Coupled GUE-matrices & Aztec diamonds

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1. Eigenvalues of principal minors of GUE-matrices

2. Domino tilings of Aztec Diamonds & Random Surfaces

3. Coupled GUE-matrices

4. Domino tiling of a double Aztec diamond

Joint work with Mark Adler

Adler, Johansson & PvM: *Double Aztec Diamonds and the Tacnode Process, 2011.* (arXiv:1112.5532)

Mark Adler, Sunil Chhita, Kurt Johansson & PvM: *Tacnode GUE-minor Processes and Double Aztec Diamonds*, 2013. (arXiv:1303.5279)

Mark Adler & PvM: *The Double GUE-minor process*, 2013.

1. Eigenvalues of principal minors of GUE-matrices

$$A = \begin{pmatrix} A^{(1)} & * & * & * & * & \cdots \\ * & A^{(2)} & * & * & * & \\ * & * & A^{(3)} & * & * & \\ * & * & * & * & * & \\ * & * & * & * & * & \\ \vdots & & & & & \ddots \end{pmatrix} \in \mathrm{GUE}(n)$$

- Eigenvalues $\mathbf{x} = (x_1, \ldots, x_n)$ of A:

$$\rho_n^{\mathsf{GUE}}(\mathbf{x}) := c_n \Delta_n^2(\mathbf{x}) \prod_1^n e^{-x_i^2}$$

$$\Delta_n(\mathbf{x}) = \prod_{1 \le i < j \le n} (x_i - x_j) = \text{Vandermonde}$$

$$A = \begin{pmatrix} A^{(1)} & * & * & * & * & \dots \\ * & A^{(2)} & * & * & * & \\ * & * & A^{(3)} & * & * & \\ * & * & * & * & * & \\ * & * & * & * & * & \\ \vdots & & & & & \ddots \end{pmatrix} \in \mathrm{GUE}(n)$$

- Eigenvalues $\mathbf{x} = (x_1, \dots, x_n)$ of A:

$$\rho_n^{\mathsf{GUE}}(\mathbf{x}) := c_n \Delta_n^2(\mathbf{x}) \prod_1^n e^{-x_i^2}$$

- Eigenvalues of principal minors (*INTERLACING*):

$\mathbf{x} = \mathbf{x}^{(n)} \succ \mathbf{x}^{(n-1)} \succ \dots \succ \mathbf{x}^{(1)}$ (Baryshnikov 2001)

$$\mathbb{P}\left(\bigcap_{k=1}^{n-1} \{\mathbf{x}^{(k)} \in d\mathbf{x}^{(k)}\} \;\middle|\; \mathbf{x}^{(n)} = \mathbf{x} \right) = \frac{d\mu_{\mathbf{x}}}{\mathsf{Vol}(\mathcal{C}_{\mathbf{x}})}.$$

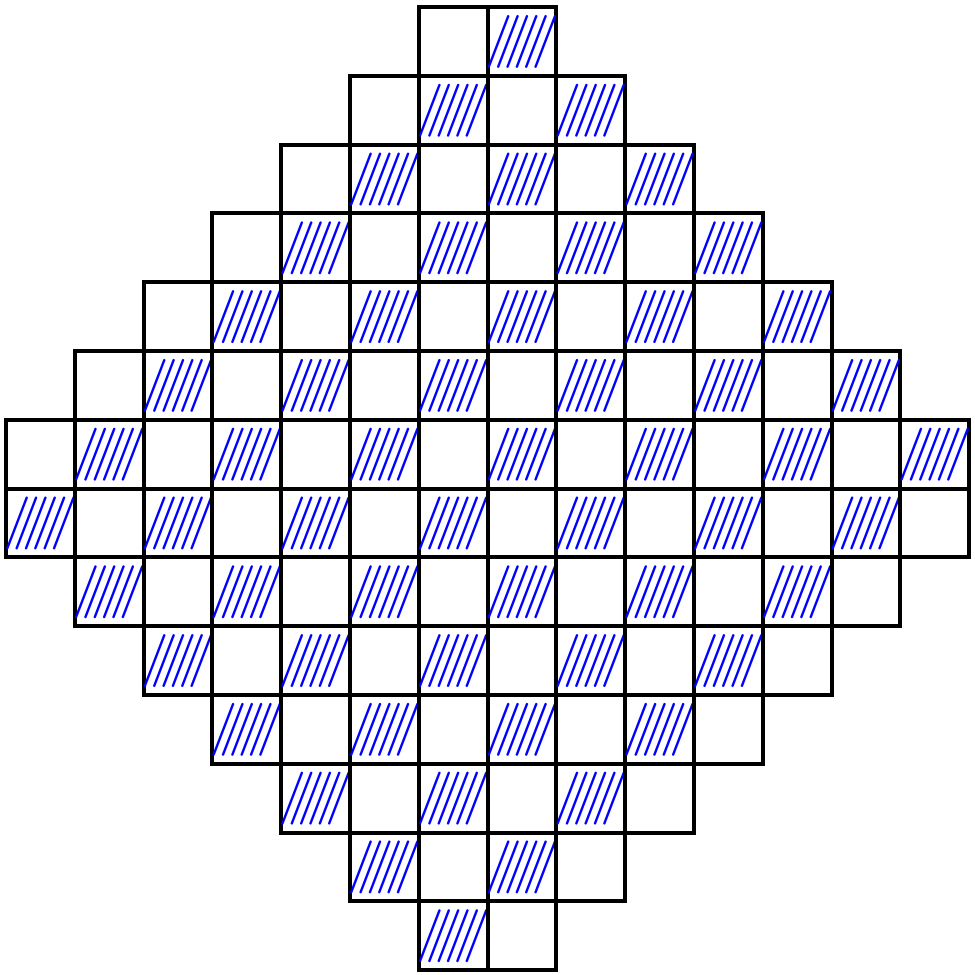
where

- $d\mu_{\mathbf{x}} := \prod_1^{n-1} d\mathbf{x}^{(k)} \mathbb{1}_{\mathbf{x}^{(k+1)} \succ \mathbf{x}^{(k)}} =$ uniform measure on $\mathcal{C}_{\mathbf{x}}$
- Cone $\mathcal{C}_{\mathbf{x}} = \{(\mathbf{x}^{(n-1)}, \dots, \mathbf{x}^{(1)}) \in \mathbb{R}^{n(n-1)/2}$ with $\mathbf{x} \succ \mathbf{x}^{(n-1)} \succ \dots \succ \mathbf{x}^{(1)}\}$
- $\mathsf{Vol}(\mathcal{C}_{\mathbf{x}}) := \int_{\mathbb{R}^{n(n-1)/2}} d\mu_{\mathbf{x}} = \dfrac{\Delta_n(\mathbf{x})}{\prod_1^{n-1} k!}$, .

