Y}} P^{(z,z',t)}(\lambda) = 1$ is a variant of Cauchy identity for Schur functions [Okounkov, '99]
$$\sum_{\lambda\in\mathbb{Y}} s_\lambda(\mathbf{x})\, s_\lambda(\mathbf{y}) = \prod_{i,j}\left(1-x_i y_j\right)^{-1}$$

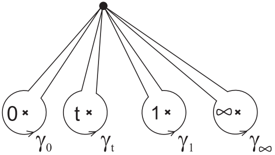
Our aim is to establish a similar series representation for the **general solution of Painlevé VI**.

Painlevé VI and isomonodromy

PVI describes monodromy preserving deformations of rank 2 linear systems on $\mathbb{P}^1$ with 4 regular singular points $0, t, 1, \infty$:

$$\frac{d\Phi}{dz} = \mathcal{A}(z)\Phi, \qquad \mathcal{A}(z) = \frac{\mathcal{A}_0}{z} + \frac{\mathcal{A}_t}{z-t} + \frac{\mathcal{A}_1}{z-1}$$

- matrices $\mathcal{A}_\nu$ are 2×2, traceless, with eigenvalues $\pm\theta_\nu$
- $\mathcal{A}_0 + \mathcal{A}_t + \mathcal{A}_1 \overset{def}{=} -\mathcal{A}_\infty = \operatorname{diag}\{-\theta_\infty, \theta_\infty\}$
- 3 monodromy matrices $\mathcal{M}_{0,t,1} \in G = SL(2,\mathbb{C})$ (note $\mathcal{M}_\infty\mathcal{M}_1\mathcal{M}_t\mathcal{M}_0 = \mathbf{1}$)
- monodromy manifold $\mathcal{M} = G^3/G$, $\dim \mathcal{M} = 6$



Painlevé VI and isomonodromy (continued)

Schlesinger equations:

$$\frac{d\mathcal{A}_0}{dt} = \frac{[\mathcal{A}_t, \mathcal{A}_0]}{t}, \qquad \frac{d\mathcal{A}_1}{dt} = \frac{[\mathcal{A}_t, \mathcal{A}_1]}{t-1}$$

- Lax form $\Rightarrow$ $\theta_{0,t,1,\infty}$ are conserved
- remains 2 degrees of freedom (recall that $\mathcal{A}_0 + \mathcal{A}_t + \mathcal{A}_1 = -\mathcal{A}_\infty$)
- $\left(\frac{\mathcal{A}_0}{z} + \frac{\mathcal{A}_t}{z-t} + \frac{\mathcal{A}_1}{z-1}\right)_{12} = \frac{k(t)(z-w(t))}{z(z-t)(z-1)}$ $\Rightarrow$ standard form of PVI for $w(t)$
- $\sigma = (t-1)\,\mathrm{Tr}\,\mathcal{A}_0\mathcal{A}_t + t\,\mathrm{Tr}\,\mathcal{A}_t\mathcal{A}_1$ $\Rightarrow$ sigma form of PVI for $\sigma(t)$

Painlevé VI and isomonodromy (continued)

Monodromy data:

- to any solution corresponds (the conjugacy class of) a triple $(\mathcal{M}_0, \mathcal{M}_t, \mathcal{M}_1)$
- $p_\nu = 2\cos 2\pi\theta_\nu = \operatorname{tr}\mathcal{M}_\nu$ (with $\nu = 0, t, 1, \infty$) give four PVI parameters
- remaining two coordinates $\Rightarrow$ integration constants
- introduce $p_{\mu\nu} = 2\cos 2\pi\sigma_{\mu\nu} = \operatorname{tr}\mathcal{M}_\mu\mathcal{M}_\nu$, then [Jimbo, '82]

$$p_{0t}p_{1t}p_{01} + p_{0t}^2 + p_{1t}^2 + p_{01}^2 - \omega_{0t}p_{0t} - \omega_{1t}p_{1t} - \omega_{01}p_{01} + \omega_4 - 4 = 0, \qquad (1)$$

where $\omega_4 = p_0^2 + p_t^2 + p_1^2 + p_\infty^2 + p_0 p_t p_1 p_\infty$ and

$$\omega_{0t} = p_0 p_t + p_1 p_\infty, \quad \omega_{1t} = p_1 p_t + p_0 p_\infty, \quad \omega_{01} = p_0 p_1 + p_t p_\infty$$

The triple $\boldsymbol{\sigma}$ satisfying (1) can be interpreted as a pair of PVI integration constants. Our task is: given $\boldsymbol{\sigma}$, to obtain the corresponding solution.

Jimbo's formula ['82]

- expresses the asymptotics of $\tau(t)$ as $t \to 0$, 1, or ∞ in terms of monodromy
- e.g. for $t \to 0$, denote $\sigma = \sigma_{0t}$ and choose $0 < |\mathrm{Re}\,\sigma| < \frac{1}{2}$
- also denote $\Delta_\nu = \theta_\nu^2$ $(\nu = 0, t, 1, \infty)$ and $\Delta_\sigma = \sigma^2$; then

$$\tau(t) = \mathrm{const} \cdot \left(t^{\Delta_\sigma - \Delta_0 - \Delta_t} + C_{\pm 1} t^{\Delta_\sigma \pm 1 - \Delta_0 - \Delta_t} + \text{ smaller terms } \right),$$

with

$$C_{\pm 1} = \frac{\Gamma^2(1 \mp 2\sigma)}{\Gamma^2(1 \pm 2\sigma)} \prod_{\epsilon = \pm} \frac{\Gamma(1 + \epsilon\theta_0 + \theta_t \pm \sigma)\,\Gamma(1 + \epsilon\theta_\infty + \theta_1 \pm \sigma)}{\Gamma(1 + \epsilon\theta_0 + \theta_t \mp \sigma)\,\Gamma(1 + \epsilon\theta_\infty + \theta_1 \mp \sigma)} \times$$
$$\times \frac{\left(\theta_0^2 - (\theta_t \mp \sigma)^2\right)\left(\theta_\infty^2 - (\theta_1 \mp \sigma)^2\right)}{4\sigma^2 (1 \pm 2\sigma)^2} (-s_{0t})^{\pm 1},$$

and

$$\begin{aligned} s_{0t}^{\pm 1} & \left(\cos 2\pi(\theta_t \mp \sigma) - \cos 2\pi\theta_0\right)\left(\cos 2\pi(\theta_1 \mp \sigma) - \cos 2\pi\theta_\infty\right) = \\ &= \left(\cos 2\pi\theta_t \cos 2\pi\theta_1 + \cos 2\pi\theta_0 \cos 2\pi\theta_\infty \pm i \sin 2\pi\sigma \cos 2\pi\sigma_{01}\right) - \\ &- \left(\cos 2\pi\theta_0 \cos 2\pi\theta_1 + \cos 2\pi\theta_t \cos 2\pi\theta_\infty \mp i \sin 2\pi\sigma \cos 2\pi\sigma_{1t}\right) e^{\pm 2\pi i \sigma}. \end{aligned}$$

