The String Equation and Scalar PDEs in the semiclassical regime

$\langle \phi | \mu \rangle$ *Grupo de Física Matemática da Universidade de Lisboa*

Davide Masoero

Hamiltonian PDEs, Frobenius Manifolds and Moduli Spaces,
SISSA 16-20 September 2013

Three Years as a Scholiast (2011-2013)

A scholiast (see picture) was a medieval monk writing annotations and comments (scholia) on ancient manuscripts.

A medieval scholiast

Modern scholiasts

The 'Manuscript':

B. Dubrovin 'On Hamiltonian perturbations of hyperbolic systems of conservation laws, II: universality of critical behaviour'. Comm Math Phys. (2006)

The annotations (scholia):

- D. M., A. Raimondo, 'Semiclassical limit for generalized KdV equations before the gradient catastrophe'. Lett. Math. Phys. (2011 [2013])

- D. M., A. Raimondo, 'A deformation of the method of characteristics and the Cauchy problem for Hamiltonian PDEs in the small dispersion limit'. (2012)

- P. Antunes, D.M., A. Raimondo, 'Universality classes of scalar nonlinear PDEs', In preparation

Short Resume of The Manuscript.

Conjecture: Given a generic Hamiltonian perturbation of the Hopf equation (e.g. KdV)

$$u_t = uu_x + \varepsilon^2 u_{xxx},$$

then

- Before the breaking of Hopf solution:

  the $\epsilon \to 0$ limit converges.

- At the breaking point (x_c, t_c):

  the (leading term in the) multiscale limit $(x, t, \epsilon) \to (x_c, t_c, 0)$ is universally described by a special solution of Painleve I (2).

Plan: Report on our annotations

- Semiclassical limit before the 'gradient catastrophe'

- The String Equation

- Classification of Universality Classes in the Scalar Case. A novel Painleve equation?

Semiclassical Limit of Generalized KdV - 2011

We studied the (Hamiltonian) Cauchy problems:

$$u_t = a(u)\, u_x + \sum_{i=1}^{n} \varepsilon^{2i}\, c_i \partial_x^{2i+1} u, \qquad u(x,0) = \varphi(x),$$

where

- $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$,
- $a(u)$ smooth,
- φ is independent of ε.

Let $u(\varepsilon)$ be the solution of

$$u_t = a(u)u_x + c_1\varepsilon^2 u_{xxx}, \qquad u(x,0) = \varphi(x) \in H^s,$$

Theorem [D.M., A. Raimondo, 2011]

Fix $s \geq 3K$, the Cauchy problem of GKdV has a unique strong solution

$$u(\varepsilon) \in C([0,T], H^s)\ .$$

Moreover, the map

$$u : \mathbb{R} \to C([0,T], H^{s-3K}), \qquad \varepsilon \mapsto u(\varepsilon)$$

is $2K$-times differentiable.

Remark: if the perturbed equation is globally well-posed, T can be chosen to be any time smaller than the critical time.

Remarks/Corollaries

- If the initial data belongs to $\cap_{s\geq 0} H^s$ then $u(x,t,\varepsilon)$ is a smooth function of the three variables $\to$ the perturbative expansion is asymptotic.

- The methods of the proof work for any perturbation of the Hopf equation by a (sensible) linear pseudo-differential operator.

- Conjecture: T can be always chosen to be any time smaller than the critical time.

- (Partial) Proof of the 'Main Conjecture, Part 1' of The Manuscript.

Second Year: the String Equation

Consider again the general Hamiltonian perturbation (e.g. KdV)

$$u_t = uu_x + \varepsilon^2 u_{xxx}\,, \qquad u(x, t = 0, \varepsilon) = \varphi(x)\ .$$

The Hopf equation, as any scalar PDE of the first order, can be solved by the method of characteristics.

Can we deform this method? Yes, if ε is small.

Why? Perturbative corrections and Critical Behaviour.

How? Look at the 'manuscript'.

String Equation II

For the Hopf equation the method of characteristics boils down to the hodograph functional equation:

$$x + ut - f(u) = 0\,, f(u) \text{ a local inverse of the initial data .}$$

Remark: $u_s = \partial_x\Big(x + ut - f(u)\Big)$ is a symmetry of Hopf $u_t = uu_x$.

The functional equation = equation for the fixed point of this symmetry.