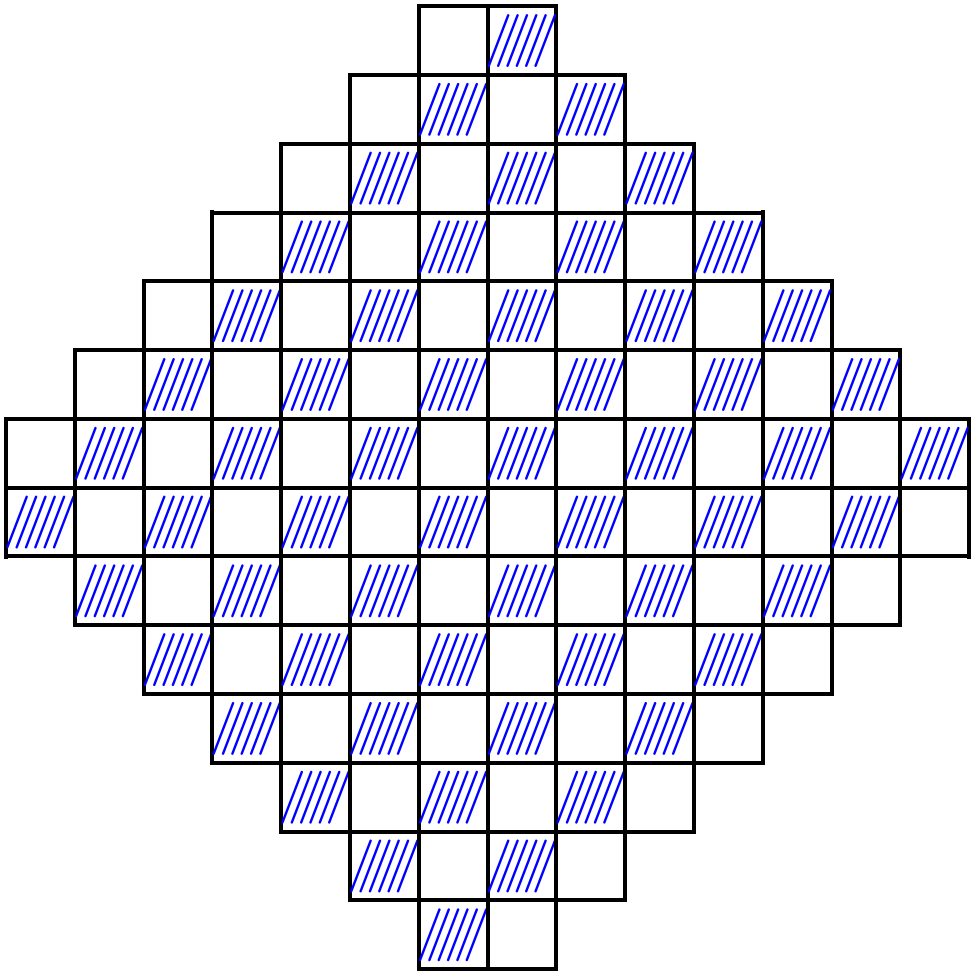
2. Domino tilings of Aztec Diamonds

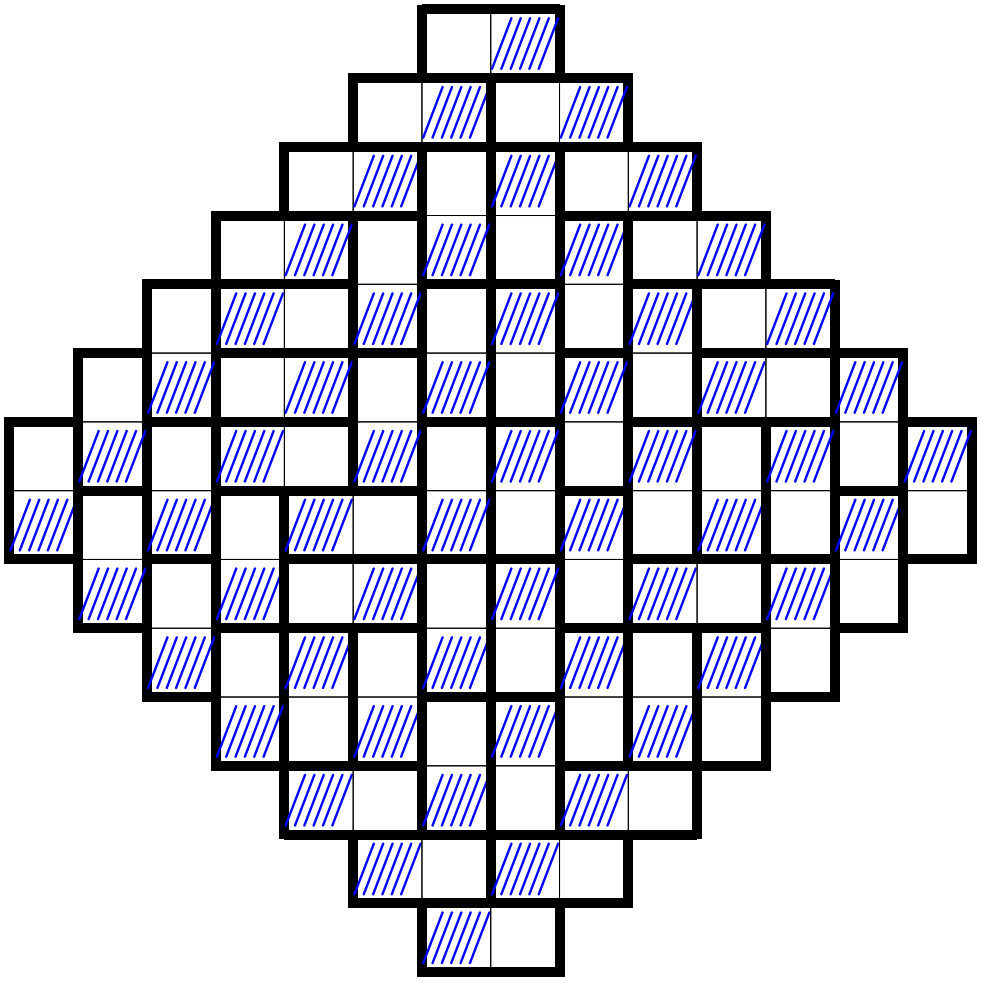
& Random Surfaces

An Aztec diamond (size n):



Domino tilings of an Aztec diamond :



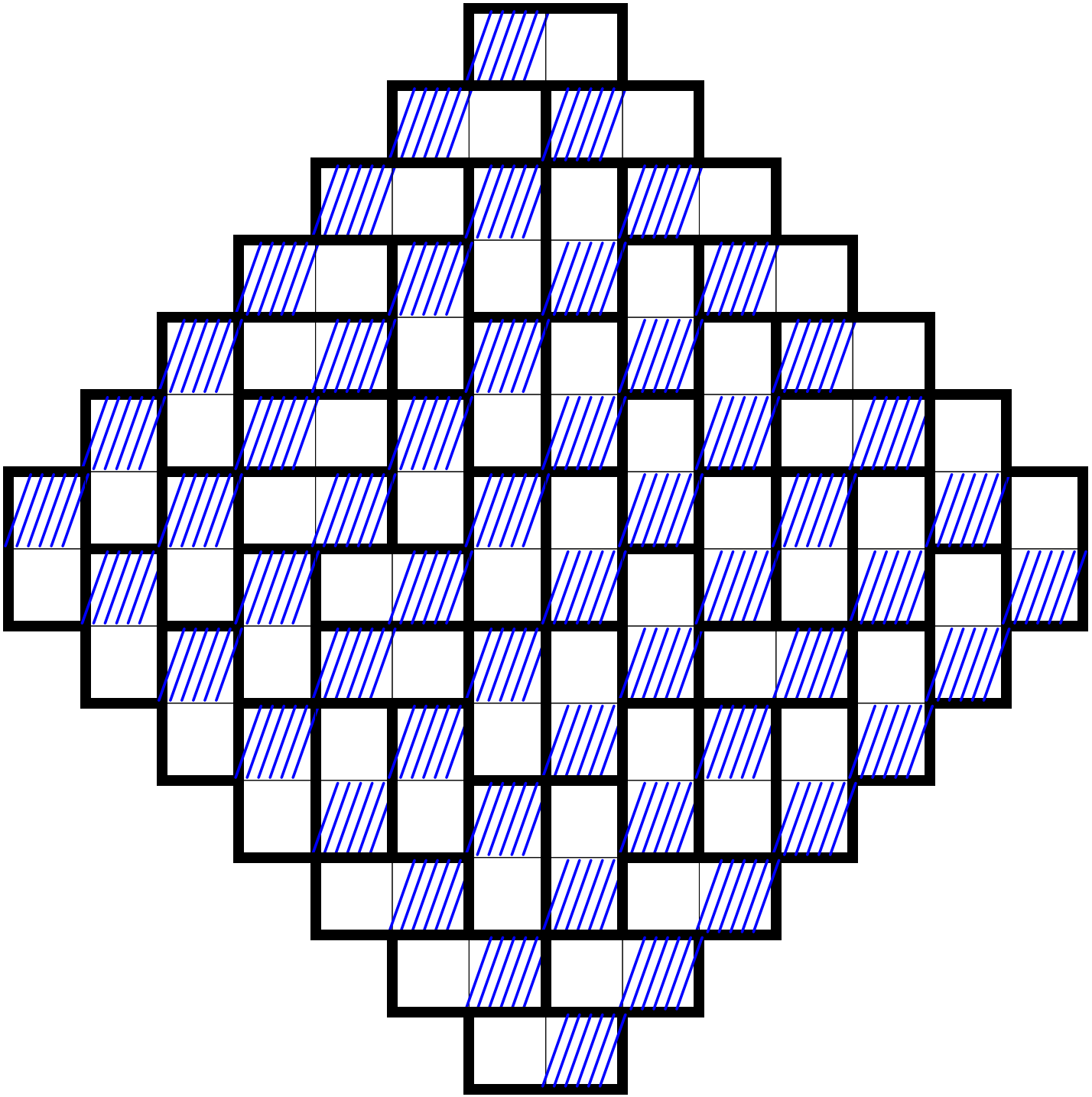


4 types of situations for the domino's!

Kasteleyn 1961, Elkies, Kuperberg, Larsen, Propp '92

Cohn, Elkies, Propp '96, Johansson '00, '03, '05

M. Fulmek, C. Krattenthaler '00, Johansson-Nordenstam '06



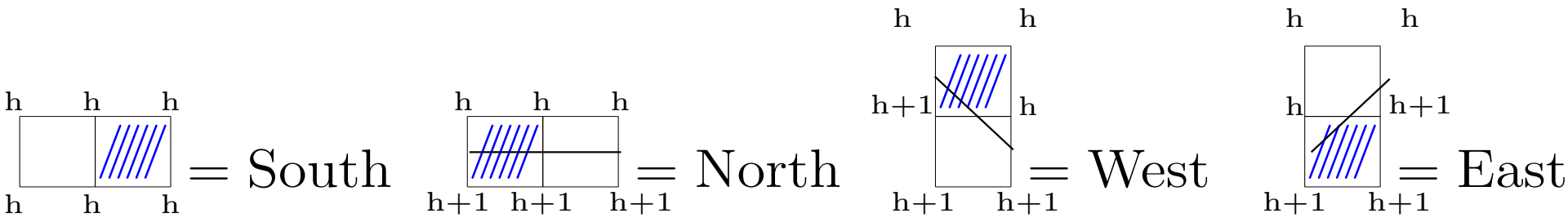
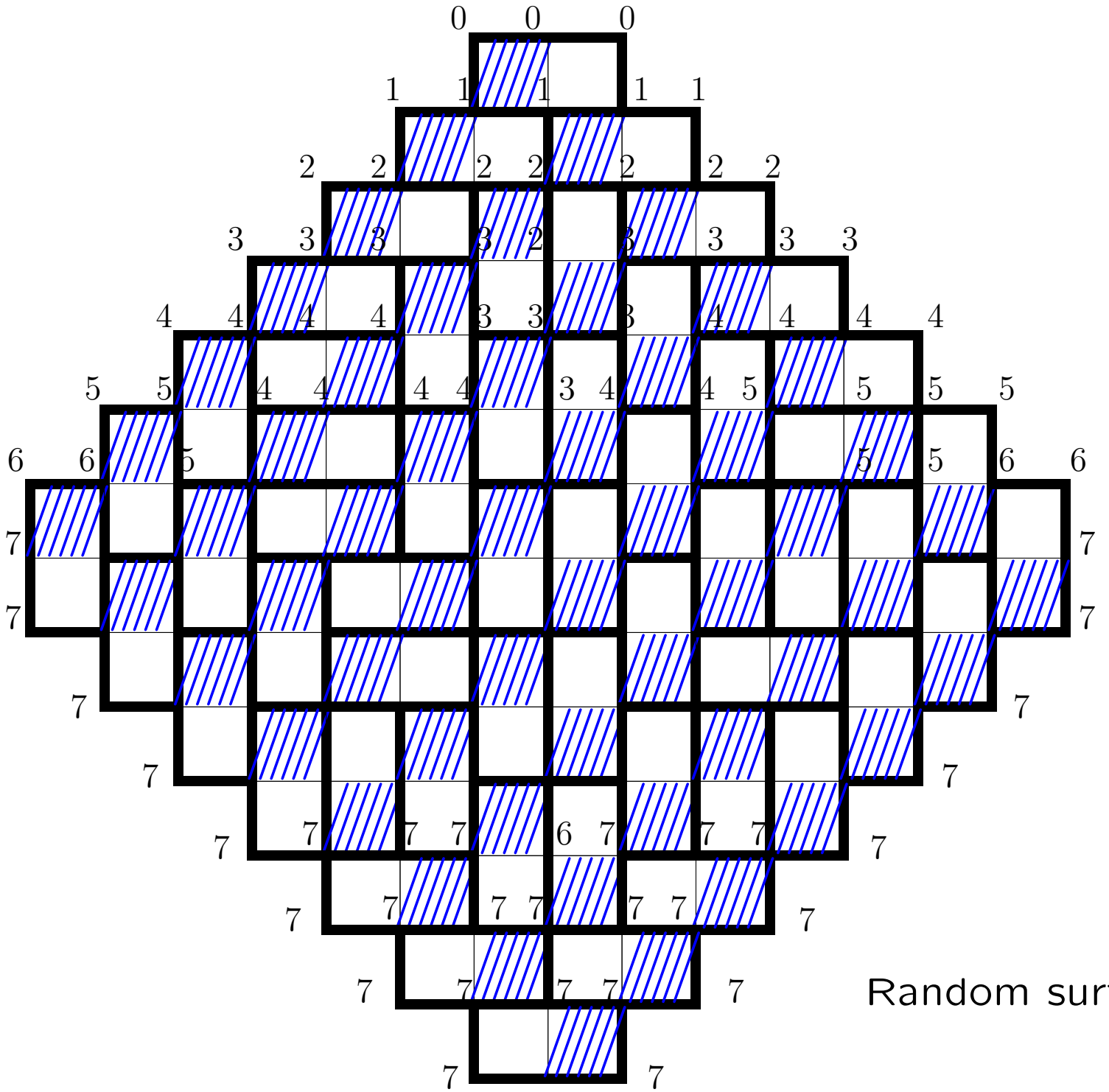
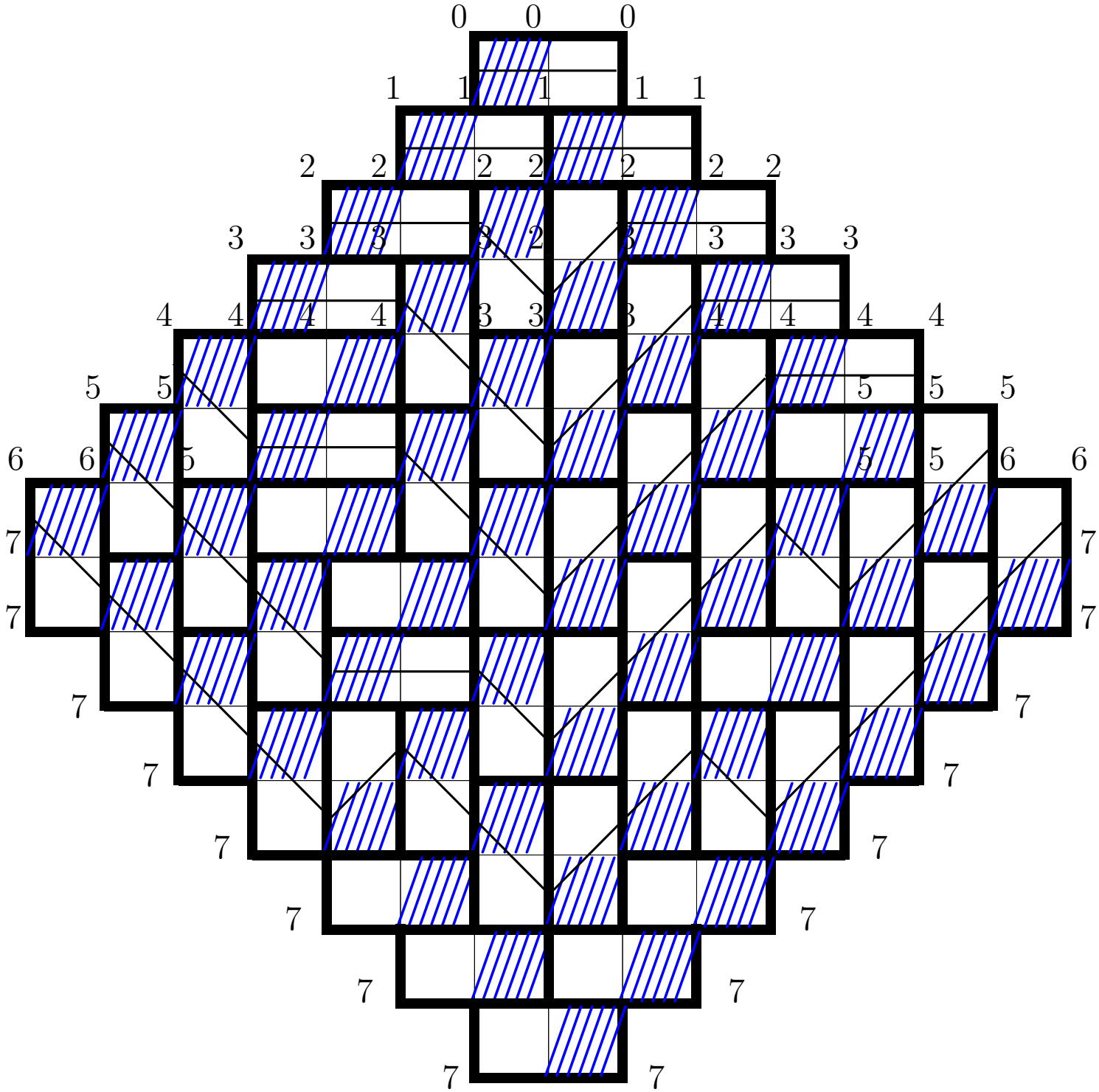