- higher-order corrections can be determined recursively from σPVI (in principle)

Higher order corrections

$$\tau(t) \sim t^{\Delta_\sigma - \Delta_0 - \Delta_t}\Big(1 + \mathcal{B}_1(\boldsymbol{\theta}, \sigma)t + \qquad \ldots\Big)$$

$$+C_{\pm 1}\ t^{\Delta_{\sigma\pm 1} - \Delta_0 - \Delta_t}$$

with

$$\mathcal{B}_1(\boldsymbol{\theta}, \sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_\infty + \Delta_1)}{2\Delta_\sigma},$$

Higher order corrections

$$\tau(t) \sim t^{\Delta_\sigma - \Delta_0 - \Delta_t}\left(1 + \mathcal{B}_1(\boldsymbol{\theta},\sigma)t + \mathcal{B}_2(\boldsymbol{\theta},\sigma)t^2 \ldots\right) +$$
$$+ C_{\pm 1}\, t^{\Delta_{\sigma\pm 1} - \Delta_0 - \Delta_t}\left(1 + \mathcal{B}_1^{(\pm 1)}(\boldsymbol{\theta},\sigma)t + \ldots\right) + C_{\pm 2} t^{\Delta_{\sigma\pm 2} - \Delta_0 - \Delta_t}\left(1 + \ldots\right)$$

with

$$\mathcal{B}_1(\boldsymbol{\theta},\sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_\infty + \Delta_1)}{2\Delta_\sigma},$$

$$\mathcal{B}_2(\boldsymbol{\theta},\sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_0 + \Delta_t + 1)(\Delta_\sigma - \Delta_\infty + \Delta_1)(\Delta_\sigma - \Delta_\infty + \Delta_1 + 1)}{4\Delta_\sigma(2\Delta_\sigma + 1)}$$
$$+ \frac{\left[(1+2\Delta_\sigma)(\Delta_0 + \Delta_t) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_0 - \Delta_t)^2\right]\left[(1+2\Delta_\sigma)(\Delta_\infty + \Delta_1) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_\infty - \Delta_1)^2\right]}{2(2\Delta_\sigma + 1)(4\Delta_\sigma - 1)^2},$$

$$\mathcal{B}_1^{(\pm 1)}(\boldsymbol{\theta},\sigma) = \mathcal{B}_1(\boldsymbol{\theta},\sigma \pm 1).$$

Higher order corrections

$$\tau(t) \sim t^{\Delta_\sigma - \Delta_0 - \Delta_t}\Big(1 + \mathcal{B}_1(\boldsymbol{\theta},\sigma)t + \mathcal{B}_2(\boldsymbol{\theta},\sigma)t^2 \ldots\Big) +$$

$$+ C_{\pm 1}\, t^{\Delta_{\sigma \pm 1} - \Delta_0 - \Delta_t}\Big(1 + \mathcal{B}_1^{(\pm 1)}(\boldsymbol{\theta},\sigma)t + \ldots\Big) + C_{\pm 2} t^{\Delta_{\sigma \pm 2} - \Delta_0 - \Delta_t}\Big(1 + \ldots\Big)$$

with

$$\mathcal{B}_1(\boldsymbol{\theta},\sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_\infty + \Delta_1)}{2\Delta_\sigma},$$

$$\mathcal{B}_2(\boldsymbol{\theta},\sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_0 + \Delta_t + 1)(\Delta_\sigma - \Delta_\infty + \Delta_1)(\Delta_\sigma - \Delta_\infty + \Delta_1 + 1)}{4\Delta_\sigma(2\Delta_\sigma + 1)}$$

$$+ \frac{\Big[(1+2\Delta_\sigma)(\Delta_0 + \Delta_t) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_0 - \Delta_t)^2\Big]\Big[(1+2\Delta_\sigma)(\Delta_\infty + \Delta_1) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_\infty - \Delta_1)^2\Big]}{2\,(2\Delta_\sigma + 1)\,(4\Delta_\sigma - 1)^2},$$

$$\mathcal{B}_1^{(\pm 1)}(\boldsymbol{\theta},\sigma) = \mathcal{B}_1(\boldsymbol{\theta},\sigma \pm 1).$$

Observation. PVI tau function is a linear combination of $c = 1$ conformal blocks:

$$\tau(t) = \sum_{n\in\mathbb{Z}} C_n t^{\Delta_{\sigma+n} - \Delta_0 - \Delta_t} \mathcal{B}\left(\boldsymbol{\theta}, \sigma + n, t\right)$$

Higher order corrections (continued)

$$\mathcal{B}_3(\boldsymbol{\theta},\sigma)=\frac{(\Delta_\sigma-\Delta_0+\Delta_t+2)(\Delta_\sigma-\Delta_\infty+\Delta_1+2)}{24\Delta_\sigma(4\Delta_\sigma-1)^2(\Delta_\sigma-1)^2}\times$$

