Darboux transformations and random point processes.

Mattia Cafasso
(joint work with Marco Bertola)

LAREMA – Université d'Angers

Hamiltonian PDEs, Frobenius manifolds and Deligne-Mumford moduli spaces.
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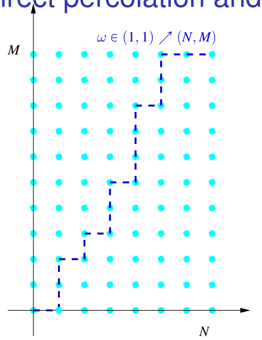
Plan of the talk (joint work with Marco Bertola)

- The Baik-Ben Arous-Peché distribution and Baik's formula.
- More examples: the Airy process with two sets of parameters and the Pearcey process with inliers.
- The general setting leading to Darboux transformations, general Baik's formula.
- Some examples
- Open questions.

Direct percolation and the Tracy–Widom distribution



For any (i,j) assign $X(i,j)$ exponential random variable of mean $1/M$ (density function $= M\exp(-Mx),\ x \geqslant 0$).

$$L(N,M) := \max_{\omega \in (1,1) \nearrow (N,M)} \sum_{(i,j)\in\omega} X(i,j);$$

Connection with Random Matrices (Johansson, 2000)

$$\mathbb{P}(L(N,M) \leqslant s) = \mathbb{P}(\lambda_{\max} \leqslant s)$$

λ_{max} the largest eigenvalue of a random matrix taken from the complex Wishart ensemble

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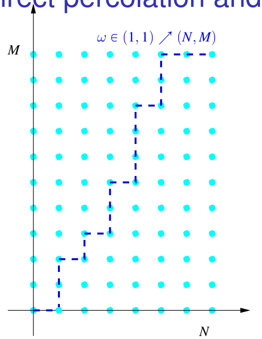

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$$\Downarrow \left(\gamma := \sqrt{M/N} \geqslant 1,\ M \longmapsto \infty\right)$$

$$\mathbb{P}\left[\left(L(N,M) - \left(\frac{1+\gamma}{\gamma}\right)^2\right)\frac{\gamma M^{2/3}}{(1+\gamma)^{4/3}} \leqslant s\right] \longrightarrow F_{TW}(s) := \det\left(\mathbf{1} - K_{\mathrm{Ai}}\chi_{[s,\infty)}\right)$$

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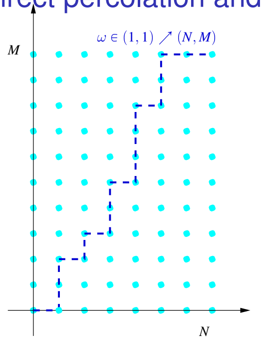

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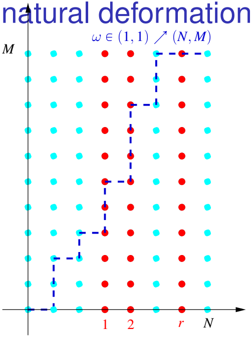
$$F_{TW}(s) = \exp\left(-\int_s^{\infty}(x-s)q(x)^2dx\right)$$

$$q''(s) = 2q^3(s) + sq(s), \qquad q(s) \sim e^{-\frac{2}{3}s^{\frac{3}{2}}} \quad (\exists!)$$

(Tracy–Widom, 1993.)

A natural deformation: what if few columns slow down the percolation?



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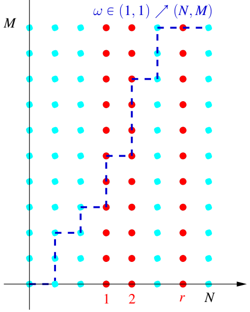

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If $\ell_i < 1 + \gamma^{-1}\ \forall i = k_1, \ldots, k_r$, (Peché, 2003)

$$\mathbb{P}\left[\left(L(N,M) - \left(\frac{1+\gamma}{\gamma}\right)^2\right)\frac{\gamma M^{2/3}}{(1+\gamma)^{4/3}} \leqslant s\right] \longrightarrow F_{TW}(s)$$

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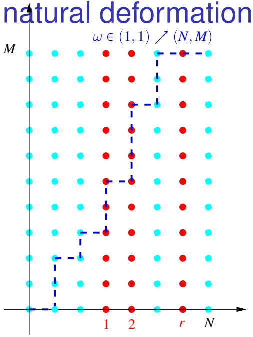

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Theorem (Baik–Ben Arous–Peché, 2005)

$$\ell_j := 1 + \gamma^{-1} - \frac{(1+\gamma)^{2/3} b_j}{\gamma M^{1/3}},\ j = k_1, \ldots, k_r;$$

$$\Downarrow$$

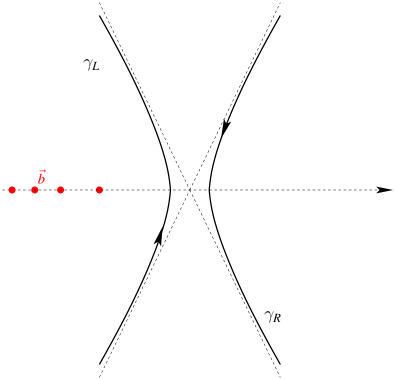
$$\mathbb{P}\left[\left(L(N,M) - \left(\frac{1+\gamma}{\gamma}\right)^2\right)\frac{\gamma M^{2/3}}{(1+\gamma)^{4/3}} \leqslant s\right] \longrightarrow F_{BBP}(s; b_1, \ldots, b_r)$$

Applications to RMT (Wishart ensemble), Dyson Brownian motions, percolations, TASEP, KPZ ...