Figure 4. Height function on domino's and level-lines.

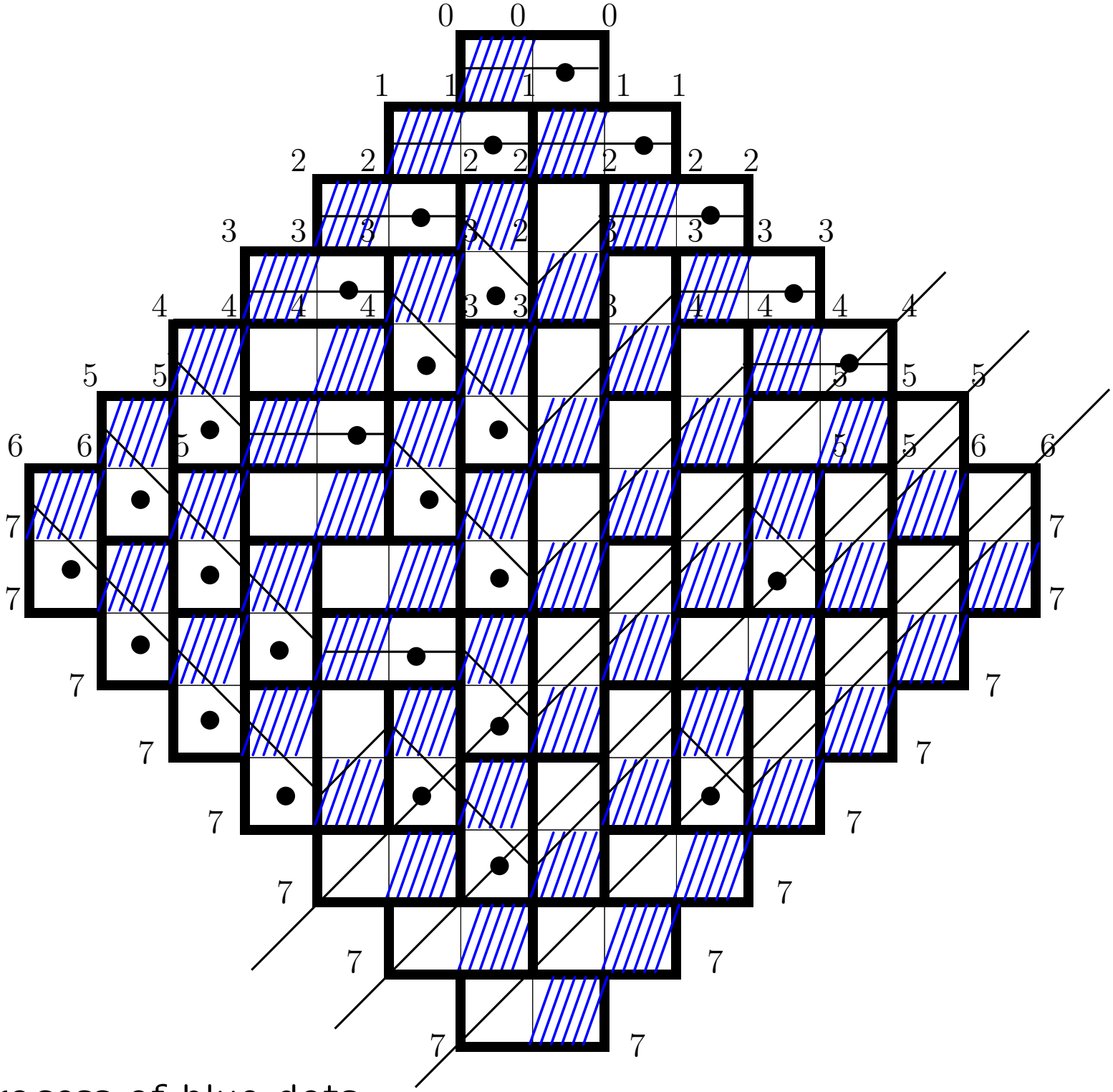
Assume vertical and horizontal domino are equally probable!



Random surface



Level lines of height $\{\frac{1}{2}, \frac{3}{2}, \ldots, n - \frac{1}{2}\}$



Point process of blue dots, .

$$\left\{ \begin{array}{l} \textit{The statistics of the eigenvalues of} \\ \textit{the principal minors of GUE-matrix} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \textit{The statistics of the} \\ \textit{dot-particles in the} \\ \textit{Aztec diamond, in the} \\ \textit{scale } n^{-1/2}\textit{, for } n \to \infty \end{array} \right\}$$

Johansson-Nordenstam '06

3. Coupled GUE-matrices

$$-\beta\mathbb{1}_n + \mathrm{GUE}(n) \ni \begin{pmatrix} \ddots & & & & & \vdots \\ & * & * & * & * & * \\ & * & * & * & * & * \\ & * & * & B^{(3)} & * & * \\ & * & * & * & B^{(2)} & * \\ \dots & * & * & * & * & B^{(1)} \end{pmatrix} \iff \iff \iff \begin{pmatrix} A^{(1)} & * & * & * & * & \dots \\ * & A^{(2)} & * & * & * & \\ * & * & A^{(3)} & * & * & \\ * & * & * & * & * & \\ * & * & * & * & * & \\ \vdots & & & & & \ddots \end{pmatrix} \in \mathrm{GUE}(n) + \beta\mathbb{1}_n$$

spectra $\dots \succ y^{(4)} \succ y^{(3)} \succ \dots \succ y^{(1)}$

spectra $\mathbf{x}^{(1)} \prec \dots \prec \mathbf{x}^{(3)} \prec \mathbf{x}^{(4)} \prec \dots\dots$

COUPLING: $\rho = 3$ *inequalities between spectra of three principal minors*

$$-\beta\mathbb{1}_n + \mathrm{GUE}(n) \ni \begin{pmatrix} \ddots & & & & & \vdots \\ & * & * & * & * & * \\ & * & * & * & * & * \\ & * & * & B^{(3)} & * & * \\ & * & * & * & B^{(2)} & * \\ \dots & * & * & * & * & B^{(1)} \end{pmatrix} \iff \begin{pmatrix} A^{(1)} & * & * & * & * & \dots \\ * & A^{(2)} & * & * & * & \\ * & * & A^{(3)} & * & * & \\ * & * & * & * & * & \\ * & * & * & * & * & \\ \vdots & & & & & \ddots \end{pmatrix} \in \mathrm{GUE}(n) + \beta\mathbb{1}_n$$

$$\iff$$

$$\iff$$

spectra $\dots \succ y^{(4)} \succ y^{(3)} \succ \dots \succ y^{(1)}$

spectra $\mathbf{x}^{(1)} \prec \dots \prec \mathbf{x}^{(3)} \prec \mathbf{x}^{(4)} \prec \dots\dots$