$$\times\Big\{(8\Delta_\sigma^2-5\Delta_\sigma+3)(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+1)(\Delta_\sigma-\Delta_\infty+\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+1)$$
$$-4(9\Delta_\sigma^2-4\Delta_\sigma+1)(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+1)(\Delta_\sigma-\Delta_\infty+2\Delta_1)$$
$$-4(9\Delta_\sigma^2-4\Delta_\sigma+1)(\Delta_\sigma-\Delta_\infty+\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+1)(\Delta_\sigma-\Delta_0+2\Delta_t)$$
$$+8(6\Delta_\sigma^3+11\Delta_\sigma^2-6\Delta_\sigma+1)(\Delta_\sigma-\Delta_0+2\Delta_t)(\Delta_\sigma-\Delta_\infty+2\Delta_1)\Big\}$$
$$+\frac{1}{6\Delta_\sigma(\Delta_\sigma-1)^2}\Big\{(\Delta_\sigma^2+3\Delta_\sigma+2)(\Delta_\sigma-\Delta_0+3\Delta_t)(\Delta_\sigma-\Delta_\infty+3\Delta_1)$$
$$+(\Delta_\sigma-\Delta_0+3\Delta_t)(\Delta_\sigma-\Delta_\infty+\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+1)(\Delta_\sigma-\Delta_\infty+\Delta_1+2)$$
$$+(\Delta_\sigma-\Delta_\infty+3\Delta_1)(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+1)(\Delta_\sigma-\Delta_0+\Delta_t+2)$$
$$-2(\Delta_\sigma+1)(\Delta_\sigma-\Delta_0+3\Delta_t)(\Delta_\sigma-\Delta_\infty+2\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+2)$$
$$-2(\Delta_\sigma+1)(\Delta_\sigma-\Delta_\infty+3\Delta_1)(\Delta_\sigma-\Delta_0+2\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+2)\Big\}$$

- more terms can be checked using computer algebra

Conformal blocks

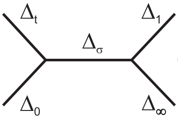
- OPE of two primaries:

$$\phi_0(0)\phi_t(t) = \sum_p C_{0t}^p t^{\Delta_p - \Delta_0 - \Delta_t} \sum_{\lambda \in \mathbb{Y}} \beta_\lambda(\Delta_0, \Delta_t, \Delta_p, c) t^{|\lambda|} \left(L_{-\lambda}\phi_p\right)(0)$$

  with β_λ completely fixed by Virasoro commutation relations

- any correlator reduces to 2- and 3-point functions
- 4-point conformal block on $\mathbb{P}^1$:

$$\mathcal{B}(t) =$$



$$(t) = 1 + \sum_{k \geq 1} \mathcal{B}_k t^k.$$

- coefficients $\mathcal{B}_k$ are rational functions of 4 external dimensions $\Delta_{0,t,1,\infty}$, one intermediate dimension Δ_σ, and Virasoro central charge c

Computation of conformal blocks

1. direct (inversion of Kac-Shapovalov matrix)

$$\mathcal{B}(t) = \sum_{\lambda,\mu\in\mathbb{Y}} \gamma_\lambda\left(\Delta,\Delta_1,\Delta_\infty\right)\left[Q\left(\Delta\right)\right]^{-1}_{\lambda\mu}\gamma_\mu\left(\Delta,\Delta_t,\Delta_0\right)t^{|\lambda|}$$

Here

$$\gamma_\mu\left(\Delta,\Delta_t,\Delta_0\right) = \prod_{j=1}^{\ell(\mu)}\left(\Delta-\Delta_0+\mu_j\Delta_t+\sum_{k=1}^{j-1}\mu_k\right)$$

and $Q_{\lambda\mu}\left(\Delta\right)=\langle\Delta|L_\lambda L_{-\mu}|\Delta\rangle$, so that e.g

$$Q_{\yng(3,1)\,\yng(2,2)}\left(\Delta\right)=\langle\Delta|L_3L_1L^2_{-2}|\Delta\rangle$$

2. recursion relation [Zamolodchikov, '84]
3. AGT correspondence [Alday, Gaiotto, Tachikawa, '09]:

$$\mathcal{B}(t)=\mathcal{Z}_{\text{inst}}(t)=\begin{array}{c}\text{combinatorial sum}\\ \text{over tuples of partitions}\end{array}$$

Proved in [Alba, Fateev, Litvinov, Tarnopolsky, '10]

Structure constants

Jimbo's asymptotic formula can be interpreted as a recurrence relation

$$\frac{C_{n\pm1}}{C_n}=\frac{\Gamma^2\left(1\mp2(\sigma_{0t}+n)\right)}{\Gamma^2\left(1\pm2(\sigma_{0t}+n)\right)}\prod_{\epsilon=\pm}\frac{\Gamma\left(1+\epsilon\theta_0+\theta_t\pm(\sigma_{0t}+n)\right)\Gamma\left(1+\epsilon\theta_\infty+\theta_1\pm(\sigma_{0t}+n)\right)}{\Gamma\left(1+\epsilon\theta_0+\theta_t\mp(\sigma_{0t}+n)\right)\Gamma\left(1+\epsilon\theta_\infty+\theta_1\mp(\sigma_{0t}+n)\right)}\times$$
$$\times\frac{\left(\theta_0^2-(\theta_t\mp(\sigma_{0t}+n))^2\right)\left(\theta_\infty^2-(\theta_1\mp(\sigma_{0t}+n))^2\right)}{4\left(\sigma_{0t}+n\right)^2\left(1\pm2(\sigma_{0t}+n)\right)^2}\left(-s_{0t}\right)^{\pm1}$$

with the solution in terms of Barnes functions

$$C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right)=s_{0t}^n\frac{\prod_{\epsilon,\epsilon'=\pm}G\left(1+\theta_t+\epsilon\theta_0+\epsilon'(\sigma_{0t}+n)\right)G\left(1+\theta_1+\epsilon\theta_\infty+\epsilon'(\sigma_{0t}+n)\right)}{G\left(1+2(\sigma_{0t}+n)\right)G\left(1-2(\sigma_{0t}+n)\right)}$$

Recursion relation for Barnes G-function: $G(z+1)=\Gamma(z)G(z)$

Main claim

Complete expansion of Painlevé VI tau function near $t=0$ can be written as

$$\tau(t) = \text{const} \cdot \sum_{n\in\mathbb{Z}} C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right) t^{(\sigma_{0t}+n)^2-\theta_0^2-\theta_t^2}\mathcal{B}(\boldsymbol{\theta},\sigma_{0t}+n;t).$$