The Baik–Ben Arous– Peché distribution

$$F_{BBP}(s; b_1, \dots b_r) := \det(\mathrm{I} - K_{BBP}^{(\vec{b})}\chi_{[s,\infty)}) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{[s,\infty)^k} \det\left(K_{BBP}^{(\vec{b})}(x_i, x_j)\right)_{i,j=1}^{k} dx_1 \dots dx_k$$

$$K_{BBP}^{(\vec{b})}(x,y) := \frac{1}{(2\pi i)^2} \int_{\gamma_L} dw \int_{\gamma_R} dz \frac{\mathrm{e}^{\frac{z^3}{3} - \frac{w^3}{3} - zx + wy}}{w - z} \prod_{k=1}^{r} \left(\frac{z - b_k}{w - b_k}\right).$$



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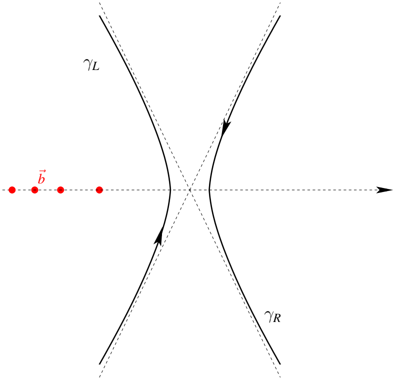

What about Tracy–Widom type result?

The Riemann–Hilbert problem for the Hasting-McLeod solution of PII

Find the sectionally analytic function $\Gamma(\lambda) \in GL(2,\mathbb{C})$ on $\mathbb{C}/\{\gamma_R \cup \gamma_L\}$ s. t.

$$\begin{cases} \Gamma_+(\lambda) = \Gamma_-(\lambda) \begin{bmatrix} 1 & -e^{\frac{\lambda^3}{3} - s\lambda}\chi_{\gamma_R} \\ -e^{-\frac{\lambda^3}{3} + s\lambda}\chi_{\gamma_L} & 1 \end{bmatrix} \\ \Gamma(\lambda) \sim \mathbf{1} + \Gamma_1\lambda^{-1} + \mathcal{O}(\lambda^{-1}), \quad \lambda \to \infty; \quad \Gamma_1 = p(s)\sigma_3 + iq(s)\sigma_2 \end{cases}.$$

$$\Downarrow \Psi(s;\lambda) := \Gamma(s;\lambda)\exp\left(\lambda^3/6 - s/2\right)\sigma_3 \quad \text{solves}$$

$$\begin{cases} \partial_s\Psi(s;\lambda) = U(s;\lambda)\Psi(s;\lambda) \\ \partial_\lambda\Psi(s;\lambda) = V(s;\lambda)\Psi(s;\lambda) \end{cases} \Longrightarrow \partial_s V - \partial_\lambda U = [V, U]$$

$$\Longrightarrow q := (\Gamma_1)_{1,2} \quad \text{satisfies}$$

$$q''(s) = 2q^3(s) + sq(s), \qquad q(s) \sim e^{-\frac{2}{3}s^{\frac{3}{2}}}; \quad s \to +\infty.$$

Hasting–McLeod solution of PII

A Painlevé like formula for $F_{BBP}(s,\vec{b})$

Given $F_{BBP}(s;\vec{b}) := \det(\mathrm{I} - K^{(\vec{b})}_{BBP}\chi_{[s,\infty)})$ with

$$K^{(\vec{b})}_{BBP}(x,y) := \frac{1}{(2\pi i)^2}\int_{\gamma_L} dw \int_{\gamma_R} dz \frac{e^{\frac{z^3}{3}-\frac{w^3}{3}-zx+wy}}{w-z}\prod_{k=1}^{r}\left(\frac{z-b_k}{w-b_k}\right), \quad \text{we have}$$

$$F_{BBP}(s;\vec{b}) = F_{TW}(s)\frac{\det\left(\left(-\partial_s + b_j\right)^{\ell-1}p(b_j)\right)^{r}_{\ell,j=1}}{\Delta(\vec{b})}, \quad (\text{Baik, } 2005)$$

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Baik's approach: Orthogonal polynomials on the unit circle and Toeplitz determinants

Aim : giving an interpretation of this formula in terms of Darboux transformations for a given integrable system.

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Baik's approach: Orthogonal polynomials on the unit circle and Toeplitz determinants

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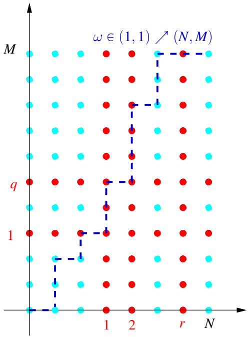
Besides, there are other examples of similar kernels obtained from "classical ones" adding some rational deformations (see below): we will see that a similar formula apply also to these cases.

The Airy kernel with two sets of parameters I



Now take

$$\begin{cases} \{i_1, i_2, \ldots, i_r\} & \subseteq & \{1, \ldots, N\}, \\ \{j_1, j_2, \ldots, j_q\} & \subseteq & \{1, \ldots, M\}, \end{cases}$$

$X_{(i_k, j_t)}$ exponential random variable of mean $(\ell_k + m_t)/M,\ k = 1, \ldots, r; \quad t = 1, \ldots, q.$

$$\begin{cases} \ell_j & := & 1 + \gamma^{-1} - \dfrac{(1+\gamma)^{2/3} b_j}{\gamma M^{1/3}},\ j = k_1, \ldots, k_r; \\ m_t & := & 1 + \gamma^{-1} + \dfrac{(1+\gamma)^{2/3} a_t}{\gamma M^{1/3}},\ t = s_1, \ldots, s_q. \end{cases}$$

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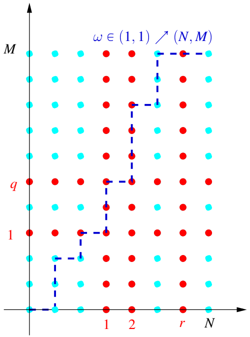