COUPLING: $\rho = 3$ inequalities between spectra of three principal minors

$\max sp\ (B^{(3)}) \leq \min sp\ (A^{(1)})$

$\max sp\ (B^{(2)}) \leq \min sp\ (A^{(2)})$

$\max sp\ (B^{(1)}) \leq \min sp\ (A^{(3)})$

INTERLACING of Spectra!

$$\dots \succ y^{(4)} \succ y^{(3)} \preccurlyeq \mathbf{y}^{(2)} \cup \mathbf{x}^{(1)} \preccurlyeq \mathbf{y}^{(1)} \cup \mathbf{x}^{(2)} \preccurlyeq \mathbf{x}^{(3)} \prec \mathbf{x}^{(4)} \prec \dots$$

COUPLING: ρ inequalities between spectra of ρ principal minors

$$\max spec(B^{(i)}) \leq \min spec(A^{(\rho-i+1)}) \text{ , for } 1 \leq i \leq \rho.$$

$$\Updownarrow$$

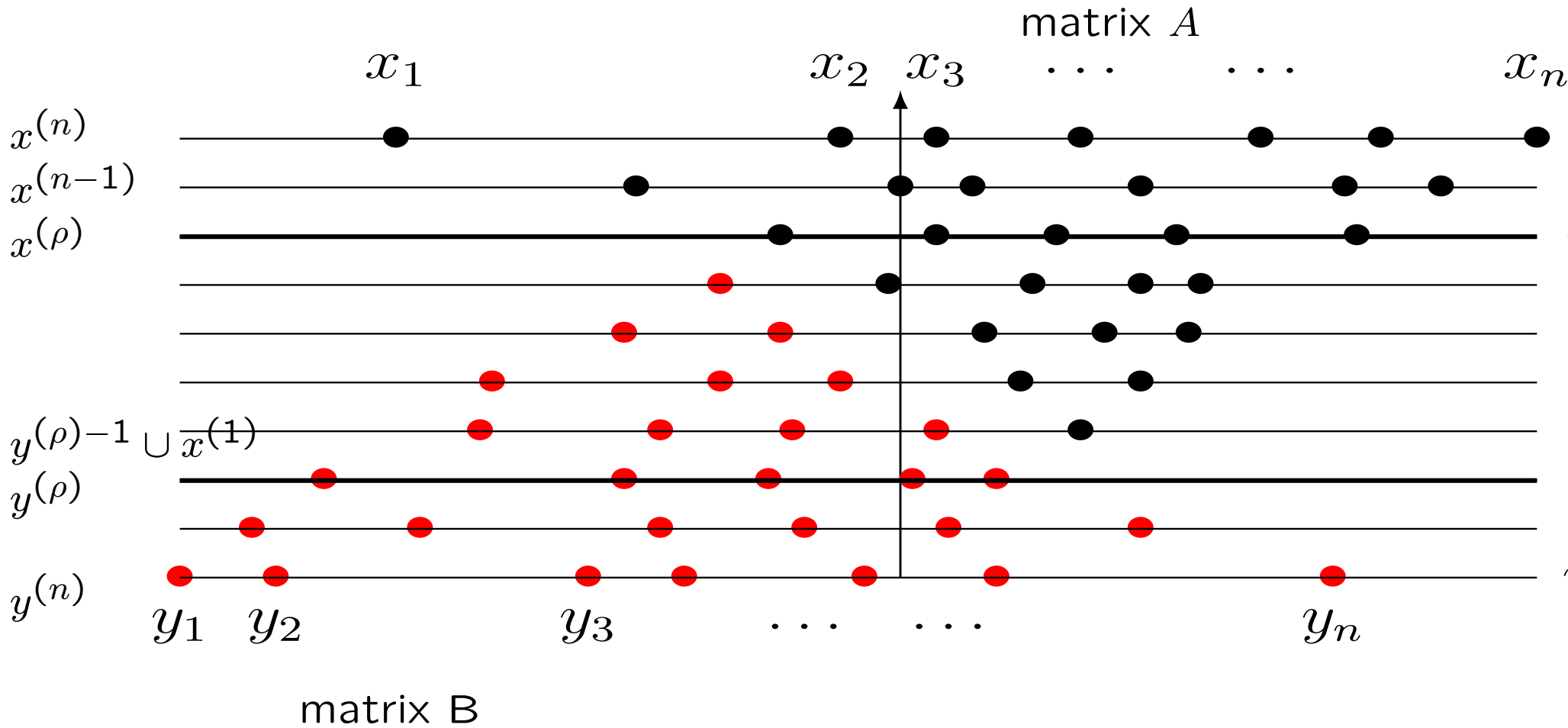
$$\max(\mathbf{y}^{(i)}) = y_i^{(i)} \leq x_1^{(\rho-i+1)} = \min(\mathbf{x}^{(\rho-i+1)})$$

Interlacing of spectra of consecutive minors?

$$\mathbf{y}^{(n)} \succ \ldots \succ \mathbf{y}^{(\rho)} \preccurlyeq \underbrace{\mathbf{y}^{(\rho-1)} \cup \mathbf{x}^{(1)} \preccurlyeq \ldots \preccurlyeq \mathbf{y}^{(1)} \cup \mathbf{x}^{(\rho-1)}}_{\rho-1} \preccurlyeq \mathbf{x}^{(\rho)} \prec \ldots \prec \mathbf{x}^{(n)}$$

$$B^{(\rho-1)} \cup A^{(1)} \qquad\qquad B^{(1)} \cup A^{(\rho-1)}$$

Coupled GUE-matrices for a given $\rho \geq 2$



$\rho = 5$

$$\mathbf{y}^{(n)} \succ \ldots \succ \mathbf{y}^{(\rho)} \preccurlyeq \mathbf{y}^{(\rho-1)} \cup \mathbf{x}^{(1)} \preccurlyeq \ldots \preccurlyeq \mathbf{y}^{(1)} \cup \mathbf{x}^{(\rho-1)} \preccurlyeq \mathbf{x}^{(\rho)} \prec \ldots \prec \mathbf{x}^{(n)}$$

Questions:
- What is the joint distribution of the spectra of A and B ?
- What is the distribution of the spectrum of A ?

Theorem : Given the eigenvalues $\mathbf{x}$ of A and the eigenvalues $\mathbf{y}$ of B, the density of the eigenvalues of the intermediate minors is given by:

$$\mathbb{P}\left(\bigcap_{1}^{n-1}\{\mathbf{x}^{(k)} \in d\mathbf{x}^{(k)},\ \mathbf{y}^{(k)} \in d\mathbf{y}^{(k)}\}\middle|\ \mathbf{x}^{(n)} \in d\mathbf{x},\ \mathbf{y}^{(n)} \in d\mathbf{y}\}\right) = \frac{d\mu_{\mathbf{x},\mathbf{y}}}{\mathrm{Vol}(\mathcal{C}_{\mathbf{xy}})}$$

Theorem : Given the eigenvalues $\mathbf{x}$ of A and the eigenvalues $\mathbf{y}$ of B, the density of the eigenvalues of the intermediate minors is given by:

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where

- $d\mu_{\mathbf{x},\mathbf{y}} =$ uniform measure on the *double cone* $\mathcal{C}_{\mathbf{x},\mathbf{y}}$

- $\mathcal{C}_{\mathbf{x},\mathbf{y}} = \left\{\begin{matrix}\text{double}\\ \text{cone}\end{matrix}\right\} = \left\{\begin{matrix}\mathbf{y} = \mathbf{y}^{(n)} \succ \ldots \succ \mathbf{y}^{(\rho)} \preccurlyeq \mathbf{y}^{(\rho-1)} \cup \mathbf{x}^{(1)} \preccurlyeq \ldots \\ \preccurlyeq \mathbf{y}^{(1)} \cup \mathbf{x}^{(\rho-1)} \preccurlyeq \mathbf{x}^{(\rho)} \prec \ldots \prec \mathbf{x}^{(n)} = \mathbf{x}\end{matrix}\right\} \subset \mathbb{R}^{n(n-1)}$

- $\mathrm{Vol}(\mathcal{C}_{\mathbf{x},\mathbf{y}}) = \int_{\mathbb{R}^{n(n-1)}} d\mu_{\mathbf{x},\mathbf{y}}$

(1)

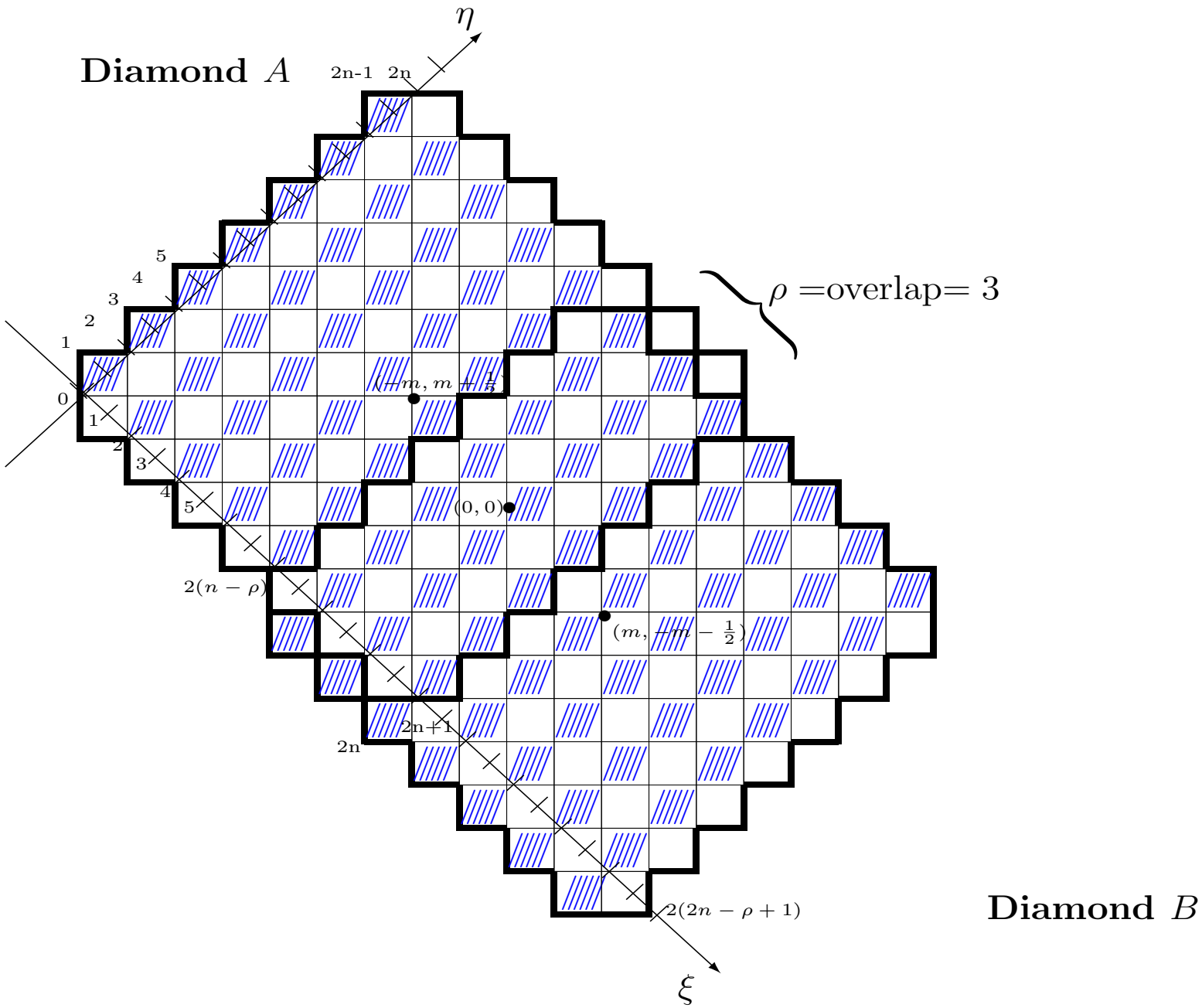
It follows that:

$$\mathbb{P}\left(\bigcap_{1}^{n}\{\mathbf{x}^{(k)} \in d\mathbf{x}^{(k)},\ \mathbf{y}^{(k)} \in d\mathbf{y}^{(k)}\}\right) = \frac{\rho_n^{\mathsf{GUE}}(\mathbf{x}-\beta)}{Vol(\mathcal{C}_{\mathbf{x}})}\frac{\rho_n^{\mathsf{GUE}}(\mathbf{y}+\beta)}{Vol(\mathcal{C}_{\mathbf{y}})}\frac{d\mathbf{x}d\mathbf{y}d\mu_{\mathbf{x},\mathbf{y}}}{\det(\mathbb{1}-\mathcal{K}^{\beta})_{[1,\rho]}},$$

where

- $\mathcal{K}^{\beta}(\lambda,\kappa) := \oint_{\Gamma_0}\frac{d\zeta}{(2\pi\mathsf{i})^2}\int_{0^+ + i\mathbb{R}\uparrow}\frac{d\omega}{\omega-\zeta}\frac{e^{-2\zeta^2+4\beta\zeta}}{e^{-2\omega^2+4\beta\omega}}\frac{\zeta^{\kappa}}{\omega^{\lambda+1}} \qquad \lambda,\kappa \in \mathbb{Z}$

4. Domino tiling of a double Aztec diamond



Double Aztec diamond of type $n = 7$, with overlap $=\rho = 3$

Coordinate system: (ξ, η)

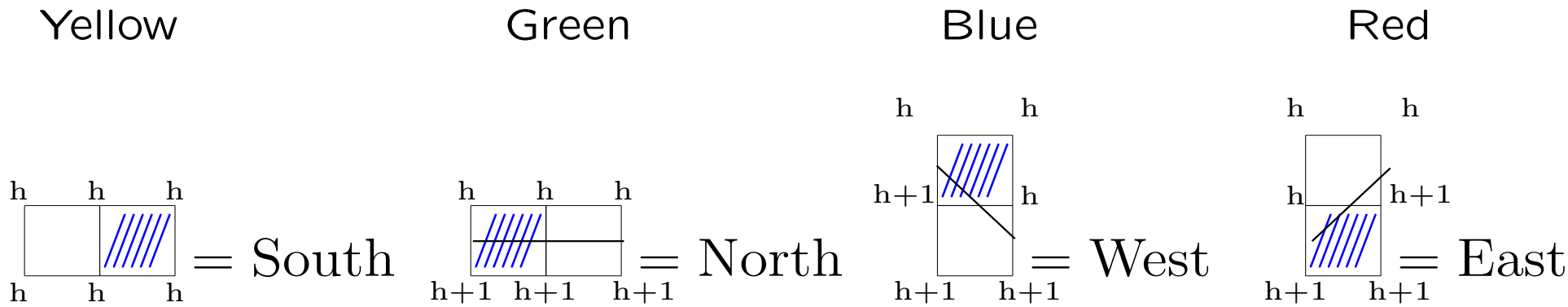


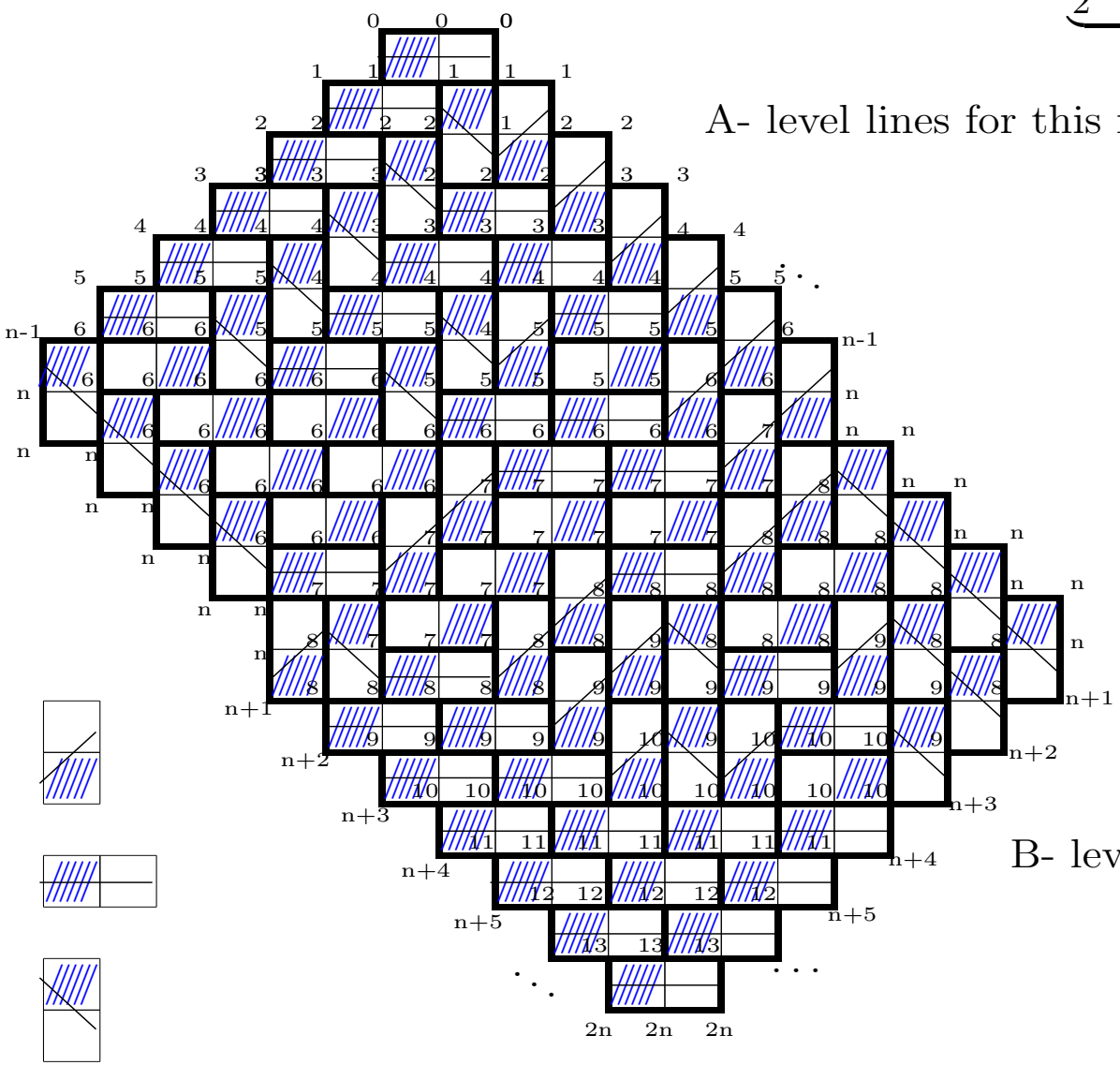
Figure 4. Height function on domino's and level-lines.

Assume

- *weight* $= a > 0$ *on horizontal domino's*
- *weight* $= 1$ *on vertical domino's*

Random cover with domino's: two groups of non-intersecting paths

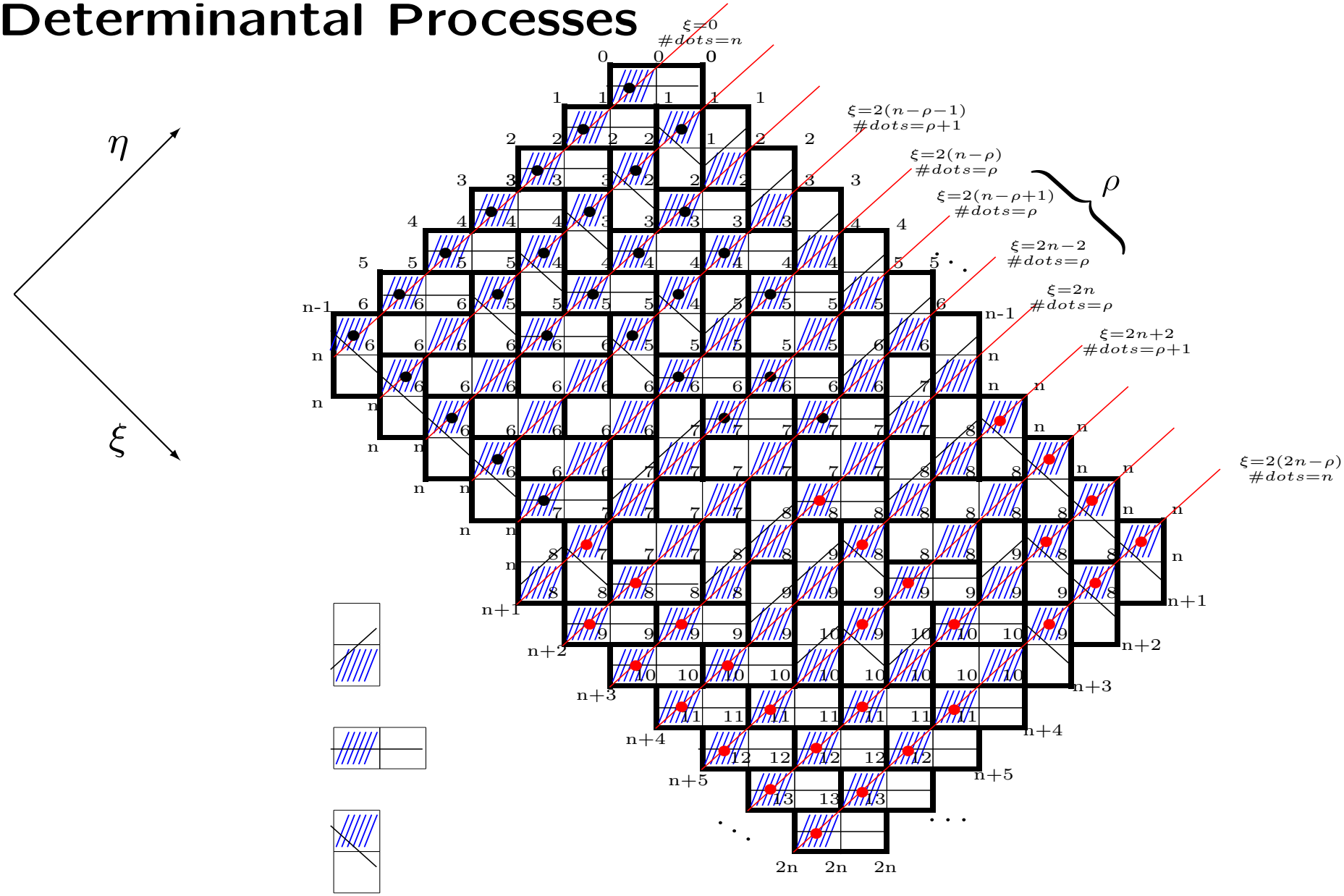
level lines corresponding to height $\underbrace{\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ldots,n-\frac{1}{2}}_{A-\text{level lines}},\ \underbrace{n+\frac{1}{2},\ldots,\ 2n-\frac{1}{2}}_{B-\text{level lines}}$