Idea: We have a symmetry and a small parameter ε^2, let us deform the symmetry!

The String Equation = fixed point equation of the deformed symmetry.

How does it look like?

Method of Characteristics $\varepsilon = 0$

$$x + ut = f(u)\ .$$

The String Equation for KdV of order $O(\varepsilon^4)$

$$\begin{aligned} x + t\,u = f + \varepsilon^2 \left(\frac{1}{2} f^{(3)} u_x^2 + f'' u_{xx} + f_1\right) + \varepsilon^4 \Bigg(\frac{3}{5} f^{(5)} u_x^2 u_{xx} \\ + \frac{9}{10} f^{(4)} u_{xx}^2 + \frac{6}{5} f^{(4)} u_x u_{xxx} + \frac{3}{5} f^{(3)} u_{xxxx} - \frac{1}{24} f^{(6)} u_x^4 + f_2\Bigg) \\ + O(\varepsilon^6)\,, \qquad f_1 = -\frac{1}{2}\frac{f''}{(f')^2}, \qquad f_2 = \frac{1}{8}\frac{f'''}{(f')^4} - \frac{1}{5}\frac{(f'')^2}{(f')^5}\ . \end{aligned}$$

The choice of the initial data determines the String equation uniquely.

We chose the functions f_1, f_2 to fix the initial data independent of ε.

Our Results (Sloppy formulation)

Theorem:

Take any Hamiltonian PDE and suppose the deformation of the symmetry is found up to order $O(\varepsilon^{2N})$, the String equation looks like

$$\mathcal{S}^N(u, u_x, \dots, \varepsilon) = O(\varepsilon^{2N+2}) \; .$$

Call $\sigma(x, t, \varepsilon)$ the evaluation of $\mathcal{S}^N$ at the solution of the Cauchy problem. If $D \subset \mathbb{R}^3$ is a compact domain on which σ is *enough differentiable*, then

$$\sigma(x, t, \varepsilon) = O(\varepsilon^{2N+2}) \text{ on } D \; .$$

Proof required some technical innovations because the String equation is local in the space-time.

Our Results II

- We construct the String equation explicitly up to $O(\varepsilon^4)$ for the general Hamiltonian perturbations $\longrightarrow$ we compute explicitly the first two non trivial perturbative corrections.

  The generic second correction is a differential expression with more than one thousand terms.

- Quasi-triviality for solutions of KdV with ε independent initial data.

  Smooth (rapidly decaying) solutions of KdV admit an asymptotic expansion in ε and every term of the expansion is a polynomial differential in the solution of Hopf equation.

Example: First Corrections for KdV

Let u the solution of semiclassical KdV and v^0 the solution of Hopf then if $u(t=0) \in H^9$,

$$u = v^0 + \varepsilon^2 v^1 + \varepsilon^4 v^2 + O(\varepsilon^6)$$

where

$$v^1 = \frac{\varepsilon^2}{2}\partial_x^2\left(\log\left(1 + t\, v_x^0\right) - \frac{1}{(1 + t\, v_x^0)}\right)$$

$$v^2 = \frac{\varepsilon^4\, t^2}{8}\partial_x^2\left[t\,\partial_x\left(\frac{3}{10}\frac{20 + 15\, t\, v_x^0 + 3\, t^2 (v_x^0)^2}{(1 + t\, v_x^0)^5}\right)\right.$$
$$\left. - \frac{t^2}{10}\frac{(5 + t\, v_x^0)(v_{xx}^0)^3}{(1 + t\, v_x^0)^5} + \partial_x^2\left(\frac{(2 + t\, v_x^0)^2}{(1 + t\, v_x^0)^4}\, v_{xx}^0\right)\right]$$

Remarks

- The hypothesis of the proof are verified for GKdV due to our previous results.

- The proof breaks down at the critical time.

- There is no obstacle to introducing the String equation for non Hamiltonian perturbations [Arsie,Lorenzoni, Moro]. Hamiltonian equations are more manageable.

- If we admit symmetries non-polynomial in the derivatives, it should be possible to extend the String equation up to any order in ε^2 for any smooth perturbation (issue related with quasi-triviality: [Dubrovin, Wu, Zhang ...]).