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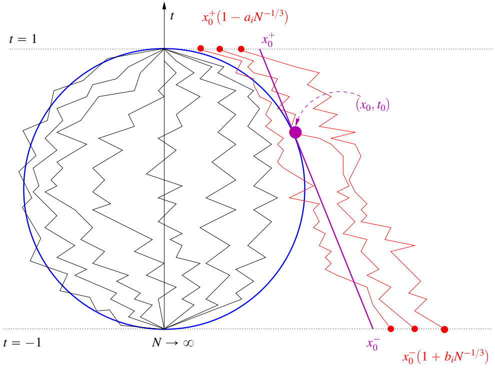
Theorem (Borodin–Peché, 2008) [also multi–time case]

$$\mathbb{P}\left[\left(L(N,M) - \left(\frac{1+\gamma}{\gamma}\right)^2\right)\frac{\gamma M^{2/3}}{(1+\gamma)^{4/3}} \leqslant s\right] \longrightarrow F_{\mathrm{Ai}}(s; \vec{a}; \vec{b}) = \det(\mathrm{I} - K_{\mathrm{Ai}}^{(\vec{a},\vec{b})}\chi_{[s,\infty)})$$

$$K_{\mathrm{Ai}}^{(\vec{a},\vec{b})}(x,y) := \frac{1}{(2\pi i)^2}\int_{\gamma_L} dw \int_{\gamma_R} dz \frac{\mathrm{e}^{\frac{z^3}{3} - \frac{w^3}{3} - zx + wy}}{w - z} \prod_{k=1}^{r}\left(\frac{z - b_k}{w - b_k}\right)\prod_{j=1}^{q}\left(\frac{w - a_j}{z - a_j}\right).$$

The Airy kernel with two sets of parameters II



N non–intersecting Brownian particles $\{\lambda_i(t)\}$ with transition probability

$$p(x, y; t) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$$

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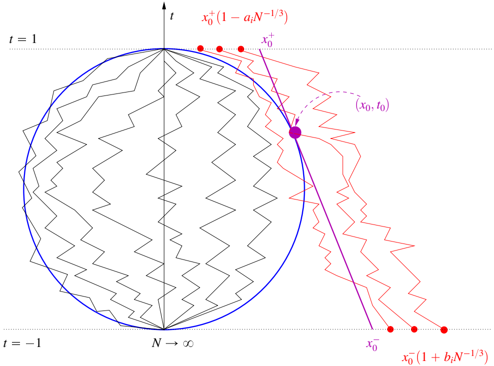

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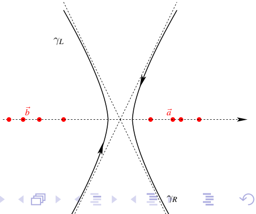
Theorem (Adler–Ferrari–van Moerbeke, 2010) [also multi–time and multi–interval cases]

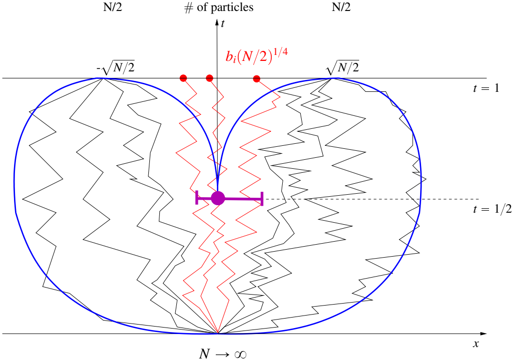
$$\lim_{N\to\infty} \mathbb{P}\left(\lambda_{max}(t_0) < x_0 \left(1 + \frac{s}{2N^{2/3}}\right)\right) = F_{\mathrm{Ai}}(s; \vec{a}; \vec{b})$$

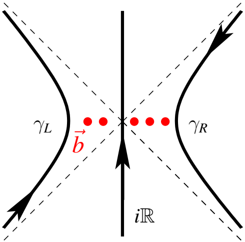
$$K_{\mathrm{Ai}}^{(\vec{a},\vec{b})}(x, y) :=$$

$$\frac{1}{(2\pi i)^2} \int_{\gamma_L} dw \int_{\gamma_R} dz \frac{e^{\frac{z^3}{3} - \frac{w^3}{3} - zx + wy}}{w - z} \prod_{k=1}^{r} \left(\frac{z - b_k}{w - b_k}\right) \prod_{j=1}^{q} \left(\frac{w - a_j}{z - a_j}\right).$$



The Pearcey kernel with inliers





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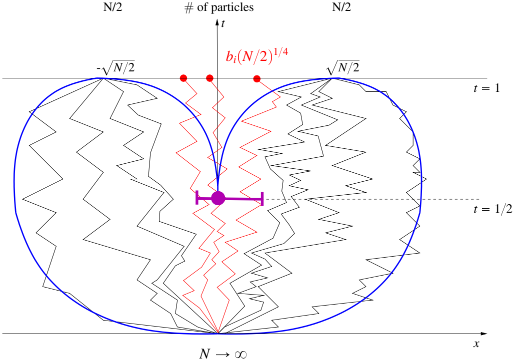

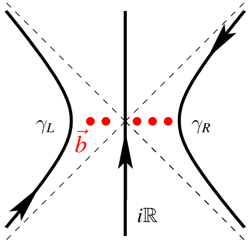

$$\lim_{N\to\infty}\mathbb{P}\left(x_i\left(\frac{1}{2}+\frac{\tau}{4\sqrt{2N}}\right)\notin\frac{E}{4(N/2)^{1/4}};\ i=1,\dots,N\right)=\det\left(\mathrm{I}-K_P^{(\vec{b})}\chi_E\right)$$

$$K_P^{(\vec{b})}(x,y):=\frac{1}{(2\pi i)^2}\int_{i\mathbb{R}}dw\int_{\gamma}dz\,\frac{e^{\theta_\tau(x;z)-\theta_\tau(y;w)}}{w-z}\prod_{k=1}^{r}\left(\frac{w-b_k}{z-b_k}\right);\ \theta_\tau(x;z):=\frac{z^4}{4}-\tau\frac{z^2}{2}-xz.$$