A- level lines for this random surface

B- level lines for this random surface

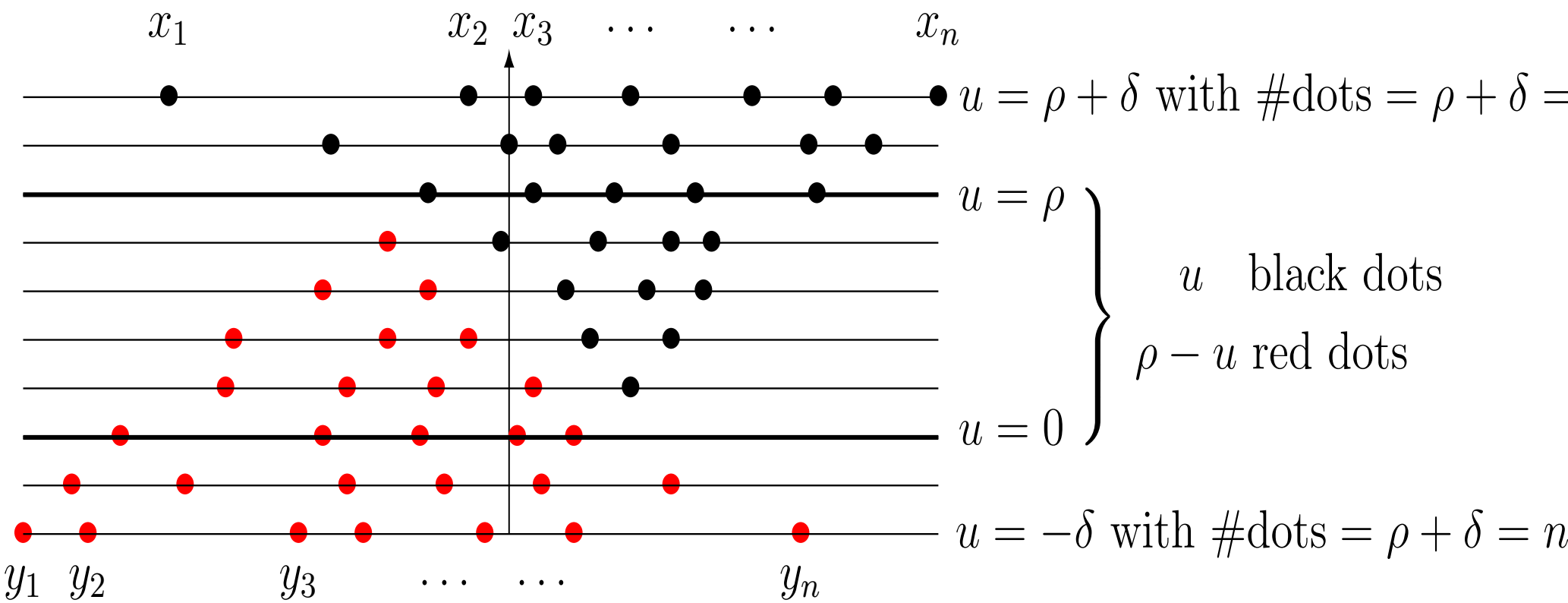
Determinantal Processes



Each time a line $\xi = 2k$ intersects an

$$\left.\begin{array}{l}\text{(1) } A\text{-level line: put a blue dot in that square}\\ \text{(2) } B\text{-level line: put a red dot in that square}\end{array}\right\} \textit{interlacing}!$$

The (discrete) interlacing can schematically be represented by the same picture as before:



$x_i,\ y_i \in \mathbb{Z}$

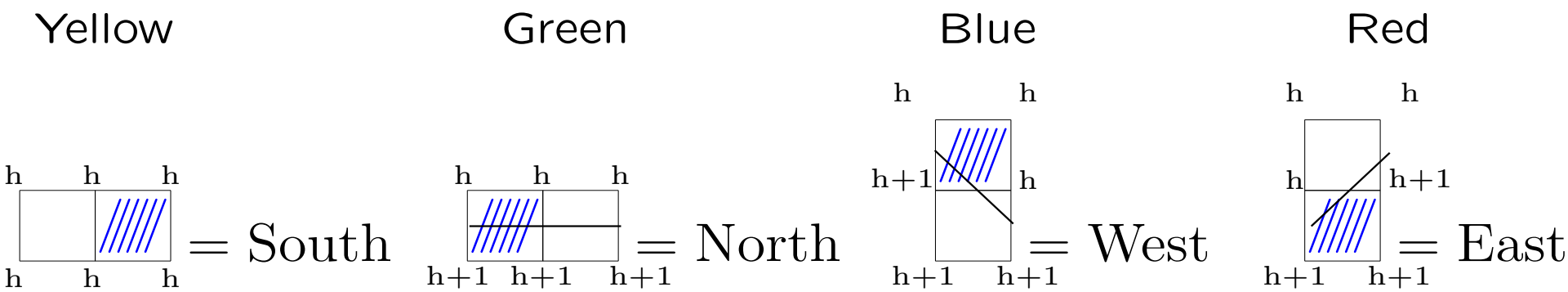
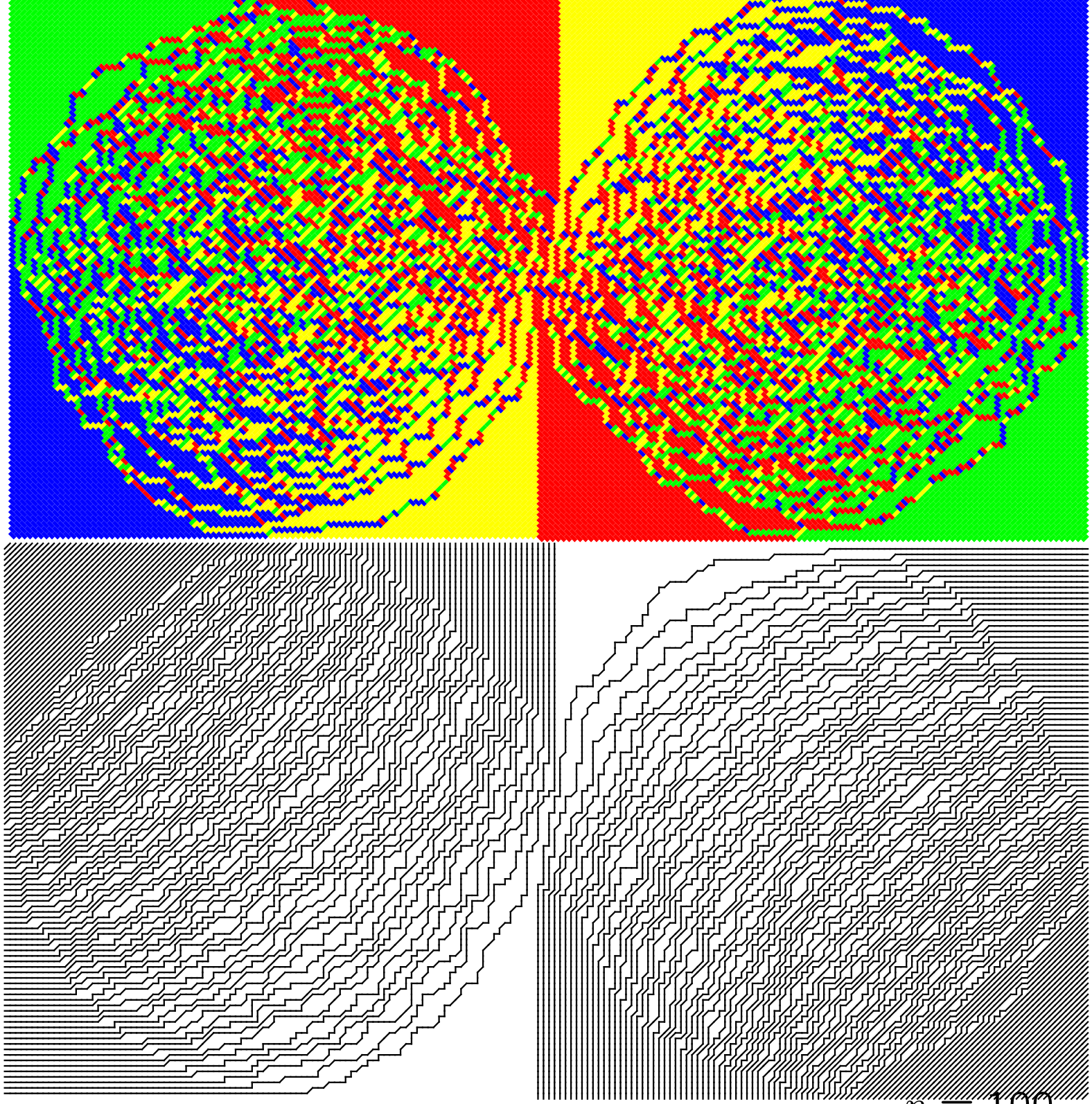
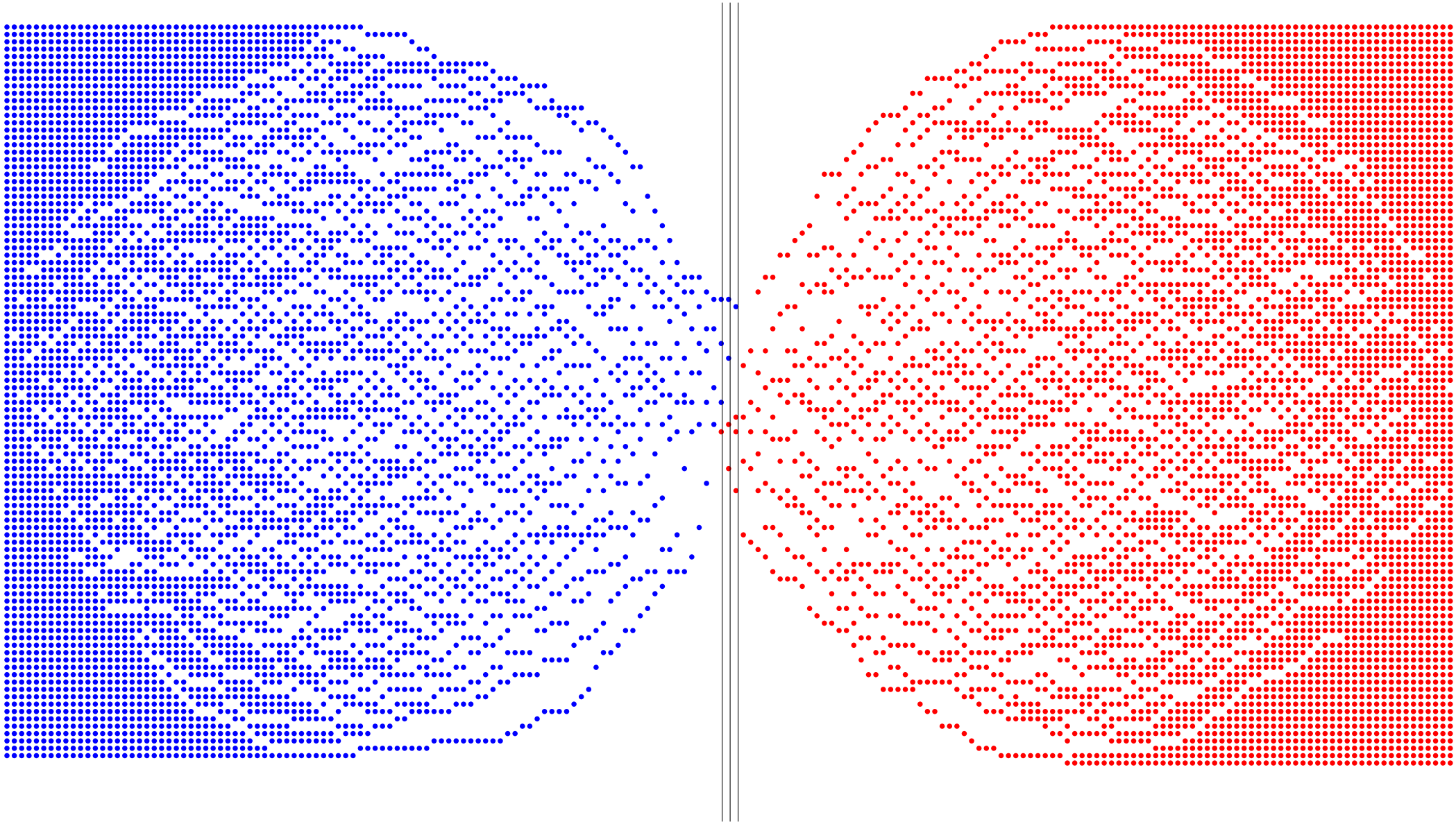


