Confluence of the Painlevé equations and q-Askey scheme

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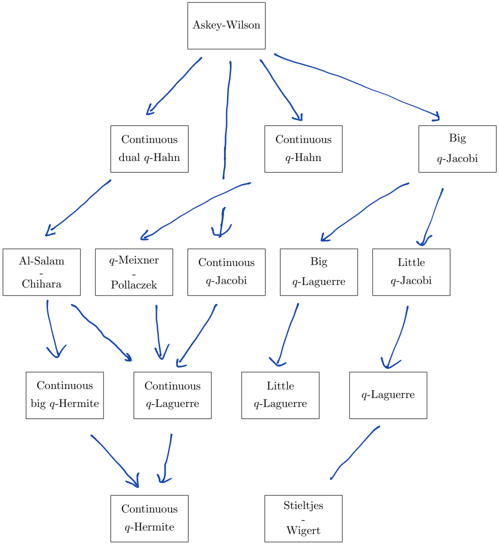
q-Askey scheme

Koekoek, Lesky, Swarttouw 2010



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This relation relies on the τ-function

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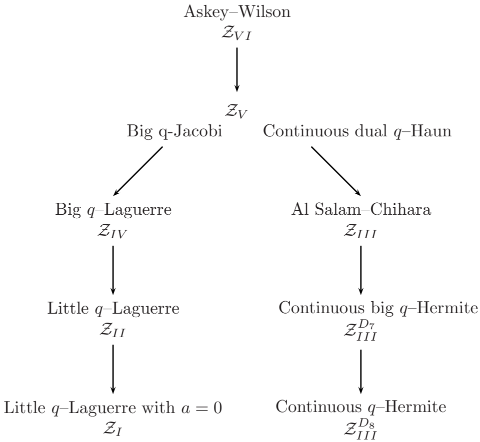
- Elements in the q–Askey scheme span eigenspaces in this representation.

q-Askey scheme



M.M. arXiv:1307.6140

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The monodromy group of PVI admits a **natural quantisation** to the **Cherednik algebra $\mathcal{H}$ of type $\check{C}_1 C_1$**.

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We obtain 7 **new algebras** as confluences such that their spherical subalgebra is the quantisation of the monodromy manifold of each Painlevé equation.

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Cherednik algebra of type $\check{C}_1 C_1$

Algebra generated by $V_0, V_1, \check{V}_0, \check{V}_1$:

$$
\begin{aligned}
(V_0 - k_0)(V_0 + k_0^{-1}) &= 0\\
(V_1 - k_1)(V_1 + k_1^{-1}) &= 0\\
(\check{V}_0 - u_0)(\check{V}_0 + u_0^{-1}) &= 0\\
(\check{V}_1 - u_1)(\check{V}_1 + u_1^{-1}) &= 0\\
\check{V}_1 V_1 V_0 \check{V}_0 &= q^{-1/2},
\end{aligned}
$$

k_0, k_1, u_0, u_1 constants.

Cherednik '92, Sahi '99

Zhedanov algebra:

generated by: $e = \frac{1+\check{V}_1}{1+u_1^2}$,

$X_1 = \frac{1+\check{V}_1}{1+u_1^2}\left(\check{V}_1 V_1 + (\check{V}_1 V_1)^{-1}\right)$,

$X_2 = \frac{1+\check{V}_1}{1+u_1^2}\left(\check{V}_1 V_0 + (\check{V}_1 V_0)^{-1}\right)$,

$X_3 = \frac{1+\check{V}_1}{1+u_1^2}\left(q^{1/2} V_1 V_0 + q^{-1/2}(V_1 V_0)^{-1}\right)$

$$q^{-1/2}X_1X_2 - q^{1/2}X_2X_1 = (q^{-1}-q)X_3 + (q^{-1/2}-q^{1/2})\omega_3\frac{1+\check{V}_1}{1+u_1^2}$$

$$q^{-1/2}X_2X_3 - q^{1/2}X_3X_2 = (q^{-1}-q)X_1 + (q^{-1/2}-q^{1/2})\omega_1\frac{1+\check{V}_1}{1+u_1^2}$$

$$q^{-1/2}X_3X_1 - q^{1/2}X_1X_3 = (q^{-1}-q)X_2 + (q^{-1/2}-q^{1/2})\omega_2\frac{1+\check{V}_1}{1+u_1^2}$$

$$q^{\frac{1}{2}}X_2X_1X_3 - qX_2^2 - q^{-1}X_1^2 - qX_3^2 + q^{\frac{1}{2}}\omega_2X_2 + q^{-\frac{1}{2}}\omega_1X_1 + q^{\frac{1}{2}}\omega_3X_3 = \omega_4.$$

Zhedanov '91, Oblomkov '04

Representation theory:

Cherednik algebra on Laurent polynomials

$$V_1[f] = \frac{z\left[u_1 z^2 - \frac{1}{u_1} + \left(k_1 - \frac{1}{k_1}\right) z\right]}{z^2-1} f\left(\frac{1}{z}\right) - \frac{z\left(u_1 - \frac{1}{u_1}\right) + k_1 - \frac{1}{k_1}}{z^2-1} f(z)$$

$$V_0[f] = \frac{q k_0 - \frac{z^2}{k_0} + \sqrt{q}\left(u_0 - \frac{1}{u_0}\right) z}{z^2-1} f\left(\frac{q}{z}\right) - \frac{z\left[\left(k_0 - \frac{1}{k_0}\right) + \sqrt{q}\left(u_0 - \frac{1}{u_0}\right)\right]}{z^2-1} f(z)$$

The operator $\frac{1+\check{V}_1}{1+u_1^2}$ acts as symmetrizer $\Rightarrow$ representation of the Zhedanov algebra on symmetric Laurent polynomials.

Zhedanov algebra and Askey Wilson polynomials

$$ {}_r\varphi_{r-1}\left(\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_{r-1} \end{array}; q, x\right) := \sum_{k=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_k}{(b_1,\ldots,b_{r-1};q)_k} x^k $$

$$ (a;q)_k := \Pi_{j=0}^{k-1}(1 - a\, q^j) $$

$$ (a_1,\ldots,a_r;q)_k := (a_1;q)_k (a_2;q)_k \cdots (a_r;q)_k $$

Askey-Wilson polynomials:

$$ p_n\,(z;a,b,c,d \mid q) := {}_4\varphi_3\left(\begin{array}{c} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{array}; q, q\right) $$

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$$ X_1\left(p_n(z,a,b,c,d|q)\right) = (z + \tfrac{1}{z}) p_n(z,a,b,c,d|q), $$

$$ X_2\left(p_n(z,a,b,c,d|q)\right) = \left(q^{-n} + abcdq^{n-1}\right)\left(p_n(z,a,b,c,d|q)\right). $$

Koornwinder '07

Sixth Painlevé equation

$$
\begin{aligned}
y_{tt} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)y_t^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)y_t+\\
&+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right].
\end{aligned}
$$

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Singular points at $0, 1, \infty$.

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&+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right].
\end{aligned}
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Singular points at $0, 1, \infty$.
Parameters $\alpha, \beta, \gamma, \delta$.

PVI as isomonodromic deformations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} Y = \sum_{k=1}^{3} \frac{A_k}{\lambda - a_k} Y, \qquad \lambda \in \mathbb{C} \backslash \{a_1, a_2, a_3\}$$

$A_1, A_2, A_3 \in \mathfrak{sl}(2, \mathbb{C}), \qquad \sum_{k=1}^{3} A_k = -A_\infty$, diagonal.

- Fundamental matrix: $Y_\infty(\lambda) = \left(1 + O(\frac{1}{\lambda})\right) \lambda^{-A_\infty}$
- Monodromy matrices: $\gamma_i(Y_\infty) = Y_\infty M_i$

$\mathrm{eigenv}(M_i) = \exp(\pm i\, p_i), \quad i = 1, 2, 3, \infty,$

$M_\infty M_1 M_2 M_3 = 1.$

Jimbo, Miwa '81

Cherednik algebra as quantisation of the PVI monodromy group

$$
\begin{array}{rr}
(V_0 - k_0)(V_0 + k_0^{-1}) = 0, & (M_3 - e^{i\,p_3})(M_3 - e^{-i\,p_3}) = 0, \\
(V_1 - k_1)(V_1 + k_1^{-1}) = 0, & (M_2 - e^{i\,p_2})(M_2 - e^{-i\,p_2}) = 0, \\
(\check{V}_1 - u_1)(\check{V}_1 + u_1^{-1}) = 0, & (M_1 - e^{i\,p_1})(M_1 - e^{-i\,p_1}) = 0, \\
(\check{V}_0 - u_0)(\check{V}_0 + u_0^{-1}) = 0, & (M_\infty - e^{i\,p_\infty})(M_\infty - e^{-i\,p_\infty}) = 0, \\
\check{V}_1 V_1 V_0 \check{V}_0 = q^{-1/2}, & M_\infty M_3 M_2 M_1 = 1.
\end{array}
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\check{V}_1 V_1 V_0 \check{V}_0 &= q^{-1/2}, & M_\infty M_3 M_2 M_1 &= 1.
\end{aligned}
$$

There is a natural quantisation which works.