(Adler–Delépine–van Moerbeke–Vanhaecke, 2011)

The general problem

Given a kernel of type

$$K^{(\vec{a},\vec{b})}(x,y) := \frac{1}{(2\pi i)^2}\int_{\gamma_2} dw \int_{\gamma_1} dz \frac{e^{\theta(x;z)-\theta(y;w)}}{w-z} \prod_{k=1}^{r}\left(\frac{z-b_k}{w-b_k}\right)\prod_{k=1}^{q}\left(\frac{w-a_k}{z-a_k}\right).$$

such that

1. $\theta(x;z)$ is a polynomial in z linear in x; more precisely $\theta(x;z)=\theta(0;z)-xz$.
2. γ_1 and γ_2 are oriented contours s. t. the integrand is convergent, $\gamma_1\cap\gamma_2=\varnothing$
3. γ_2 can be deformed to the imaginary axes.
4. ...

We want to find a "Riemann–Hilbert" like expression for $\frac{F^{(\vec{a},\vec{b})}(\vec{s})}{F^{\varnothing}(\vec{s})}$; where

$$F^{(\vec{a},\vec{b})}(\vec{s}) := \det(\mathrm{I}-K^{(\vec{a},\vec{b})}\chi_E) := 1+\sum_{k=1}^{\infty}(-1)^k\int_{I^k}\det\left(K^{(\vec{a},\vec{b})}(x_i,x_j)\right)_{i,j=1}^{k} dx_1\dots dx_k,$$

E is a (multi)–interval with endpoints $\vec{s}:=\{s_i,\ i=1,\dots,N\}$,
$\vec{a}:=\{a_i, i=1,\dots,q\},\ \vec{b}:=\{b_i, i=1,\dots r\}$ are two sets of (real) parameters.

A short reminder about Fredholm determinants

Given an integral operator

$$\mathcal{K} : L^2(\Sigma) \to L^2(\Sigma)$$
$$(\mathcal{K}f)(x) = \int_{\Sigma} K(x,y)f(y)dy$$

its Fredholm determinant is defined as

$$\det(\mathrm{I} - z\mathcal{K}) := 1 + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!} \int_{\Sigma^n} \det\left[K(x_j, x_k)\right]_{j,k\leqslant n} dx_1 \ldots dx_n.$$

The series defines an entire function of z as long as $\mathcal{K}$ is trace-class. For sufficiently small z (less than the spectral radius of $\mathcal{K}$) then the following can be used equivalently

$$\ln\det(\mathrm{I} - z\mathcal{K}) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}\mathcal{K}^n$$

If $\mathcal{K}$ depends on some parameter (say s) we have (Jacobi formula)

$$\partial_s \ln\det(\mathrm{I} - \mathcal{K}) = -\mathrm{Tr}\Big[\,(\mathrm{I} + \mathcal{R})\,\partial_s\mathcal{K}\Big]$$

where $\mathcal{R}$ is the resolvent

$$\mathcal{R} = \mathcal{K} \circ (\mathrm{I} - \mathcal{K})^{-1}$$

Our tools: IIKS (Its-Izergin-Korepin-Slavnov) theory in a nutshell

Let $N : L^2(\Sigma, \mathbb{C})^{\circlearrowleft}$ be an operator with kernel given by ("integrable form")

$$N(\lambda, \mu) := \frac{\mathbf{f}^T(\lambda)\mathbf{g}(\mu)}{\lambda - \mu} \qquad \mathbf{f}^T(\lambda)\mathbf{g}(\lambda) \equiv \mathbf{0}\ , \quad \mathbf{f}, \mathbf{g} : \Sigma \to \mathbb{C}^k$$

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Then the resolvent operator is also of integrable form:

$$\mathcal{R}(\lambda, \mu) = N \circ (\mathbf{I} - N)^{-1}(\lambda, \mu) = \frac{\mathbf{f}^T(\lambda)\Theta^T(\lambda)\Theta^{-T}(\mu)\mathbf{g}(\mu)}{\lambda - \mu}$$

where $\Theta(\lambda)$ is the $k \times k$ matrix bounded solution of the following Riemann–Hilbert problem

$$\begin{cases} \Theta_+(\lambda) & = & \Theta_-(\lambda)\left(\mathbf{1}_k - 2i\pi\mathbf{f}(\lambda)\mathbf{g}^T(\lambda)\right) \\ \Theta(\lambda) & = & \mathbf{1}_k + O(\lambda^{-1})\ , \quad \lambda \to \infty \end{cases}$$

Furthermore the solution of the RHP exists if and only if $\det(\mathrm{I} - N) \neq 0$.

Why is this helpful?

Its Izergin Korepin and Slavnov developed their theory to establish a connection between certain Fredholm determinants representing quantum correlation functions for Bose gas and the Painlevé V equation (1990). Lately their theory has been used extensively in the theory of random matrices and random processes. Indeed these are some of its general features:

- The RHP typically has jumps which are conjugated to constant jumps $\Rightarrow$ the solution of the RHP solves an ODE with meromorphic coefficients $\Rightarrow$ Jimbo–Miwa–Ueno theory of isomonodromic deformations
- the Fredholm determinant coincides (in interesting cases) with the isomonodromic tau function of JMU;

$$\partial_s \log\det(\mathrm{I} - N) = \int_\Sigma \mathrm{Tr}\left(\Theta_-^{-1}(\partial_\lambda \Theta_-)(\partial_s M)M^{-1}\right)\frac{d\lambda}{2i\pi}$$

$$M := \mathbf{1}_k - 2i\pi \mathbf{f}(\lambda)\mathbf{g}^T(\lambda)$$

Main theorem I (M. Bertola, M.C.)