Figure 4. Height function on domino's and level-lines.



$n = 100\ ,\ \ \rho = 4,\ \ a = 1$



$n = 100$, $\rho = 4$: two circles touching ("tacnode") on the overlap of the two diamonds ! Exactly three lines containing a mix of red and blue dots.

Theorem 1. **The dot-process for the double Aztec diamond is determinantal**

Theorem 2. **The scaling limit of this dot-process is determinantal in the limit $n \to \infty$, in the $n^{-1/2}$-scale along the lines**

("Tacnode kernel")

Theorem 3.

$$\left\{\begin{array}{l}\textbf{The statistics of the eigenvalues}\\ \textbf{of the intermediate minors } A^{(i)} \textbf{ and } B^{(i)}\\ \textbf{of the coupled GUE-matrices } A \textbf{ and } B\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}\textbf{The statistics of the}\\ \textbf{dot-particles in the}\\ \textbf{double-Aztec diamonds}\\ \textbf{in the } n^{-1/2}\textbf{-scale,}\\ \textbf{when } n \to \infty\end{array}\right.$$

Tacnode kernel, two parameters: $\begin{cases} \beta = \text{speed at which } \ a \to 1 \\ \rho = \text{overlap of two diamonds} \end{cases}$

$$\mathbb{K}^{\mathsf{TAC}}_{\beta,\rho}(u_1, y_1; u_2, y_2) = \mathbb{K}^{\mathsf{OneAztec}}(u_1, \beta - y_1;\ u_2, \beta - y_2) + 2\Big\langle (\mathbb{1} - \mathcal{K}^{\beta}(\lambda, \kappa))^{-1}\ \mathcal{A}^{\beta, y_1 - \beta}_{u_1}(\kappa), \mathcal{B}^{\beta, y_2 - \beta}_{u_2}(\lambda) \Big\rangle_{\geq -\rho}$$

with $u_i \in \mathbb{Z}, \qquad y_i \in \mathbb{R}, \qquad \lambda, \mu \in \mathbb{Z}$

Tacnode kernel, two parameters: $\begin{cases} \beta = \text{such that} \quad a = 1 - \beta/\sqrt{n/2} \\ \rho = \text{overlap of two diamonds} \end{cases}$

$$\mathbb{K}^{\mathsf{TAC}}_{\beta,\rho}(u_1, y_1; u_2, y_2) = \mathbb{K}^{\mathsf{minor}}(u_1, \beta - y_1;\ u_2, \beta - y_2)$$

$$+ 2\Big\langle (\mathbb{1} - \mathcal{K}^{\beta})^{-1}\ \mathcal{A}^{\beta, y_1 - \beta}_{u_1}(\lambda), \mathcal{B}^{\beta, y_2 - \beta}_{u_2}(\lambda) \Big\rangle_{\geq -\rho}$$

$$\uparrow$$

(Finite rank perturbation of GUE-minor kernel!) $\lambda, \mu \in \mathbb{Z}$

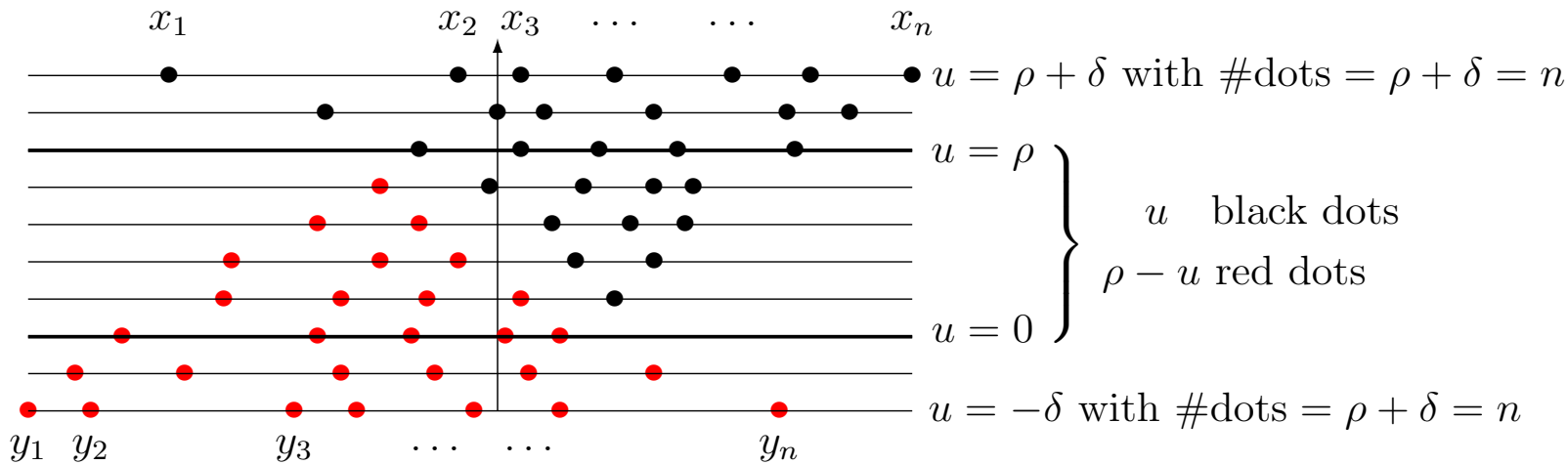
$$\mathbb{K}^{\mathsf{minor}}(n, x; n', x') := -\mathbb{1}_{n > n'} 2^{n-n'} \mathbb{H}^{n-n'}(x - x')$$

$$+ \frac{2}{(2\pi \mathsf{i})^2} \int_{\Gamma_0} dz \int_L \frac{dw}{w - z} \frac{e^{-z^2 + 2zx}}{e^{-w^2 + 2wx'}} \frac{w^{n'}}{z^n}$$

$$\mathcal{K}^{(\beta)}(\lambda, \kappa) := \oint_\Gamma \frac{d\zeta}{(2\pi \mathsf{i})^2} \int_L \frac{d\omega}{\omega - \zeta} \frac{e^{-2\zeta^2 + 4\beta\zeta}}{e^{-2\omega^2 + 4\beta\omega}} \frac{\zeta^\kappa}{\omega^{\lambda+1}}, \qquad \lambda, \kappa \in \mathbb{Z}$$

$$\mathcal{A}^{\beta, y}_v(\kappa) := \oint_\Gamma \frac{d\zeta}{(2\pi \mathsf{i})^2} \int_L \frac{d\omega}{\zeta - \omega} \frac{e^{-\zeta^2 - 2y\zeta}}{e^{-2\omega^2 + 4\beta\omega}} \frac{\zeta^{-v}}{\omega^{\kappa+1}} + \int_L \frac{d\zeta}{2\pi \mathsf{i}} \frac{e^{\zeta^2 - 2\zeta(y + 2\beta)}}{\zeta^{v+\kappa+1}}$$