The function $\mathcal{B}(\boldsymbol{\theta},\sigma;t)$ is a power series in t which coincides with the general $c=1$ conformal block and is explicitly given by

$$\mathcal{B}\left(\boldsymbol{\theta},\sigma;t\right) = (1-t)^{2\theta_t\theta_1}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}_{\lambda,\mu}\left(\boldsymbol{\theta},\sigma\right)t^{|\lambda|+|\mu|},$$

$$\mathcal{B}_{\lambda,\mu}\left(\boldsymbol{\theta},\sigma\right) = \prod_{(i,j)\in\lambda}\frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2}\times$$

$$\times\prod_{(i,j)\in\mu}\frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2}.$$

The structure constants $\{C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right)\}_{n\in\mathbb{Z}}$ can be written in terms of Barnes G-function,

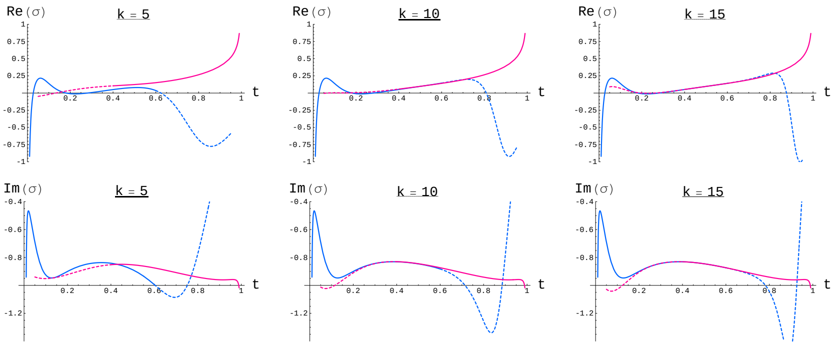
$$C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right) = s_{0t}^n\frac{\prod_{\epsilon,\epsilon'=\pm}G\left(1+\theta_t+\epsilon\theta_0+\epsilon'(\sigma_{0t}+n)\right)G\left(1+\theta_1+\epsilon\theta_\infty+\epsilon'(\sigma_{0t}+n)\right)}{G\left(1+2(\sigma_{0t}+n)\right)G\left(1-2(\sigma_{0t}+n)\right)}$$

Remarks

- checked about 30 first terms in the asymptotic expansion of τ (up to level 10, ~ 500 bipartitions) in full generality
- to prove rigorously, it is sufficient to demonstrate two bilinear relations satisfied by $c = 1$ conformal blocks
- expansions at $1, \infty$ are obtained by parameter change; for example, near $t = 1$

$$\theta_0 \leftrightarrow \theta_1, \quad \sigma_{0t} \leftrightarrow \sigma_{1t}, \qquad p'_{01} = \omega_{01} - p_{01} - p_{0t}p_{1t}.$$

- series representations suitable for numerical evaluation of PVI functions



$$\begin{pmatrix} \theta_0 \\ \theta_t \\ \theta_1 \\ \theta_\infty \end{pmatrix} = \begin{pmatrix} 0.501790 + 0.216884i \\ 0.382251 + 0.723641i \\ 0.152700 + 0.358959i \\ 0.158518 + 0.674992i \end{pmatrix}, \quad \begin{pmatrix} \sigma_{0t} \\ \sigma_{1t} \end{pmatrix} = \begin{pmatrix} 0.837497 + 0.943080i \\ 0.411398 + 0.480375i \end{pmatrix}$$

Riccati solutions

- parameters satisfy

$$\begin{cases} \omega_{0t} = 2p_{0t} + p_{1t}p_{01}, \\ \omega_{1t} = 2p_{1t} + p_{0t}p_{01}, \\ \omega_{01} = 2p_{01} + p_{0t}p_{1t}. \end{cases}$$

- simplest case $\theta_0 + \theta_t + \theta_1 + \theta_\infty = 0$, $\boldsymbol{\sigma} = (\theta_0 + \theta_t, \theta_1 + \theta_t, \theta_0 + \theta_1)$:

$$\tau(t) = \mathrm{const} \cdot t^{2\theta_0\theta_t}(1-t)^{2\theta_t\theta_1}.$$

- four-point correlator $\langle \mathcal{V}_{\theta_0}(0)\mathcal{V}_{\theta_t}(t)\mathcal{V}_{\theta_1}(1)\mathcal{V}_{\theta_\infty}(\infty)\rangle$ of chiral vertex operators $\mathcal{V}_\theta(z) = :e^{i\sqrt{2}\theta\phi(z)}:$ (only one conformal block!)
    - can add screenings
    - transformation s_δ maps CBs of exponential fields (with screening insertions) to CBs with degenerate external dimensions

Riccati/Chazy solutions (continued)

More general situation: for

$$\boldsymbol{\theta} = \frac{1}{2}\left(\eta, N, N-z-z', z'-z+\eta\right), \qquad N \in \mathbb{Z}_{>0},$$

$$\boldsymbol{\sigma} = \frac{1}{2}\left(N+\eta, z+z', z'-z+N\right) \mod \mathbb{Z}$$

PVI admits a 4-parameter family of solutions [Forrester, Witte, '02]

$$\tau(t) = t^{\frac{N\eta}{2}}\,(1-t)^{\frac{N(z+z'-N)}{2}}\,\det\left[f_{j-k}\right]_{j,k=1,\ldots,N}$$

$$f_\ell = \frac{\Gamma(1-z')}{\Gamma(1-\ell+\eta)\,\Gamma(1+\ell-\eta-z')}\,{}_2F_1\left(z,-\ell+\eta+z',1-\ell+\eta,t\right)+$$
$$+\frac{\xi\,\Gamma(1-z)}{\Gamma(1+\ell-\eta)\,\Gamma(1-\ell+\eta-z)}\,t^{\ell-\eta}\,{}_2F_1\left(z',\ell-\eta+z,1+\ell-\eta,t\right).$$

- dimension $\Delta_t = \frac{N^2}{4}$ is degenerate (level $N+1$)
- $\xi = 0 \Rightarrow$ single conformal block $\mathcal{B}\left(\frac{\eta}{2},\frac{N}{2},\frac{N-z-z'}{2},\frac{z'-z+\eta}{2},\frac{N+\eta}{2},t\right)$
- subsequently $\eta \to 0 \Rightarrow$ Toeplitz determinant $D_N^{(z,z')}$ for z-measures

Riccati/Chazy solutions (continued)

How to recover Gessel from AGT?