The Fredholm determinant $F^{(\vec{a},\vec{b})}(\vec{s}) = \det(\mathrm{I} - K^{(\vec{a},\vec{b})}\chi_E)$ associated to the kernel

$$K^{(\vec{a},\vec{b})}(x,y) := \frac{1}{(2\pi i)^2}\int_{\gamma_2} dw \int_{\gamma_1} dz \frac{e^{\theta(x;z)-\theta(y;w)}}{w-z}\prod_{k=1}^{r}\left(\frac{z-b_k}{w-b_k}\right)\prod_{k=1}^{q}\left(\frac{w-a_k}{z-a_k}\right).$$

coincides with the isomonodromic tau function of the following RH problem

(General RH$(\vec{a},\vec{b})$)

$$\begin{cases} \Gamma_+^{(\vec{a},\vec{b})}(\lambda) = \Gamma_-^{(\vec{a},\vec{b})}(\lambda)\begin{bmatrix} 1 & -e^{\theta(s_1;\lambda)}C(\lambda)\chi_1 & \dots & (-)^N e^{\theta(s_N;\lambda)}C(\lambda)\chi_1 \\ -\dfrac{e^{-\theta(s_1;\lambda)}}{C(\lambda)}\chi_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\dfrac{e^{-\theta(s_N;\lambda)}}{C(\lambda)}\chi_2 & 0 & \dots & 1 \end{bmatrix} \\ \Gamma^{(\vec{a},\vec{b})}(\lambda) \sim \mathbf{1} + \Gamma_1^{(\vec{a},\vec{b})}\lambda^{-1} + \mathcal{O}(\lambda^{-2}), \quad \lambda\to\infty; \quad C(\lambda) := \dfrac{\prod_{k=1}^{r}(\lambda-b_k)}{\prod_{j=1}^{q}(\lambda-a_j)}. \end{cases}$$

In particular $\partial_{s_i}\log F^{(\vec{a},\vec{b})} = -\left(\Gamma_1^{(\vec{a},\vec{b})}\right)_{(i+1,i+1)}, \quad \forall i = 1,\dots,N.$

How the proof begins

We start observing that, using the Cauchy residue's theorem, we have that
$K^{(\vec{a},\vec{b})}(x,y)\chi_E(x) =$

$$\frac{1}{(2\pi i)^3}\int_{i\mathbb{R}} d\xi \sum_{\ell=1}^{N}(-1)^{\ell+1}e^{\xi(s_\ell - x)}\int_{\gamma_2} dw \int_{\gamma_1} dz \frac{e^{\theta(s_\ell;z)-\theta(0;w)+yw}}{(z-w)(\xi-z)}\prod_{k=1}^{r}\left(\frac{z-b_k}{w-b_k}\right)\prod_{j=1}^{q}\left(\frac{w-a_j}{z-a_j}\right).$$

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Hence, in particular, $K^{(\vec{a},\vec{b})} = \mathrm{T}K^{(\vec{a},\vec{b};\vec{s})}\mathrm{T}^{-1}$, where $\mathrm{T} : L^2(\gamma_2) \longrightarrow L^2(\mathbb{R})$ is the Fourier transform and

$$K^{(\vec{a},\vec{b};\vec{s})}(\xi,w) := \sum_{\ell=1}^{N}(-1)^{\ell+1}\int_{\gamma_1}\frac{dz}{2\pi i}\frac{\mathrm{e}^{\theta(s_i;z)-\theta(0;w)+\xi s}}{(z-w)(\xi-z)}\prod_{k=1}^{r}\left(\frac{z-b_k}{w-b_k}\right)\prod_{j=1}^{q}\left(\frac{w-a_j}{z-a_j}\right).$$

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It is easy to observe that $K^{(\vec{a},\vec{b};\vec{s})}$ is the composition of two integrable operators, namely

$$K^{(\vec{a},\vec{b};\vec{s})}(\xi,w) = \mathcal{G}_{\vec{s}}(\xi,z) * \mathcal{F}(z,w)$$

with

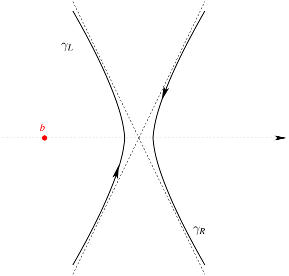
$$\mathcal{G}_{\vec{s}}(\xi,z) := \frac{1}{2\pi i}\sum_{\ell=1}^{N}(-1)^{\ell+1}\frac{\mathrm{e}^{\frac{1}{2}\theta(0;z)-s_\ell(z-\xi)}}{\xi-z}; \qquad \mathcal{G}_{\vec{s}} : L^2(\gamma_1) \longrightarrow L^2(\gamma_2),$$

$$\mathcal{F}(z,w) := \frac{1}{2\pi i}\frac{\mathrm{e}^{\frac{1}{2}\theta(0;z)-\theta(0,w)}}{z-w}\prod_{k=1}^{r}\left(\frac{z-b_k}{w-b_k}\right)\prod_{j=1}^{q}\left(\frac{w-a_j}{z-a_j}\right); \quad \mathcal{F} : L^2(\gamma_2) \longrightarrow L^2(\gamma_1).$$

Example: BBP distribution with one parameter

$$F(s;b) := \det(\mathrm{I} - K_{BBP}^{(b)}\chi_{[s,\infty)}); \quad K_{BBP}^{(b)}(x,y) := \frac{1}{(2\pi i)^2}\int_{\gamma_L} dw \int_{\gamma_R} dz \frac{e^{\frac{z^3}{3} - zx - \frac{w^3}{3} + wy}}{w-z}\frac{z-b}{w-b}.$$