Monodromy manifold for PVI

Riemann Hilbert correspondence: $\forall (M_1, M_2, M_3)/Mat_2(\mathbb{C})$ there exists a unique local solution to PVI.
We can describe the monodromy manifold by

$$x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4.$$

where

$$x_i := \mathrm{Tr}(M_j M_k), \quad i, j, k = 1, 2, 3, \quad k, j \neq i.$$

This cubic is the moduli space of monodromy representations:

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}) \to SL_2(\mathbb{C}).$$

Jimbo '81, Iwasaki '03

Monodromy manifold and Zhedanov algebra

$$\varphi := x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 - \omega_4,$$

defines a natural Poisson bracket:

$$\{x_1, x_2\} = \frac{\partial\varphi}{\partial x_3}, \quad \{x_2, x_3\} = \frac{\partial\varphi}{\partial x_1}, \quad \{x_3, x_1\} = \frac{\partial\varphi}{\partial x_2}.$$

This looks a lot like the classical version of the Zhedanov algebra:

$$q^{-1/2}X_1X_2 - q^{1/2}X_2X_1 = (q^{-1} - q)X_3 + (q^{-1/2} - q^{1/2})\omega_3 \tfrac{1+\check{V}_1}{1+u_1^2}$$

....cyclic...

$$q^{\frac{1}{2}}X_2X_1X_3 - qX_2^2 - q^{-1}X_1^2 - qX_3^2 + q^{\frac{1}{2}}\omega_2X_2 + q^{-\frac{1}{2}}\omega_1X_1 + q^{\frac{1}{2}}\omega_3X_3 = \omega_4.$$

Teichmüller space of a Riemann sphere with 4 singularities

Interpret each x_i as a geodesic length:

$$x_1 = e^{s_2+s_3} + e^{-s_2-s_3} + e^{-s_2+s_3} + (e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}})e^{s_3} + (e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}})e^{-s_2}$$
$$x_2 = e^{s_3+s_1} + e^{-s_3-s_1} + e^{-s_3+s_1} + (e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}})e^{s_1} + (e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}})e^{-s_3}$$
$$x_3 = e^{s_1+s_2} + e^{-s_1-s_2} + e^{-s_1+s_2} + (e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}})e^{s_2} + (e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}})e^{-s_1}$$

they satisfy the cubic relation for PVI:

$$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1x_1 + \omega_2x_2 + \omega_3x_3 = \omega_4.$$

The Goldman bracket $\{s_1, s_2\} = \{s_2, s_3\} = \{s_3, s_1\} = 1$ gives rise to the correct Poisson bracket on it (L. Chekhov and M.M. J.Phys A 2010).

Quantisation:

$s_i \to$ quantum operator $s_i^\hbar$ with commutation relation

$$[s_i^\hbar, s_j^\hbar] = i\pi\hbar\{s_i, s_j\}.$$

$\Rightarrow$ Weyl ordering:

$$\exp\left(a s_i^\hbar\right)\exp\left(b s_j^\hbar\right) = \exp\left(a s_i^\hbar + b s_i^\hbar + \frac{ab}{2}[s_i^\hbar, s_j^\hbar]\right),$$

Quantum algebra:

$$\begin{array}{rcl} q^{-1/2}x_1^\hbar x_2^\hbar - q^{1/2}x_2^\hbar x_1^\hbar & = & (q^{-1}-q)x_3^\hbar + (q^{-1/2}-q^{1/2})\omega_3 \\ q^{-1/2}x_2^\hbar x_3^\hbar - q^{1/2}x_3^\hbar x_2^\hbar & = & (q^{-1}-q)x_1^\hbar + (q^{-1/2}-q^{1/2})\omega_1 \\ q^{-1/2}x_3^\hbar x_1^\hbar - q^{1/2}x_1^\hbar x_3^\hbar & = & (q^{-1}-q)x_2^\hbar + (q^{-1/2}-q^{1/2})\omega_2 \end{array}$$

(L. Chekhov and M.M. J.Phys A 2010).

Quantise the monodromy matrices: each monodromy matrix corresponds to a half–geodesic on our Riemann surface.

Quantise them in the same way: we obtain the Cherednik algebra of type $\check{C}_1 C_1$ (M.M. (2013))

$$M_1 \to i\check{V}_1, \quad M_2 \to iV_1, \quad M_3 \to iV_0, \quad M_\infty \to i\check{V}_0.$$

$$V_0 = \begin{pmatrix} k_0 - k_0^{-1} - ie^{-s_3} & -i\, e^{-s_3} \\ k_0^{-1} - k_0 + i\, e^{-s_3} + i\, e^{s_3} & i\, e^{-s_3} \end{pmatrix},$$

$$V_1 = \begin{pmatrix} k_1 - k_1^{-1} - i\, e^{s_2} & k_1 - k_1^{-1} - i\, e^{-s_2} - i\, e^{s_2} \\ i\, e^{s_2} & i\, e^{s_2} \end{pmatrix},$$

$$\check{V}_1 = \begin{pmatrix} 0 & -ie^{s_1} \\ i\, e^{-s_1} & u_1 - u_1^{-1} \end{pmatrix}, \qquad \check{V}_0 = \begin{pmatrix} u_0 & 0 \\ s & -\frac{1}{u_0} \end{pmatrix},$$

Embedding of the Cherednik algebra of type $\check{C}_1 C_1$ into $Mat(2, \mathbb{T}_q)$.