- We have $\sigma_{0t} = \theta_0 + \theta_t$. But $\mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta}, \sigma)$ contains the product
$$\prod_{(i,j)\in\mu} (\theta_0 + \theta_t - \sigma + i - j)$$
It vanishes for any non-empty μ as it contains $(i,j) = (1,1)$.
- remaining AGT sum over λ simplifies to
$$\sum_{\lambda\in\mathbb{Y}} t^{|\lambda|} \prod_{(i,j)\in\lambda} \frac{i-j+N}{i-j+N+\eta} \frac{(i-j+z)(i-j+z'+\eta)}{h_\lambda^2(i,j)},$$
and can be restricted to λ with $\lambda_1 \leq N$ thanks to $\theta_t - \theta_0 + \sigma_{0t} = N$
- finally letting $\eta \to 0$ we get
$$D_N^{(z,z')} = \sum_{\lambda\in\mathbb{Y},\lambda_1\leq N} t^{|\lambda|} \prod_{(i,j)\in\lambda} \frac{(i-j+z)(i-j+z')}{h_\lambda^2(i,j)}$$

Transformation $t \leftrightarrow 1-t$:

- $\xi = 0$:

1 CB at $t = 0$ (internal dimension $\sigma = \frac{N}{2}$) $\rightarrow$ $N+1$ CBs at $t = 1$ (internal dimensions $\sigma_k = \theta_1 - \frac{N}{2} + k$, $k = 0, \ldots, N$)

- $\xi \neq 0$: $N+1$ CBs at $t = 0$ (internal dimensions $\sigma_k = \theta_0 + \frac{N}{2} - k$, $k = 0, \ldots, N$)

Picard solutions

- $\omega_{0t} = \omega_{1t} = \omega_{01} = \omega_4 = 0$
- parameters can be Backlund transformed to $\boldsymbol{\theta}_{\text{Picard}} = \left(\frac14, \frac14, \frac14, \frac14\right)$
- dimensions $\Delta_\nu = \theta_\nu^2$ correspond to Ashkin-Teller conformal block [Zamolodchikov, '86]

$$\mathcal{B}\left(\boldsymbol{\theta}_{\text{Picard}}, \sigma; t\right) = \frac{\left(16t^{-1}q\right)^{\sigma^2}}{(1-t)^{\frac18}\vartheta_3(0|\tau)}$$

where $q = e^{i\pi\tau}$, $\tau = \dfrac{iK'(t)}{K(t)}$ and

$$K(t) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-tx^2)}}, \qquad K'(t) = K(1-t).$$

- structure constants C_n and parameter s_{0t} simplify to

$$C_n \sim 2^{-4(\sigma_{0t}+n)^2}(-s_{0t})^n, \qquad s = -e^{\pm 2\pi i \sigma_{1t}}$$

- conformal expansion $\tau(t) = \sum_{n\in\mathbb{Z}} C_n t^{(\sigma_{0t}+n)^2-\theta_0^2-\theta_t^2}\mathcal{B}(\boldsymbol{\theta}, \sigma_{0t}+n, t)$ then gives theta function series so that finally

$$\tau_{\text{Picard}}(t) = \text{const}\cdot \frac{q^{\sigma_{0t}^2}}{t^{\frac18}(1-t)^{\frac18}}\frac{\vartheta_3\left(\sigma_{0t}\pi\tau \pm \sigma_{1t}\pi|\tau\right)}{\vartheta_3(0|\tau)}.$$

(this indeed coincides with Picard tau function [Kitaev, Korotkin, '98])

Coalescence diagram revisited

$$\begin{tikzcd}
\mathrm{VI} \arrow[r] & \mathrm{V} \arrow[r] \arrow[d] & \mathrm{III}_1 \arrow[r] \arrow[d] & \mathrm{III}_2 \arrow[r] \arrow[d] & \mathrm{III}_3 \\
 & \mathrm{IV} \arrow[r] & \mathrm{II} \arrow[r] & \mathrm{I} &
\end{tikzcd}$$

- can easily write similar expansions for Painlevé V and $\mathrm{III}_{1,2,3}$
- coalescence corresponds to decoupling of matter hypermultiplets

$$\begin{matrix} N_f = 4 \\ (P_{\mathrm{VI}}) \end{matrix} \xrightarrow{\mu_4 \to \infty} \begin{matrix} N_f = 3 \\ (P_{\mathrm{V}}) \end{matrix} \xrightarrow{\mu_3 \to \infty} \begin{matrix} N_f = 2 \\ (P_{\mathrm{III}_1}) \end{matrix} \xrightarrow{\mu_2 \to \infty} \begin{matrix} N_f = 1 \\ (P_{\mathrm{III}_2}) \end{matrix} \xrightarrow{\mu_1 \to \infty} \begin{matrix} \text{pure gauge theory} \\ (P_{\mathrm{III}_3}) \end{matrix}$$

Painlevé III_3:

$$\left(D^4 + (1-2\delta)D^2 + 4t\right)\tau \cdot \tau = 0.$$

where $\delta = t\dfrac{d}{dt}$ and D is the associated Hirota derivative

Conformal expansion:

$$\tau(t) = \sum_{n\in\mathbb{Z}} C_{\sigma+n}\, s^n\, \mathcal{B}_{\sigma+n}(t)\,,$$

- integration constants σ, s
- $C_\sigma = [G(1+2\sigma)G(1-2\sigma)]^{-1}$
- AGT representation:

$$\mathcal{B}_\sigma(t) = \sum_{\lambda,\mu\in\mathbb{Y}} t^{\sigma^2+|\lambda|+|\mu|}\Bigg[\prod_{(i,j)\in\lambda} h_\lambda(i,j)\left(\lambda'_j + \mu_i - i - j + 1 + 2\sigma\right)\times$$
$$\times\prod_{(i,j)\in\mu} h_\mu(i,j)\left(\lambda_i + \mu'_j - i - j + 1 - 2\sigma\right)\Bigg]^{-2}$$

Painlevé III_3 (continued):

It suffices to prove

$$\sum_{n\in\mathbb{Z}} \chi_{\sigma,n} \left(D^4 + (1-2\delta)D^2 + 4t\right) \mathcal{B}_{\sigma+n} \cdot \mathcal{B}_{\sigma-n} = 0$$

with

$$\chi_{\sigma,n} = \prod_{k=1-2n}^{2n-1} (2\sigma - k)^{-2(2n-|k|)}$$

and the same relation with $n \in \mathbb{Z} + \frac{1}{2}$.