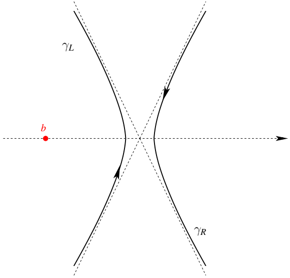
$$\begin{cases} \Gamma_+^{(b)}(\lambda) = \Gamma_-^{(b)}(\lambda)\begin{bmatrix} 1 & -e^{\frac{\lambda^3}{3} - s\lambda}(\lambda - b)\chi_R \\ -\dfrac{e^{-\frac{\lambda^3}{3}+s\lambda}}{\lambda - b}\chi_L & 1 \end{bmatrix} \\ \Gamma^{(b)}(\lambda) \sim \mathbf{1} + \Gamma_1^{(b)}\lambda^{-1} + \mathcal{O}(\lambda^{-2}), \quad \lambda \to \infty. \end{cases}$$



Example: BBP distribution with one parameter

$$F(s;b) := \det(\mathrm{I} - K^{(b)}_{BBP}\chi_{[s,\infty)}); \quad K^{(b)}_{BBP}(x,y) := \frac{1}{(2\pi i)^2}\int_{\gamma_L} dw \int_{\gamma_R} dz \frac{e^{\frac{z^3}{3} - zx - \frac{w^3}{3} + wy}}{w-z}\frac{z-b}{w-b}.$$

$$\begin{cases} \Gamma^{(b)}_+(\lambda) = \Gamma^{(b)}_-(\lambda)\begin{bmatrix} 1 & -e^{\frac{\lambda^3}{3} - s\lambda}(\lambda - b)\chi_R \\ -\frac{e^{-\frac{\lambda^3}{3} + s\lambda}}{\lambda - b}\chi_L & 1 \end{bmatrix} \\ \Gamma^{(b)}(\lambda) \sim \mathbf{1} + \Gamma^{(b)}_1\lambda^{-1} + \mathcal{O}(\lambda^{-2}), \quad \lambda \to \infty. \end{cases}$$



It is easy to verify that, if Γ solves the RH problem for the HmcL solution of PII, then

$$\Gamma^{(b)}(\lambda) := \begin{pmatrix} \frac{1}{\lambda - b} & -\frac{\Gamma_{12}(b)}{(\lambda - b)\Gamma_{22}(b)} \\ -\frac{(\Gamma_1)_{21}}{\lambda - b} & 1 + \frac{(\Gamma_1)_{21}\Gamma_{12}(b)}{(\lambda - b)\Gamma_{22}(b)} \end{pmatrix}\Gamma(\lambda)\mathrm{diag}\Big((\lambda - b), 1\Big)$$

solves the Riemann–Hilbert problem above

More generally:

Let Γ the solution of General RH $(\varnothing, \varnothing)$. Then, $\Gamma^{(\vec{b})}$ solution of General RH $(\varnothing, \vec{b})$ is obtained solving the following :

Find a matrix $R(\lambda)$ rational in λ such that

$$\Gamma^{(\vec{b})}(\lambda) := R(\lambda)\Gamma(\lambda)\mathrm{diag}\Big(\prod_k(\lambda - b_k), 1, \ldots, 1\Big)$$

is bounded everywhere and $\Gamma^{(\vec{b})}(\lambda) = \mathrm{I} + \mathcal{O}(\lambda^{-1}), \quad \lambda \to \infty.$

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is bounded everywhere and $\Gamma^{(\vec{b})}(\lambda) = \mathrm{I} + \mathcal{O}(\lambda^{-1}), \quad \lambda \to \infty.$

Proposition (Jimbo–Miwa–Ueno; M. Bertola–M.C.):
The solution to the problem above is given by ($\{b_i\}$ distincts)

$$R(\lambda) = \mathbf{1} - E_{1,1} + \sum_{j=1}^{r} \frac{v_j \mathrm{e}_1^{\mathrm{T}} \Gamma^{-1}(b_j)}{z - b_j}$$

where $\{v_1, \ldots, v_r\}$ are given by the unique solution to the non–homogeneous linear problem

$$\sum_{j=1}^{r} G_{i,j} v_j = \sum_{k \geqslant 2} (\Gamma_i)_{k,1} \mathrm{e}_k - \delta_{i,r}\mathrm{e}_1, \quad i = 1 \ldots, r.$$

$$G_{i,j} := \mathrm{res}_{\lambda=\infty} \frac{\left[\lambda^{i-1}\Gamma^{-1}(b_j)\Gamma(\lambda)\right]_{1,1}}{\lambda - b_j}; \quad \mathbb{G} := \left(G_{i,j}\right)_{i,j=1}^{r}.$$

For the general case ($\vec{a} \neq \emptyset$) the matrix $\mathbb{G}$ is given by

$$\mathbb{G} := \left(G_{i,j}\right)_{i,j=1}^{r}; \qquad G_{i,j} := \frac{\left[\Gamma^{-1}(b_j)\Gamma(a_i)\right]_{11}}{b_j - a_i}$$

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Main Theorem II (M. Bertola–M.C): Given

$$F^{(\vec{a},\vec{b})}(\vec{s}) := \det(\mathrm{I} - K^{(\vec{a},\vec{b})}\chi_E) := 1 + \sum_{k=1}^{\infty}(-1)^k \int_{E^k} \det\left(K^{(\vec{a},\vec{b})}(x_i, x_j)\right)_{i,j=1}^{k} dx_1 \dots dx_k,$$

with

$$K^{(\vec{a},\vec{b})}(x,y) := \frac{1}{(2\pi i)^2} \int_{\gamma_2} dw \int_{\gamma_1} dz \frac{e^{\theta(x;z)-\theta(y;w)}}{w - z} \prod_{k=1}^{r}\left(\frac{z - b_k}{w - b_k}\right) \prod_{k=1}^{q}\left(\frac{w - a_k}{z - a_k}\right).$$

we have

$$\frac{F^{(\vec{a},\vec{b})}(\vec{s})}{F^{\emptyset}(\vec{s})} = C \det(\mathbb{G}),$$

where Γ is the solution of General RH$(\vec{a},\vec{b})$ for $\vec{a} = \vec{b} = \emptyset$.

$$C = \frac{\prod(b_i - a_j)}{\Delta(\vec{a})\Delta(\vec{b})}$$

Using isomonodromic ODEs associated to the relevant RH problems, we obtain as examples the following results...