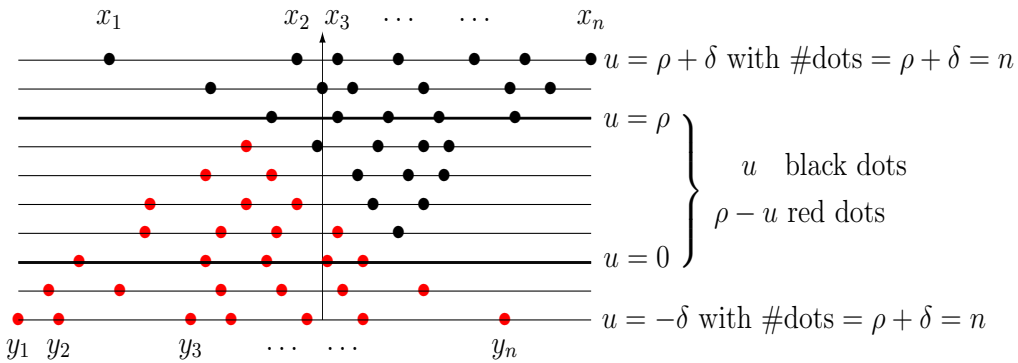
$$\mathcal{B}^{\beta, y}_u(\lambda) := \oint_\Gamma \frac{d\zeta}{(2\pi \mathsf{i})^2} \int_L \frac{d\omega}{\zeta - \omega} \frac{e^{-2\zeta^2 + 4\zeta\beta}}{e^{-\omega^2 - 2\omega y}} \frac{\zeta^\lambda}{\omega^{-u}} + \oint_\Gamma \frac{d\omega}{2\pi \mathsf{i}} \frac{\omega^{u+\lambda}}{e^{\omega^2 - 2\omega(y + 2\beta)}}.$$



$$\mathbb{P}(\mathbf{x}^{(n)} \in d\mathbf{x}) = \det\left(\mathbb{K}^{\mathsf{TAC}}_{\beta,\rho}(-\delta, y_i; -\delta, y_j)\right)_{1\le i,j\le n}$$

$$\mathbb{P}\left(\mathbf{x}^{(n)} \in d\mathbf{x},\ \mathbf{y}^{(n)} \in d\mathbf{y}\}\right)$$

$$= \det\left(\begin{array}{c|c} \left(\mathbb{K}^{\mathsf{TAC}}_{\beta,\rho}(-\delta, y_i; -\delta, y_j)\right)_{1\le i,j\le \rho+\delta} & \left(\mathbb{K}^{\mathsf{TAC}}_{\beta,\rho}(-\delta, y_i; n, x_j)\right)_{1\le i,j\le n} \\ \hline \left(\mathbb{K}^{\mathsf{TAC}}_{\beta,\rho}(n, x_i; -\delta, y_j)\right)_{1\le i,j\le n} & \left(\mathbb{K}^{\mathsf{TAC}}_{\beta,\rho}(n, x_i; n, x_j)\right)_{1\le i,j\le n} \end{array}\right) d\mathbf{x}d\mathbf{y}$$



Distribution of dots at a distance δ away from overlap:

density given by: with $\mathbf{y} = (y_1, \ldots, y_n)$ and $n = \rho + \delta$

$$\mathbb{P}(\mathbf{y}^{(n)} \in d\mathbf{y}) = C_{\rho,\delta} \Delta_n(\mathbf{y}) \widetilde{\Delta}_n(\mathbf{y}) \prod_1^n \frac{e^{-(y_i+\beta)^2}}{\sqrt{\pi}}$$

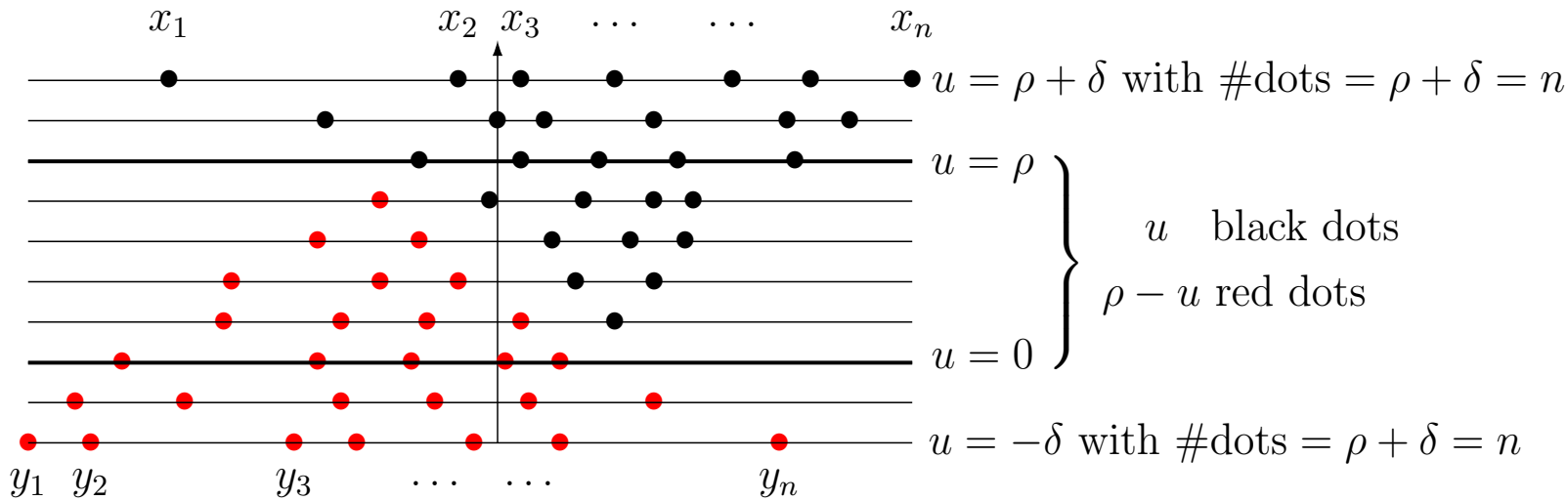
where

$$\Delta_n(\mathbf{y}) = \det \begin{pmatrix} 1 \\ y \\ \vdots \\ y^{\delta-1} \\ y^{\delta} \\ \vdots \\ y^{\delta+\rho-1} \end{pmatrix}_{y=y_1,\ldots,y_n} \qquad \widetilde{\Delta}_n(\mathbf{y}) := \det \begin{pmatrix} 1 \\ y \\ \vdots \\ y^{\delta-1} \\ \Phi_\delta(\beta - y) \\ \vdots \\ \Phi_{\delta+\rho-1}(\beta - y) \end{pmatrix}_{y=y_1,\ldots,y_n}$$

This formula also applies to the coupled random matrices !

Define for $k \geq 0$

$$\Phi_k(\eta) := \frac{2^k}{\sqrt{\pi} k!} \int_0^\infty \xi^k e^{-(\xi-\eta)^2} d\xi, \qquad k \geq 0,$$

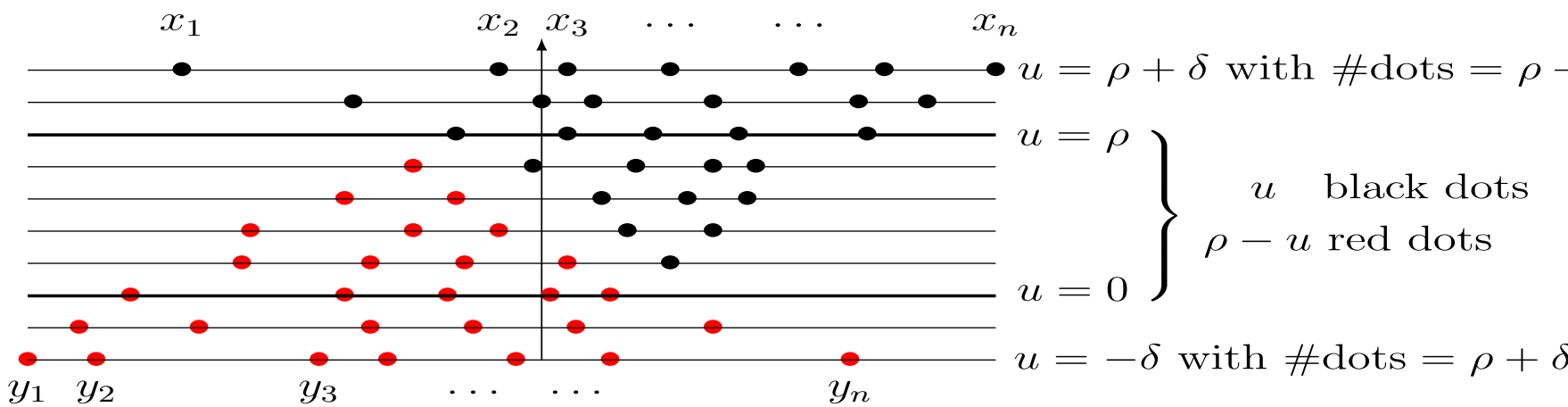


In the overlap $(1 \leq \delta \leq \rho - 1)$ the joint density of the points $\mathbf{x}^{(\delta)}, \mathbf{y}^{(\rho-\delta)}$, taken together, is given by

$$\mathbb{P}(\{(\mathbf{x}^{(\delta)}, \mathbf{y}^{(\rho-\delta)}) \in d\mathbf{z}\})$$

$$= c_{\rho,\delta} \det \begin{pmatrix} \frac{y^{\delta-1}}{\sqrt{\pi}} e^{-(\beta-y)^2} \\ \vdots \\ \frac{1}{\sqrt{\pi}} e^{-(\beta-y)^2} \\ \Phi_0(\beta - y) \\ \vdots \\ \Phi_{\rho-\delta-1}(\beta - y) \end{pmatrix}_{y=z_1,\ldots,z_\rho} \times \det \begin{pmatrix} \frac{y^{\rho-\delta-1}}{\sqrt{\pi}} e^{-(y+\beta)^2} \\ \vdots \\ \frac{1}{\sqrt{\pi}} e^{-(y+\beta)^2} \\ \Phi_0(y + \beta) \\ \vdots \\ \Phi_{\delta-1}(y + \beta) \end{pmatrix}_{y=z_1,\ldots,z_\rho} d\mathbf{z}.$$

This formula also applies to the coupled random matrix problem !



Joint density of dots at the same distance δ below and above the overlap:

$$\mathbb{P}\left(\mathbf{x}^{(n)} \in d\mathbf{x},\ \mathbf{y}^{(n)} \in d\mathbf{y}\right) = \frac{\rho_n^{\mathsf{GUE}}(\mathbf{x}-\beta)\ \rho_n^{\mathsf{GUE}}(\mathbf{y}+\beta)}{\det(\mathbb{1}-\mathcal{K}^{\beta})_{[1,\rho]}} \frac{\mathsf{Vol}\left(\mathcal{C}_{\mathbf{xy}}\right)}{Vol(\mathcal{C}_{\mathbf{x}})\,Vol(\mathcal{C}_{\mathbf{y}})} d\mathbf{x}d\mathbf{y}$$

Thank you !