Why the name 'String equation'?

$$x + ut = f(u) \sim \sum c_n u^n = \frac{\delta}{\delta u} \sum \frac{c_n \int u^{n+1} dx}{n+1} = \frac{\delta}{\delta u} \sum c_n H^{n-1}_{\varepsilon=0} \ .$$

Here $H^n_{\varepsilon=0}$ are the standard Hamiltonians of the Hopf hierarchy.

The natural deformation for KdV is

$$x + ut \sim \frac{\delta}{\delta u} \sum c_n H^{n-1}_{\varepsilon} \ .$$

Here H^n_{ε} are the standard KdV Hamiltonians.

This is not exactly the deformation we consider because it is irregular at $t = 0$ for any decaying initial data.

Moreover, it works just for KdV and the convergence of this series is more than dubious.

Why the name 'String equation'? II

Suppose the sum is finite

$$x + ut = \frac{\delta}{\delta u} \sum c_n^N H_\varepsilon^{n-1} \; .$$

take the total x-derivative

$$1 + = -u_x t + \sum_n^N c_n u_{t_{n-1}} = -u_{t_0} t + \sum_n^N c_n [L, L_+^{\frac{2n-1}{2}}]$$

This coincides with the String equation of Moore, Novikov ... :

the Painleve I hierarchy.

Pause: Ancient and Modern Times

A serious geometer from 1500

500 Years Later

Critical Behaviour after the Manuscript

At some space-time point (x_c, t_c) the solution $u(x,t)$ of the Hopf equation develops a singularity.

To unfold the singulairty we take a double scaling limit

$$u(x,t) = u(x_c,t_c) + \lambda^{1/3} U(\frac{(x-x_c)+u_c(t-t_c)}{\lambda}, \frac{t-t_c}{\lambda^{2/3}}) + O(\lambda^{2/3})$$

and $U(X,T)$ is the solution of the truncated String equation

$$X + UT = U^3 = \frac{\delta H^2_{\varepsilon=0}[U]}{\delta U} .$$

The detailed balance of the scaling and the semiclassical parameter ε:

- Multiscale expansion:

$$u(x,t,\varepsilon) \simeq u_c + \varepsilon^{\frac{2}{7}} U\left(\frac{x - x_c + u_c(t-t_c)}{\varepsilon^{\frac{6}{7}}}, \frac{t-t_c}{\varepsilon^{\frac{4}{7}}}\right) + O(\varepsilon^{\frac{4}{7}}) \ .$$

- for any (generic) Hamiltonian PDE $U(X,T)$ solves KdV.

- U satisfies the truncated KdV String equation (Painleve I(2))

$$\begin{aligned} X + UT &= \frac{\delta H^2_{\varepsilon=1}[U]}{\delta U} = U^3 + \frac{1}{4}(U'^2 + 2UU'') + \frac{1}{40}U^{IV} \,, \\ U(X,T) &\sim \text{sign}(X)|X|^{1/3} \text{ as } |X| \to \infty \ . \end{aligned}$$

Remarks

- The existence of the multiple scale limit at the critical point ($x = x_c, t = t_c, \varepsilon = 0$) proposed by Dubrovin is yet unproven in general.

- Claeys and Grava proved it for KdV and its hierarchy by the methods of nonlinear steepest descent. Numerical investigation on Kawahara equation by Dubrovin, Grava, Klein.

- String equation expands solutions of Hamiltonian PDEs in 'Painleve modes'. At the critical point, only the second mode is triggered.

A General Scalar PDE

We consider a scalar PDE of the form

$$u_t = uu_x + N[u]$$

where N is a 'general' nonlinear (possibly nonlocal) (pseudo)differential operator.

We assume that u is a slow-variable, i.e. if the initial data vary on a long scale $1/\varepsilon$, the solution evolves slowly:

$$N[u(\frac{x}{\varepsilon})] = \varepsilon^{\beta+1}\overline{N}[u(x)] + o(\varepsilon^{\beta+1})\,, \beta > 0\,, \text{ for any smooth } u\,.$$

Rich phenomenology and lot of examples. Local N: Hamiltonian PDEs, variable-depth KdV, Burgers-like equations. Nonlocal N: Benjamin-Ono, ILW, Benjamin-Bona-Mahony, Camassa-Holm ... .

Critical Behaviour

In the long-wave (semiclassical) limit, the PDE looks like

$$u_t = uu_x + \varepsilon^\beta \overline{N}[u(x)] + o(\varepsilon^\beta).$$

Call L_c the linearization of $\overline{N}$ at the constant solution $u(x) = u_c$.