Example I: Airy kernel with two sets of parameters

Let $F_{\mathrm{Ai}}^{(\vec{a},\vec{b})}(s) := \det(\mathrm{I} - K_{\mathrm{Ai}}^{(\vec{a},\vec{b})}\chi_{[s,\infty)})$ with

$$K_{\mathrm{Ai}}^{(\vec{a},\vec{b})}(x,y) := \frac{1}{(2\pi i)^2}\int_{\gamma_L} dw \int_{\gamma_R} dz \frac{\mathrm{e}^{\frac{z^3}{3}-\frac{w^3}{3}-zx+wy}}{w-z}\prod_{k=1}^{r}\left(\frac{z-b_k}{w-b_k}\right)\prod_{k=1}^{q}\left(\frac{w-a_k}{z-a_k}\right).$$

Then, for arbitrary sets of parameters $\vec{a}$ and $\vec{b}$,

$$F^{(\vec{a},\vec{b})}(s) = F_{TW}(s)\frac{\det\left[\begin{array}{c|c} \left(\left(-\partial_s + a_j\right)^{\ell-1}\Gamma_{2,2}(a_j)\right)_{\ell,j\leqslant q} & \left(\partial_s^{\ell-1}\Gamma_{1,2}(a_j)\right)_{\ell\leqslant r; j\leqslant q} \\ \hline \left(\partial_s^{\ell-1}\Gamma_{2,1}(b_j)\right)_{\ell\leqslant q; j\leqslant r} & \left(\left(-\partial_s + b_j\right)^{\ell-1}\Gamma_{1,1}(b_j)\right)_{\ell,j\leqslant r} \end{array}\right]}{\Delta(\vec{a})\Delta(\vec{b})}.$$

where Γ is the solution of the Riemann–Hilbert problem associated to the Hasting–McLeod solution of Painlevé II.

Example II: Pearcey kernel on a single interval

Let $F_{\mathrm{P}}^{(\vec{b})}(s_1,s_2) := \det\left(\mathrm{I} - K_P^{(\vec{b})}\chi_{[s_1,s_2]}\right)$ with

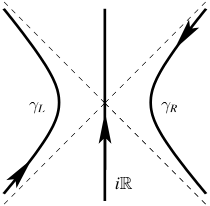
$$K_P^{(\vec{b})}(x,y) := \frac{1}{(2\pi i)^2}\int_{i\mathbb{R}} dw \int_{\gamma} dz \frac{e^{\theta_\tau(x;z)-\theta_\tau(y;w)}}{w-z}\prod_{k=1}^{r}\left(\frac{w-b_k}{z-b_k}\right);\ \theta_\tau(x;z) := \frac{z^4}{4} - \tau\frac{z^2}{2} - xz.$$

Then, for an arbitrary set of parameters $\vec{b}$,

$$F_{\mathrm{P}}^{(\vec{b})}(s_1,s_2) = F_{\mathrm{P}}(s_1,s_2)\frac{\det\left(\left(\partial_s + b_j\right)^{\ell-1} f(b_j)\right)_{\ell,j=1}^{r}}{\Delta(\vec{b})},$$

where $F_{\mathrm{P}}(s_1,s_2)$ is the gap probability for the Pearcey process on the interval $[s_1,s_2]$ and $f(\lambda) := \left(\left(\Gamma^{\varnothing}\right)^{-1}\right)_{1,1}(\lambda)$, with Γ solution of the RH problem below (M. Bertola, M.C.; '11).

$$\begin{cases} \Gamma_+(\lambda) = \Gamma_-(\lambda)\begin{bmatrix} 1 & -e^{\theta_\tau(s_1;\lambda)}\chi_\gamma & e^{\theta_\tau(s_2;\lambda)}\chi_\gamma \\ -e^{-\theta_\tau(s_1;\lambda)}\chi_{i\mathbb{R}} & 1 & 0 \\ -e^{-\theta_\tau(s_2;\lambda)}\chi_{i\mathbb{R}} & 0 & 1 \end{bmatrix} \\ \Gamma(\lambda) \sim \mathrm{I} + \mathcal{O}(\lambda^{-1}), \quad \lambda \to \infty. \end{cases}$$



Extension to the multi–time case (an example)

The Fredholm determinant $F^{(\vec{a},\vec{b})}(s_1,s_2) = \det(\mathrm{I} - K^{(\vec{a},\vec{b})}_{\tau_i,\tau_j}\chi_{(s_1,\infty];(s_2,\infty]})$ associated to the kernel $K^{(\vec{a},\vec{b})}_{\tau_i,\tau_j}(x,y) :=$

$$\frac{1}{(2\pi i)^2}\int_{\gamma_2} dw \int_{\gamma_1} dz \frac{e^{\theta_{\tau_i}(x;z)-\theta_{\tau_j}(y;w)}}{w-z}\prod_{k=1}^{r}\left(\frac{z-b_k}{w-b_k}\right)\prod_{k=1}^{q}\left(\frac{w-a_k}{z-a_k}\right) - \delta_{i<j}\int_{\gamma_2}\frac{d\lambda}{2\pi i}e^{\theta_{\tau_i}(x;\lambda)-\theta_{\tau_j}(y;\lambda)}.$$

coincides with the isomonodromic tau function of the following RH problem:

$$\begin{cases} \Gamma_+^{(\vec{a},\vec{b})}(\lambda) = \Gamma_-^{(\vec{a},\vec{b})}(\lambda)\begin{bmatrix} 1 & -e^{\theta_{\tau_1}(s_1;\lambda)}C(\lambda)\chi_1 & -e^{\theta_{\tau_2}(s_2;\lambda)}C(\lambda)\chi_1 \\ -\dfrac{e^{-\theta_{\tau_1}(s_1;\lambda)}}{C(\lambda)}\chi_2 & 1 & 0 \\ -\dfrac{e^{-\theta_{\tau_2}(s_2;\lambda)}}{C(\lambda)}\chi_2 & e^{-\theta_{\tau_1}(s_1)+\theta_{\tau_2}(s_2)}\chi_2 & 1 \end{bmatrix} \\ \Gamma^{(\vec{a},\vec{b})}(\lambda) \sim \mathbf{1} + \mathcal{O}(\lambda^{-1}), \quad \lambda\to\infty. \end{cases}$$

Applications to the Airy process, and more generally to multi–time processes.