Algebraic formulation:

- Consider the sequence of states $|n\rangle$ with $n = 0, 1, \ldots$ such that

$$|0\rangle = |\Delta\rangle, \qquad L_1|n\rangle = |n-1\rangle, \qquad L_2|n\rangle = 0.$$

- Whittaker vector $|\Delta\rangle_W = \sum_{n=0}^{\infty} t^{\frac{n+\Delta}{2}} |n\rangle$ satisfies

$$L_0|\Delta\rangle_W = 2t\frac{d}{dt}|\Delta\rangle_W, \qquad L_1|\Delta\rangle_W = \sqrt{t}\,|\Delta\rangle_W, \qquad L_2|\Delta\rangle_W = 0$$

- irregular conformal block $\mathcal{B}_\sigma(t) = {}_W\langle\Delta|\Delta\rangle_W$ with $\Delta = \sigma^2$

General isomonodromy problem

Rank N linear system with n regular singular points $a_1, \ldots, a_n$ on $\mathbb{P}^1$:

$$\partial_z \Phi = \mathcal{A}(z)\Phi, \qquad \mathcal{A}(z) = \sum_{\nu=1}^{n} \frac{\mathcal{A}_\nu}{z - a_\nu}$$

- normalization $\Phi(z_0) = \mathbf{1}_N$
- no singularity at $\infty \Rightarrow \sum_{\nu=1}^{n} \mathcal{A}_\nu = 0$
- $\mathcal{A}_\nu$'s assumed to be diagonalizable: $\mathcal{A}_\nu = \mathcal{G}_\nu \mathcal{T}_\nu \mathcal{G}_\nu^{-1}$ with some $\mathcal{T}_\nu = \operatorname{diag}\left\{\lambda_{\nu,1}, \ldots, \lambda_{\nu,N}\right\}$
- introducing $\mathcal{J}(z) = \Phi^{-1}\partial_z\Phi = \Phi^{-1}\mathcal{A}(z)\Phi$, expand $\Phi(z)$ around $z = z_0$:

$$\Phi(z \to z_0) = \mathbf{1}_N + \mathcal{J}(z_0)(z - z_0) + \left(\mathcal{J}^2(z_0) + \partial\mathcal{J}(z_0)\right)\frac{(z - z_0)^2}{2} + \ldots$$

- expansions near singular points:

$$\Phi(z \to a_\nu) = \mathcal{G}_\nu(z)\,(z - a_\nu)^{\mathcal{T}_\nu}\,\mathcal{C}_\nu$$

  - $\mathcal{G}_\nu(z)$ is holomorphic and invertible in a neighborhood of $z = a_\nu$, and satisfies $\mathcal{G}_\nu(a_\nu) = \mathcal{G}_\nu$
  - $\mathcal{C}_\nu$ are connection matrices; monodromy matrices $\mathcal{M}_\nu = \mathcal{C}_\nu^{-1} e^{2\pi i \mathcal{T}_\nu} \mathcal{C}_\nu$

General isomonodromy problem (continued)

Deformation equations:

$$\partial_{a_\nu}\Phi = -\frac{z_0 - z}{z_0 - a_\nu}\,\frac{\mathcal{A}_\nu}{z - a_\nu}\,\Phi,$$
$$\partial_{z_0}\Phi = -\mathcal{A}(z_0)\,\Phi.$$

- $\mathcal{J}(z)$ remains invariant under isomonodromic variation of z_0 !

Schlesinger equations:

$$\partial_{a_\mu}\mathcal{A}_\nu = \frac{z_0 - a_\nu}{z_0 - a_\mu}\,\frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{a_\mu - a_\nu}, \qquad \mu \neq \nu,$$
$$\partial_{a_\nu}\mathcal{A}_\nu = -\sum_{\mu\neq\nu}\frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{a_\mu - a_\nu}, \qquad \partial_{z_0}\mathcal{A}_\nu = -\sum_{\mu\neq\nu}\frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{z_0 - a_\mu}.$$

Tau function:

$$d\ln\tau = \sum_{\mu<\nu}\operatorname{tr}\mathcal{A}_\mu\mathcal{A}_\nu\; d\ln(a_\mu - a_\nu).$$

- τ does not depend on z_0 thanks to

$$\partial_{a_\mu}\ln\tau = \sum_{\nu\neq\mu}\frac{\operatorname{tr}\mathcal{A}_\mu\mathcal{A}_\nu}{a_\mu - a_\nu} = \frac{1}{2}\operatorname{res}_{z=a_\mu}\operatorname{tr}\mathcal{J}^2(z).$$

Global conformal symmetry

How does τ trasform under $f(z)=\dfrac{\alpha z+\beta}{\gamma z+\delta}$?

Example (three points): $d\ln\tau$ can be explicitly integrated to

$$\tau(a_1,a_2,a_3)=\mathrm{const}\cdot(a_1-a_2)^{\Delta_3-\Delta_2-\Delta_1}\,(a_1-a_3)^{\Delta_2-\Delta_1-\Delta_3}\,(a_2-a_3)^{\Delta_1-\Delta_2-\Delta_3},$$

with $\Delta_\nu=\frac12\operatorname{tr}\mathcal{A}_\nu^2$ and $\nu=1,2,3$. Expression for 3-point function of quasiprimary fields with dimensions $\Delta_{1,2,3}$ in 2D CFT !

Proposition: One has

$$\tau\left(f(a)\right)=\prod_{\nu=1}^{n}\left[f'\left(a_\nu\right)\right]^{-\Delta_\nu}\;\tau(a)$$

■ It suffices to consider infinitesimal transformations generated by $\left(A+Bz+Cz^2\right)\partial_z$
$\Rightarrow$ check three differential constraints

$$\sum_\nu \partial_{a_\nu}\ln\tau=0,$$

$$\sum_\nu\left(a_\nu\partial_{a_\nu}\ln\tau+\Delta_\nu\right)=0,$$

$$\sum_\nu\left(a_\nu^2\partial_{a_\nu}\ln\tau+2\Delta_\nu a_\nu\right)=0.$$

□

Ansatz for Φ

Fundamental matrix solution Φ is completely fixed by its monodromy, normalization and singular behaviour (choice of logarithm branches $\mathcal{L}_\nu = \mathcal{C}_\nu^{-1}\mathcal{T}_\nu\mathcal{C}_\nu = \frac{1}{2\pi i}\ln\mathcal{M}_\nu$).