Monodromy manifolds for the Painlevé equations

PVI	$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$
PV	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$
PIV	$x_1x_2x_3 + x_1^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_2 x_3 + 1 = \omega_4$
PIII	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 = \omega_1 - 1$
PII	$x_1x_2x_3 + x_1 + x_2 + x_3 = \omega_4$
PI	$x_1x_2x_3 + x_1 + x_2 + 1 = 0$

Saito and van der Put

The confluence from PVI to PV is realised by

$$s_3 \to s_3 - \log[\epsilon], \qquad p_3 \to p_3 - 2\log[\epsilon], \qquad \epsilon \to 0$$

$$x_1 = -e^{s_2+s_3} - e^{-s_2+s_3} - (e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}})e^{s_3} - e^{\frac{p_3}{2}} e^{-s_2}$$
$$x_2 = -e^{s_3+s_1} - e^{\frac{p_3}{2}} e^{s_1},$$
$$x_3 = -e^{s_1+s_2} - e^{-s_1-s_2} - e^{-s_1+s_2} - (e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}})e^{s_2} - (e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}})e^{-s_1}$$

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$$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$$

Quantum PV algebra:

$$\begin{aligned} q^{-1/2}x_1^\hbar x_2^\hbar - q^{1/2}x_2^\hbar x_1^\hbar &= (q^{-1/2} - q^{1/2})\omega_3 \\ q^{-1/2}x_2^\hbar x_3^\hbar - q^{1/2}x_3^\hbar x_2^\hbar &= (q^{-1} - q)x_1^\hbar + (q^{-1/2} - q^{1/2})\omega_1 \\ q^{-1/2}x_3^\hbar x_1^\hbar - q^{1/2}x_1^\hbar x_3^\hbar &= (q^{-1} - q)x_2^\hbar + (q^{-1/2} - q^{1/2})\omega_2 \end{aligned}$$

M.M. and V.Rubtsov arXiv:1212.6723

$$V_0 = \begin{pmatrix} -1 & 0 \\ 1 + i\, e^{S_3} & 0 \end{pmatrix}, \qquad V_0^2 + V_0 = 0$$

$$V_1 = \begin{pmatrix} k_1 - k_1^{-1} - i\, e^{S_2} & k_1 - k_1^{-1} - i\, e^{-S_2} - i\, e^{S_2} \\ i\, e^{S_2} & i\, e^{S_2} \end{pmatrix},$$

$$\check{V}_1 = \begin{pmatrix} 0 & -\, i e^{S_1} \\ i\, e^{-S_1} & u_1 - u_1^{-1} \end{pmatrix},$$

$$\check{V}_0 = \begin{pmatrix} 0 & 0 \\ s & -\frac{1}{u_0} \end{pmatrix}, \qquad \check{V}_0{}^2 + u_0^{-1}\check{V}_0 = 0$$

$$q^{1/2}\check{V}_1 V_1 V_0 = \check{V}_0 + u_0^{-1},$$

$$q^{1/2}\check{V}_0 \check{V}_1 V_1 = V_0 + 1.$$

From the PIV cubic:

$$V_0^2+V_0=0,$$
$$(V_1-k_1)(V_1+k_1^{-1})=0,$$
$$\check{V}_1{}^2+u_1^{-1}\check{V}_1=0,$$
$$\check{V}_0{}^2+\check{V}_0=0,$$
$$q^{1/2}\check{V}_1V_1V_0=\check{V}_0+1,$$
$$\check{V}_0\check{V}_1=0,$$
$$V_0\check{V}_0=0,$$

From the PII cubic:

$$V_0^2+V_0=0,$$
$$V_1^2+V_1=0,$$
$$\check{V}_1{}^2+u_1^{-1}\check{V}_1=0,$$
$$\check{V}_0{}^2+\check{V}_0=0,$$
$$q^{1/2}\check{V}_1V_1V_0=\check{V}_0+1,$$
$$\check{V}_0\check{V}_1=0,$$
$$V_0\check{V}_0=0,$$

From the PI cubic:

$$V_0^2=0,$$
$$V_1^2+V_1=0,$$
$$\check{V}_1{}^2+\check{V}_1=0,$$
$$\check{V}_0{}^2+\check{V}_0=0,$$
$$q^{1/2}\check{V}_1V_1V_0=\check{V}_0+$$
$$\check{V}_0\check{V}_1=0,$$
$$V_0\check{V}_0=0,$$

Conclusion

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It may be easier to study representations on the space of Laurent polynomials rather than $\mathbb{C}^2$!.