If the multiple scale expansion exists, the leading term $U(X,T)$ solves

$$U_T = UU_X + L_c[U] \text{ where } L_c[U(X/\varepsilon)] = \varepsilon^{\beta+1} L_c[U(X)] \ .$$

We say that two equations N^1, N^2 belong to the same universality class if for generic $u_c \in \mathbb{R}$ $L_c^1 = L_c^2$ up to a multiplicative constant.

Conjectural Universality Classes

Translational invariant linear pseudo-differential operator with the scaling

$$L[v(x/\varepsilon)] = \varepsilon^{\beta+1} L[v(x)]$$

is

$$L[v] = \int e^{ipx} \mu(p) \tilde{v_x}(p) dp \ ,$$

with symbol

$$\mu(p) = i\kappa \ \mathrm{sign}(p)|p|^{\beta} + \theta |p|^{\beta}, \qquad \kappa \geq 0.$$

Example: $\beta = 1, \theta = 0$: $U_T = UU_X + \kappa U_{XX}$ Burgers equation.

$\beta = 1, \kappa = 0$ $U_T = UU_X + \theta \mathcal{H}[U_{XX}]$ (Benjamin-Ono).

String Equation Again

The critical behaviour is conjecturally described by $U(X,T)$ solution of the equation $U_T = UU_X + L[U]$. Can we say something more?

Introduce $U_\mu = \mu^{1/3} U(X/\mu, T/\mu^{2/3}), \mu \leq 1$:

U_μ satisfies $U_T = UU_X + \mu^\beta L[U]$

U_0 is the solution of the cubic equation $X + UT - U^3 = 0$: the fixed point of $U_S = \partial_X (X + UT - U^3)$.

Let's look for a deformation of this symmetry:

$U_S = \partial_X (X + UT - U^3 + \mu^\beta K_1[U] + \mu^{2\beta} K_2[U] + \dots)$.

Deformed Symmetry

In general, the symmetry will be an infinite series and it will define $U(X, T)$ only asymptotically (small$\mu =$ large $|X|$).

In at least three cases the series is finite:

- Burgers: $X + UT = U^3 + 3UU_x + U_{XX}$ [Dubrovin et al., Lorenzoni et al.]

- KdV: $X + UT = \frac{\delta H^2_{KdV}}{\delta U} = U^3 + \frac{1}{4}(U'^2 + 2UU'') + \frac{1}{40}U^{IV}$, Painleve I(2) [Dubrovin]

- Benjamin-Ono: $X + UT = \frac{\delta H^2_{BO}}{\delta U} = U^3 + \frac{3}{2}(U\mathcal{H}[U_X] + \mathcal{H}[UU_X]) + U_{XX}$ [Antunes, M., Raimondo]

A (Novel) Special Function

It seems that the the critical behaviour of equations in Benjamin-Ono universality class is described by the solution of

$$X + UT = U^3 + \frac{3}{2}(U\mathcal{H}[U_X] + \mathcal{H}[UU_X]) + U_{XX}\,,$$
$$U(X,T) \sim \text{sign}(X)|X|^{1/3} \text{ as } |X| \to \infty\,.$$

- Does the solution exists and is unique? Numerically yes.

- Does it extend to a meromorphic function? ?

- Is it transcendental? Not guaranteed: In BO Whitham is 'trivial' [Miller and Xu] and traveling waves are rational [Amick and Toland]

A Novel Painleve Equation ?

Is this equation a novel Painleve-like equation?

$$X + UT = U^3 + \frac{3}{2}(U\mathcal{H}[U_X] + \mathcal{H}[UU_X]) + U_{XX}$$

- Isomonodromic system?

- Does it have other solutions, possibly a two dimensional manifold?

- Painleve property?

Conclusions

Three years as a scholiast:

- Semiclassical limit before the gradient catastrophe for non-integrable PDEs

- String equation: we proved the validity of an important heuristic tool

- 'Classified' universality classes of scalar PDEs

- A new? special function

Some (of the Many) Open Questions

- Semiclassical limit for non-integrable PDEs after the time of catastrophe? New tecnhology is needed.

- String equation at the critical point. And after.

- A mathematical theory for the analogue of Painleve I(2) for Benjamin-Ono.

- How to deform symmetries in case the perturbation is non-local?

MANY THANKS FOR THE ATTENTION!