Starting point (cf [Sato, Miwa, Jimbo, '79]):

$$\Phi_{jk}(z) = (z-z_0)^{2\Delta}\frac{\langle\mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\bar{\varphi}_j(z_0)\varphi_k(z)\rangle}{\langle\mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle}, \qquad j,k=1,\ldots,N.$$

Assumptions:

- $\{\mathcal{O}_{\mathcal{L}_\nu}\}$, $\{\bar{\varphi}_j\}$, $\{\varphi_k\}$ are primary fields in a 2D CFT
- OPEs of $\bar{\varphi}$'s with φ's contain $\mathbf{1}$ $\Rightarrow$ same dimensions Δ
- normalization
$$\bar{\varphi}_j(z_0)\varphi_k(z)\sim(z-z_0)^{-2\Delta}\,\delta_{jk}.$$
- dimensions of all other primaries in this OPE are strictly positive integers
- complete OPEs of monodromy fields with auxiliary ones:

$$\mathcal{O}_{\mathcal{L}_\nu}(a_\nu)\varphi_k(z)=\sum_{j=1}^{n}\left((z-a_\nu)^{\mathcal{L}_\nu}\right)_{jk}\sum_{\ell=0}^{\infty}\mathcal{O}_{\mathcal{L}_\nu,j,\ell}(a_\nu)\,(z-a_\nu)^{\ell},$$

If one finds a set of fields with all mentioned properties, the correlator ratio will automatically give Φ.

Tau function

Compute two more orders in the OPE $\bar{\varphi}_j(z_0)\varphi_k(z)$:

$$\bar{\varphi}_j(z_0)\varphi_k(z) = (z-z_0)^{-2\Delta}\Big[\delta_{jk} + J_{jk}(z_0)\,(z-z_0) + \\ +\Big(\frac{4\Delta}{c}\,T(z_0)\delta_{jk} + (\partial J_{jk})(z_0) + S_{jk}(z_0)\Big)\frac{(z-z_0)^2}{2} + O\left((z-z_0)^3\right)\Big].$$

- 1st order: no descendants of 1, new primary J
- 2nd order: level 2 descendant of $\mathbf{1}$, level 1 descendant of J, new primary S

Comparing with

$$\Phi(z\to z_0) = \mathbf{1}_N + \mathcal{J}(z_0)\,(z-z_0) + \left(\mathcal{J}^2(z_0) + \partial\mathcal{J}(z_0)\right)\frac{(z-z_0)^2}{2} + \ldots,$$

one can identify

$$\mathcal{J}(z) = \frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)J(z)\rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle},$$

$$\mathrm{tr}\,\mathcal{J}^2(z) = \frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)T(z)\rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle}\,\frac{4N\Delta}{c}.$$

Tau function (continued)

But $\partial_{a_\mu} \ln \tau = \frac{1}{2} \operatorname{res}_{z=a_\mu} \operatorname{tr} \mathcal{J}^2(z)$ and

$$\frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1) \ldots \mathcal{O}_{\mathcal{L}_n}(a_n) T(z) \rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1) \ldots \mathcal{O}_{\mathcal{L}_n}(a_n) \rangle} = \sum_{\nu=1}^{n} \left\{ \frac{\tilde{\Delta}_\nu}{(z-a_\nu)^2} + \frac{1}{z-a_\nu} \partial_{a_\nu} \ln \langle \mathcal{O}_{\mathcal{L}_1}(a_1) \ldots \mathcal{O}_{\mathcal{L}_n}(a_n) \rangle \right\}$$

which implies

$$\boxed{\tau(a) = \langle \mathcal{O}_{\mathcal{L}_1}(a_1) \ldots \mathcal{O}_{\mathcal{L}_n}(a_n) \rangle^{\frac{2N\Delta}{c}}}$$

- in particular, for $c = 2N\Delta$ we have $\tilde{\Delta}_\nu = \frac{1}{2} \operatorname{tr} \mathcal{A}_\nu^2$

Example [SMJ, '79; Moore, '90] (N free complex fermions $\{\bar{\psi}_j\}$, $\{\psi_k\}$)

- $c = 2 \cdot N \cdot \frac{1}{2}$, current $J_{jk} = (\bar{\psi}_j \psi_k)$, $T = \frac{1}{2} \sum_k \left[(\bar{\psi}_k \, \partial \psi_k) - (\partial \bar{\psi}_k \, \psi_k) \right]$
- monodromy fields obtained by bosonization

$$\bar{\psi}_k =: e^{-i\phi_k} :, \qquad \psi_k =: e^{i\phi_k} :,$$

$$J_{jk} = \begin{cases} : e^{i(\phi_k - \phi_j)} :, & j \neq k, \\ i \, \partial \phi_k, & j = k, \end{cases} \qquad T = -\frac{1}{2} \sum_k (\partial \phi_k \, \partial \phi_k),$$

$$\mathcal{O}_{\mathcal{L}_\nu} =: e^{i \sum_k \lambda_{\nu,k} \phi_k^{(\nu)}} : .$$

- need N distinct bosonization schemes

Back to isomonodromy problem

Decompose $\mathcal{A}_\nu$'s as $\mathcal{A}_\nu = \frac{\operatorname{tr}\mathcal{A}_\nu}{N}\,\mathbf{1}_N + \hat{\mathcal{A}}_\nu$, then

$$\Phi_{\mathcal{A}}(z) = \prod_\nu \left(\frac{z-a_\nu}{z_0-a_\nu}\right)^{\frac{\operatorname{tr}\mathcal{A}_\nu}{N}} \Phi_{\hat{\mathcal{A}}}(z),$$

$$\mathcal{J}_{\mathcal{A}}(z) = \frac{1}{N}\sum_\nu \frac{\operatorname{tr}\mathcal{A}_\nu}{z-a_\nu}\,\mathbf{1}_N + \mathcal{J}_{\hat{\mathcal{A}}}(z),$$

$$\tau_{\mathcal{A}}(a) = \prod_{\mu<\nu}(a_\mu - a_\nu)^{\frac{\operatorname{tr}\mathcal{A}_\mu \operatorname{tr}\mathcal{A}_\nu}{N}}\,\tau_{\hat{\mathcal{A}}}(a).$$