KP in Miwa variables:

KP–hierarchy:

$$\oint \frac{dz}{2\pi i} e^{(t_i - t_i')z^i} \tau\left(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, \dots\right) \tau\left(t_1' + \frac{1}{z}, t_2' + \frac{1}{2z^2}, \dots\right) = 0$$

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Introduce Miwa variables,

$$t_i := \frac{1}{i} \sum_k \left(a_k^{-i} - b_k^{-i}\right);$$

Using

$$\mathrm{e}^{\sum_n t_n z^n} = \prod_i \left(1 - \frac{z}{z_i}\right)^{-1}, \quad t_n := \frac{1}{n} \sum_i z_i^{-n}$$

We rewrite the bilinear equation for $\tau(\vec{a}, \vec{b})$ as

$$\oint \frac{dz}{2\pi i} \prod \frac{(1 - z/b_k)(1 - z/a_k')}{(1 - z/b_k')(1 - z/a_k)} \tau(\vec{a}; \vec{b}, z)\, \tau(\vec{a}', z; \vec{b}') = 0.$$

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Now suppose to have a formal series of two variables $\tau(a; b)$ with $\tau(x; x) \equiv C$.

$$C(a, b) := \frac{\tau(a; b)}{a - b}; \quad \tau(\vec{a}; \vec{b}) := \frac{\prod(a_i - b_j)}{\Delta(\vec{a})\Delta(\vec{b})} \det\left[C(a_i, b_j)\right]_{i,j=1}^{N}.$$

Theorem:

$\tau(\vec{a}; \vec{b})$ solves the KP equations in Miwa variables, and so $\frac{F(\vec{a}, \vec{b})}{F\varnothing}$ above as a formal series in $\vec{a}, \vec{b}$.

PDEs, solitonic and isomonodromic equations

$$F_{BBP}(\vec{s};b) := \det(\mathrm{I} - K_{\mathrm{Ai}}^{(b)}\chi_E); \qquad F_P(\vec{s};b) := \det(\mathrm{I} - K_P^{(b)}\chi_E)$$

solves some PDEs ($\partial := \sum \partial_{s_i}; \quad \varepsilon := \sum s_i \partial_{s_i}$):

$$\begin{aligned}
\partial^4 F_{BBP} + 6(\partial^2 F_{BBP})^2 + (2 - 4(\varepsilon - b\partial_b))\partial F_{BBP} + 3\partial_b^2 F_{BBP} &= 0;\\
(\partial_\tau \partial^2 - 2\tau\partial_\tau - 3(\varepsilon - b\partial_b) + 1)\partial F_P + 6(\partial^2 F_P)(\partial_\tau \partial F_P) - 2\partial_\tau \partial_b F_P &= 0;\\
(\varepsilon - b\partial_b - 2\tau\partial_\tau - 2)\partial^2 F_P + \partial_\tau^3 F_P + 2\{\partial_\tau \partial F_P, \partial^2 F_P\}_\partial + 2\partial_\tau \partial_b \partial F_P &= 0.
\end{aligned}$$

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$$F_{BBP}(\vec{s};b) := \det(\mathrm{I} - K_{\mathrm{Ai}}^{(b)}\chi_E); \qquad F_P(\vec{s};b) := \det(\mathrm{I} - K_P^{(b)}\chi_E)$$

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(\varepsilon - b\partial_b - 2\tau\partial_\tau - 2)\partial^2 F_P + \partial_\tau^3 F_P + 2\{\partial_\tau\partial F_P, \partial^2 F_P\}_\partial + 2\partial_\tau\partial_b\partial F_P &= 0.
\end{aligned}$$

They have been obtained combining vector–constrained KP equations with some specific Virasoro constraints

$$W \in \mathrm{Gr} \quad \text{such} \quad \text{that}$$

$$\begin{cases} z^p W' \subseteq W, \\ \left(z + \dfrac{1}{pz^p}\left(z\dfrac{\partial}{\partial z} - \dfrac{p-3}{2}\right)\right) W \subseteq W \end{cases} ; \quad W' := (z-b)W;$$

$$p = 2 \quad \text{for } F_{BBP}, \quad p = 3 \quad \text{for } F_P.$$

(M. Adler–M.C.–P. van Moerbeke (2012); M. Adler–J. Delépine–P. van Moerbeke–P. Vanhaecke (2011))

Two open questions

1. Is it possible to deduce the same PDEs starting from the relevant Riemann–Hilbert problems ?

2. The Tracy–Widom distribution satisfies (Adler–Shiota–van Moerbeke)

$$\partial^4 F_{TW} + 6(\partial^2 F_{TW})^2 + (2-4\varepsilon)\partial F_{TW} = 0;$$

reducing to the Painlevé II equation, self–similar reduction of mKdV.
Does the equation for the Baik–Ben Arous–Peché distribution

$$\partial^4 F_{BBP} + 6(\partial^2 F_{BBP})^2 + (2-4(\varepsilon - b\partial_b))\partial F_{BBP} + 3\partial_b^2 F_{BBP} = 0$$

reduce to the self–similar reduction of some solitonic equation?

Thank you!