Example (continued)

$$N \text{ complex fermions} = \hat{u}(1) \oplus \hat{su}(N)_1$$

Fermion and monodromy fields factorize

$$\bar{\psi}_k = :e^{-i\phi_0/\sqrt{N}}: \otimes\ \hat{\bar{\varphi}}_k, \qquad \psi_k = :e^{i\phi_0/\sqrt{N}}: \otimes\ \hat{\varphi}_k,$$

$$\mathcal{O}_{\mathcal{L}_\nu} = :e^{\frac{i\operatorname{tr}\mathcal{A}_\nu}{\sqrt{N}}\phi_0}: \otimes\ \mathcal{O}_{\hat{\mathcal{L}}_\nu}$$

- dimension $\Delta = \frac{N-1}{2N}$ of $\{\hat{\bar{\varphi}}_k\}$ and $\{\hat{\varphi}_k\}$ agrees with $c_{\hat{su}(N)_1} = N-1$
- dimension of $\mathcal{O}_{\hat{\mathcal{L}}_\nu}$ is equal to $\frac{1}{2}\operatorname{tr}\hat{\mathcal{A}}_\nu^2$
- tracelessness of $\mathcal{A}(z)$ corresponds to factoring out the $\hat{u}(1)$ piece

Conclusion: Isomonodromic tau function can be interpreted as a correlation function of primaries with dimensions $\Delta_\nu = \frac{1}{2}\operatorname{tr}\mathcal{A}_\nu^2$ in a CFT with $c = N - 1$.

Remark. For $N = 2$ the dimension $\Delta = \frac{1}{4}$ of φ's and $\bar{\varphi}$'s corresponds to states degenerate at level 2, and the dimension 1 of $\{J_{jk}\}$ is degenerate at level 3. Hence the correlators

$$\mathcal{P}_{jk} = \langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\bar{\varphi}_j(z_0)\varphi_k(z)\rangle,$$
$$\mathcal{Q}_{jk} = \langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)J_{jk}(z)\rangle,$$

have to satisfy linear PDEs of order 2 and 3, fixed by Virasoro symmetry.

Proposition. Under assumption $\operatorname{tr}\mathcal{A}(z) = 0$, the matrices

$$\mathcal{P} = (z - z_0)^{-\frac{1}{2}}\tau\Phi, \qquad \mathcal{Q} = \tau\Phi^{-1}\partial_z\Phi,$$

satisfy

$$\partial_{zz}\mathcal{P} = \left\{\frac{1}{z - z_0}\,\partial_{z_0} + \frac{1}{4\,(z - z_0)^2} + \sum_\nu\left(\frac{1}{z - a_\nu}\,\partial_{a_\nu} + \frac{\Delta_\nu}{(z - a_\nu)^2}\right)\right\}\mathcal{P},$$

$$\partial_{zzz}\mathcal{Q} = \left\{4\sum_\nu\left(\frac{1}{z - a_\nu}\,\partial_{a_\nu z} + \frac{\Delta_\nu}{(z - a_\nu)^2}\,\partial_z\right) + 2\sum_\nu\left(\frac{1}{(z - a_\nu)^2}\,\partial_{a_\nu} + \frac{2\Delta_\nu}{(z - a_\nu)^3}\right)\right\}\mathcal{Q}.$$

Painlevé VI

PVI tau function is a 4-point correlator of monodromy fields,

$$\tau(t) = \langle \mathcal{O}_{\mathcal{L}_0}(0)\mathcal{O}_{\mathcal{L}_t}(t)\mathcal{O}_{\mathcal{L}_1}(1)\mathcal{O}_{\mathcal{L}_\infty}(\infty)\rangle,$$

and these fields are Virasoro primaries with dimensions $\Delta_\nu = \theta_\nu^2$ in a $c = 1$ CFT.

- "conservation of monodromy" $\Rightarrow \{\varphi_k\}$ should have monodromy $\mathcal{M}_t\mathcal{M}_0$ around all fields in the OPE of $\mathcal{O}_{\mathcal{L}_0}$ and $\mathcal{O}_{\mathcal{L}_t}$
- if $\mathcal{M}_t\mathcal{M}_0 = \mathcal{C}_{0t}^{-1}\begin{pmatrix} e^{2\pi i\sigma_{0t}} & 0 \\ 0 & e^{-2\pi i\sigma_{0t}}\end{pmatrix}\mathcal{C}_{0t}$, then it is natural to expect that the set of primaries in the OPE of $\mathcal{O}_{\mathcal{L}_0}$ and $\mathcal{O}_{\mathcal{L}_t}$ consists of an infinite number of monodromy fields $\mathcal{O}_{\mathcal{L}_{0t}^{(n)}}$ with $n \in \mathbb{Z}$ and

$$\mathcal{L}_{0t}^{(n)} = \mathcal{C}_{0t}^{-1}\begin{pmatrix} \sigma_{0t}+n & 0 \\ 0 & -\sigma_{0t}-n\end{pmatrix}\mathcal{C}_{0t}$$

- inserting the OPE $\mathcal{O}_{\mathcal{L}_0}(0)\mathcal{O}_{\mathcal{L}_t}(t)$ into the correlator gives

$$\boxed{\tau(t) = \sum_{n\in\mathbb{Z}} C_n t^{\Delta_{\sigma+n}-\Delta_0-\Delta_t}\mathcal{B}(\boldsymbol{\theta}, \sigma+n, t)}$$

Conclusions

1. Painlevé VI, V, III tau functions are Fourier transforms of $c = 1$ conformal blocks and their irregular analogs
2. AGT combinatorial formulas provide series representations for general solutions of Painlevé VI, V, III and an efficient tool of numerical computation

More questions

1. connection problem for Painlevé tau functions/fusion matrix of $c = 1$ conformal blocks ✓
2. increase rank/genus/number of singular points
3. rigorous proof?